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Construction of Aharonov–Berry's superoscillations

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Abstract

A simple method is described for constructing functions that superoscillate at an arbitrarily chosen wavelength scale. Our method is based on the technique of oversampled signal reconstruction. This allows us to explicitly demonstrate that the observed fragility of superoscillating wavefunctions is indeed mathematically closely connected to what in the communication theory community is known as the instability of oversampled signal reconstruction, confirming a previous conjecture. This is of potential interest, for example, concerning the understanding of the practical difficulties in experimentally producing superoscillatory wavefunctions.

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1. Introduction

Consider a wavefunction with momentum cutoff $p_{\max}$, that is,

$$\psi(x) = \frac{1}{2\pi\hbar}\int_{-p_{\max}}^{p_{\max}} \tilde{\psi}(p)\,\mathrm{e}^{\mathrm{i}xp/\hbar}\,\mathrm{d}p.$$

Since such wavefunctions are linear combinations of plane waves with wavelengths at least as large as $\lambda_{\min} = h/p_{\max}$, the existence among the wavefunctions which oscillate arbitrarily quickly over arbitrarily large stretches may come as a surprise. Such wavefunctions have been called superoscillatory. The existence of superoscillations means that there exist functions bandlimited to a certain frequency but that oscillate at arbitrarily larger frequencies over finite but otherwise arbitrarily long intervals.

The existence of superoscillating functions was pointed out by Aharonov [1], and their physical implications have been discussed in various contexts, including the trans-Planckian problem of black holes, evanescent waves and seeming superluminal propagation. An account of superoscillatory behaviour and its physical consequences can be found in [2]. One particularly intriguing aspect, discussed in [3], is that particles of low momentum pick up large momenta when passing through a slit, if the part of their wavefunction which passes through is superoscillatory.

Superoscillating functions have been constructed by Berry [4, 5] and Qiao [6]. The construction in [3, 7] yields the function of minimum energy (square of the L^2 norm) that interpolates a given set of amplitudes. The more general procedure in [8] is able to prescribe not only amplitudes, but also linear constraints, such as derivatives and integrals.

The purpose of this paper is to describe a construction that produces superoscillatory functions which locally oscillate at a wavelength scale that is by some fixed factor $T < 1$ smaller than the cutoff scale. This allows us to explore the connection between the fine and coarse structure of band-limited functions while keeping the behaviour of the superoscillating functions under control.

In particular, we can here answer a question that has appeared in the literature. It is known, see [7], that superoscillating wavefunctions possess highly fine-tuned coefficients when expanded in any easy-to-use basis of the function space. In this sense, superoscillatory functions are 'fragile'. This indicates that extensive care will need to be taken when trying to produce superoscillatory wavefunctions in the lab. Here, we will shed new light on this fragility of superoscillating wavefunctions, namely from an information theoretic perspective. Berry has quoted Daubechies, [5], as conjecturing that superoscillations should be connected to the problem that the reconstruction of the information contained in a signal is unstable against small errors in the measurement of the signals' amplitudes if the amplitudes are measured at an oversampling rate. Here we show explicitly that the fragility of superoscillations is indeed directly connected to the instability in oversampling signals reconstruction. To this end, we here explicitly construct the superoscillatory wavefunctions in a novel way, namely by using the technique of reconstructing functions from an oversampled set of amplitudes.

We begin by recalling that a function is called bandlimited if its Fourier transform vanishes for sufficiently large values of the argument. Correspondingly, a wavefunction or field is bandlimited in space or time if it possesses a momentum or energy cutoff. Let us consider bandlimited functions whose amplitudes are determined by N coefficients c_k:

$$f(t) = \sum_{k=0}^{N-1} c_k \operatorname{sinc}(t-k),$$

where $\operatorname{sinc} x = \sin(\pi x)/(\pi x)$. The collection of all such $f(t)$ is a linear subspace of dimension N of Paley–Wiener space (the Hilbert space of square-integrable functions bandlimited to $\frac{1}{2}$). The values of the coefficients c_k and those of $f(t)$ are related in a very simple way because the sinc function has the interpolation property[4]. Plainly, $c_k = f(k), 0 \leqslant k < N$, whereas $f(k) = 0$ for any other integer k. The functions considered are therefore concentrated, as far as the values $f(k)$ are concerned, in the interval $[0, N-1]$.

The collection of values $f(k)$ can be associated with 'scale 1 behaviour', and $f(kT)$ with 'scale T behaviour'. Because $f(k) = c_k$, the coefficients c_k determine the behaviour of $f(t)$ at scale 1. As shown next, they can control the behaviour of $f(t)$ at scale T, with T as small as desired. More precisely, the c_k can be chosen so that N relations of the form $f(kT) = a_k$ are satisfied. This remains true even if the kT lie at an arbitrary distance from the interval $[0, N-1]$. The tools used are known as sampling expansions.

[4] That is, $\operatorname{sinc} k = \delta_k$ (the Kronecker symbol), for integer k.

2. The construction

2.1. The sampling expansions

Any square-integrable function f, bandlimited (without loss of generality) to $\frac{1}{2}$, satisfies the classical sampling formula [9, 10]

$$f(t) = \sum_{k=-\infty}^{+\infty} f(k)\,\mathrm{sinc}(t-k). \tag{1}$$

This can be obtained by termwise integration of the Fourier series expansion of the Fourier transform of f, which by definition is zero outside $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The coefficients of the Fourier series turn out to be the samples $f(k)$.

A simple variant of the same argument leads to the 'oversampled expansion'

$$f(t) = T \sum_{k=-\infty}^{+\infty} f(kT)\,\mathrm{sinc}(t-kT), \qquad (T < 1). \tag{2}$$

These sampling expansions are well known. In fact, replacing the Fourier transform, to which they are closely related, with other integral transforms leads to expansions that involve kernels other than the sinc, and for which oversampling is still meaningful [11].

2.2. The mappings between scale 1 and scale T

Setting $t = \ell T$ in (1) leads to

$$f(\ell T) = \sum_{k=-\infty}^{+\infty} f(k)\,\mathrm{sinc}(\ell T - k), \tag{3}$$

whereas setting $t = \ell$ in (2) yields

$$f(\ell) = T \sum_{k=-\infty}^{+\infty} f(kT)\,\mathrm{sinc}(\ell - kT). \tag{4}$$

Equation (3) specifies the mapping between the samples $f(k)$ taken at what is usually called the Nyquist rate (twice the highest frequency present in the function) and the samples $f(kT)$ taken at the higher rate $1/T$. Equation (4) specifies the mapping in the opposite direction. These mappings between the values of f at grids of size 1 and T, and which apparently have not been noticed before, will be useful later on.

2.3. Finite oversampled expansions are trivial

The coefficients $f(k)$ in equation (1) are independent of each other. It is possible to set all but finitely many of them to zero, and still obtain nonzero band-limited functions. In fact, (1) is an interpolation formula, and any set of $f(k)$, subject only to $\sum |f(k)|^2 < \infty$, determines a band-limited, square-integrable function.

In contrast, the coefficients $f(kT)$ of the oversampled expansion (2) are not independent, and cannot be arbitrarily prescribed. As is well known in the communication theory community, this implies an instability in the reconstruction of signals from oversampled amplitudes. This is because as the oversampled amplitudes are subjected to noise the amplitudes start to become independent, thereby beginning to violate the interdependence of the exact oversampled amplitudes. It has been conjectured that this instability in oversampled reconstruction is related to superoscillations.

To show that this is indeed the case, we first note that finite nontrivial oversampled expansions (2) do not exist, i.e. there is no band-limited function that would take finitely many pre-specified oversampled amplitudes while all the other of its oversampled amplitudes are zero. Concretely, let J be a finite index set, and T any real satisfying $0 < T < 1$. We claim that there exists no nonzero square-integrable function bandlimited to $\frac{1}{2}$ that satisfies the expansion (2) with a finite number of nonzero coefficients $f(kT)$.

The proof is a direct consequence of the observation that a band-limited function that vanishes on a set of points of sufficiently high density must vanish everywhere, but such results lie deeper than necessary. The following elementary argument suffices. Let $\vec{a}$ be the vector formed by the coefficients $f(kT)$ to be determined; replace t with ℓT in (2) and rearrange as in [11] to obtain an equation of the form $M\vec{a} = 0$. The matrix M is nonsingular [11], and so $\vec{a} = 0$, meaning that $f(t)$ is identically zero.

We conclude that it is impossible to construct superoscillating band-limited functions by prescribing finitely many amplitudes as in

$$f(kT) = a_k, \qquad k \in J, \qquad \mathrm{card}(J) < \infty, \quad 0 < T < 1, \tag{5}$$

and then setting all remaining samples to zero: $f(kT) = 0, k \notin J$. If this were possible, the behaviour of the superoscillating segment could be controlled simply by controlling the a_k, say, $a_k = (-1)^k$. Unfortunately, as seen, the only band-limited function that satisfies $f(kT) = 0$ for $k \notin J$ is $f = 0$.

2.4. *Method using oversampled reconstruction*

The impossibility of building a superoscillating function based on a finite oversampled series of the type (2) suggests the following question: can a superoscillating function be built by prescribing its values on a grid of size T as in (5), and simultaneously enforcing a finite expansion of type (1), rather than (2)? In other words, are there functions satisfying (5) and also

$$f(t) = \sum_{k \in I} f(k)\,\mathrm{sinc}(t-k) \tag{6}$$

for some *finite* index set I, say, $I = \{0, 1, \ldots, N-1\}$?

There is one obvious necessary condition, namely, the number of degrees of freedom in this sum (the cardinal of I) must at least equal the number of interpolatory constraints (the cardinal of J). For simplicity, we will assume that $\mathrm{card}(I) = \mathrm{card}(J)$, although $\mathrm{card}(I) \geqslant \mathrm{card}(J)$ could also have been considered.

To show that the necessary condition mentioned is also sufficient, we need to show that there are functions given by the finite expansion (6), which are by construction bandlimited to $\frac{1}{2}$, that interpolate a number of prescribed points separated by T, as in (5).

To construct them, we first show that their coefficients satisfy a set of linear equations, and then we show that the associated matrix is nonsingular. To obtain the equations, rewrite (4) as

$$f(\ell) = T \sum_{k \notin J} f(kT)\,\mathrm{sinc}(\ell - kT) + g(\ell),$$

where

$$g(\ell) = T \sum_{k \in J} a_k\,\mathrm{sinc}(\ell - kT)$$

is known (computable for any ℓ, since the a_k and J are given). Substitution of (6), which is the finite version of (3), leads to

$$f(\ell) = T \sum_{k \notin J} \sum_{m \in I} f(m) \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT) + g(\ell)$$
$$= T \sum_{m \in I} f(m) \sum_{k \notin J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT) + g(\ell).$$

Restriction to $\ell \in I$ leads to a set of linear equations of the form $\vec{f} = M\vec{f} + \vec{g}$, where $\vec{f}$ is the vector of all $f(\ell)$, $\ell \in I$, and similarly for $\vec{g}$. The square matrix M has elements

$$M_{m\ell} = T \sum_{k \notin J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT), \qquad m, \ell \in I.$$

It is possible to solve the obtained equations for the card(I) unknown quantities in the vector $\vec{f}$ if $I - M$ is nonsingular, and we will see shortly that this is indeed true. We find it convenient to rewrite the equation in a different form. Note that the replacement of $f(t)$ with $\operatorname{sinc}(t - m)$ in (2) leads to

$$T \sum_{k=-\infty}^{+\infty} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT) = \operatorname{sinc}(\ell - m) = \delta_{\ell m},$$

and consequently

$$M_{m\ell} = T \sum_{k \notin J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT)$$
$$= \delta_{\ell m} - T \sum_{k \in J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT).$$

In this form, M can be computed using a finite sum, since J has finitely many elements. The equations $\vec{f} = M\vec{f} + \vec{g}$ become

$$\sum_{m \in I} f(m) T \sum_{k \in J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT) = \sum_{k \in J} a_k \operatorname{sinc}(\ell - kT),$$

or, equivalently,

$$A\vec{f} = \vec{g}, \tag{7}$$

with

$$A_{m\ell} = T \sum_{k \in J} \operatorname{sinc}(kT - m) \operatorname{sinc}(\ell - kT), \qquad m, \ell \in I.$$

Note that A and $\vec{g}$ depend only on finitely many known quantities. Unlike $\vec{g}$, the matrix A is independent of the prescribed amplitudes $f(kT) = a_k$ and depends only on T and the index sets I and J.

We now turn to the question of the nonsingularity of the matrix A. Its elements are given by

$$A_{m\ell} = T \sum_{k \in J} \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi(kT - m)x} \, \mathrm{d}x \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi(\ell - kT)y} \, \mathrm{d}y$$

and therefore the quadratic form $\vec{v}^{\dagger} A \vec{v}$ can be written

$$T \sum_{m} \sum_{\ell} v_m^* v_\ell \sum_{k \in J} \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi(kT - m)x} \, \mathrm{d}x \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi(\ell - kT)y} \, \mathrm{d}y$$
$$= T \sum_{k \in J} \iint_{-1/2}^{1/2} \left(\sum_{m} v_m^* \mathrm{e}^{-\mathrm{i}2\pi m x} \right) \left(\sum_{\ell} v_\ell \, \mathrm{e}^{\mathrm{i}2\pi \ell y} \right) \mathrm{e}^{\mathrm{i}2\pi k T (x - y)} \, \mathrm{d}x \, \mathrm{d}y.$$

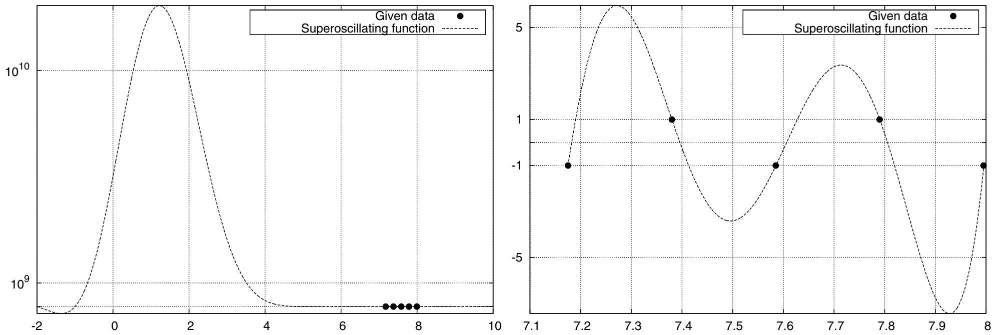

Figure 1. An example of a superoscillating function generated using the method described (and $T = 0.205$). Its global behaviour is depicted on the left. Note how the energy is concentrated near $[0, 4]$. The detail on the right corresponds to the superoscillating segment, and confirms the interpolating behaviour at the scale T, at a distance from $[0, 4]$.

Upon setting

$$V(z) = \sum_{\ell} v_\ell \, \mathrm{e}^{\mathrm{i}2\pi \ell z},$$

this becomes

$$\begin{aligned} \vec{v}^\dagger A \vec{v} &= T \sum_{k \in J} \int_{-1/2}^{1/2} V^*(x) \, \mathrm{e}^{\mathrm{i}2\pi kTx} \, \mathrm{d}x \int_{-1/2}^{1/2} V(y) \, \mathrm{e}^{-\mathrm{i}2\pi kTy} \, \mathrm{d}y \\ &= T \sum_{k \in J} \left| \int_{-1/2}^{1/2} V(x) \, \mathrm{e}^{-\mathrm{i}2\pi kTx} \, \mathrm{d}x \right|^2 \geqslant 0. \end{aligned}$$

This expression can be zero only if $V(y)$ is identically zero, and this in turn can happen only if all the v_i are zero. It follows that $v^\dagger A v > 0$ when v is a nonzero vector. Thus, A is positive definite, hence nonsingular.

An example of a superoscillating signal generated using this method is given in figure 1.

3. Conclusion

We have described a construction of superoscillating functions, through which a function can be made to oscillate at rates arbitrarily higher than any of its Fourier components. The function is subject to finitely many interpolatory constraints at scale T, with T as small as desired. The nature of the superoscillating stretch can be controlled by varying the number N of constraints, their separation and amplitudes. The superoscillating behaviour at scale T is induced by values $f(k)$, $k = 0, 1, \ldots, N - 1$, which determine the behaviour of the function at scale 1. Furthermore, the superoscillating stretch can be located at arbitrary distance from the interval $[0, N - 1]$ where the scale-1 behaviour is determined.

We note that, given that we are working here with linear constraints, the fact that such superoscillations can be constructed is guaranteed, in principle, by the results in [8]. The key novel point here is that we are explicitly constructing these superoscillations and that we do so using the method of oversampling. This allowed us, in particular, to show that, as had been conjectured in [5], the instability in oversampled reconstruction in communication theory translates into the fragility of the construction of superoscillations. Our results therefore contribute to the understanding of the difficulties of fine tuning that are to be expected when trying to experimentally produce superoscillating wavefunctions or fields.

References

[1] Aharonov Y, Anandan J, Popescu S and Vaidman L 1990 Superpositions of time evolutions of a quantum system and a quantum time-translation machine *Phys. Rev. Lett.* **64** 2965–8
[2] Kempf A 2000 Black holes, bandwidths and Beethoven *J. Math. Phys.* **41** 2360–74
[3] Kempf A and Ferreira P J S G 2004 Unusual properties of superoscillating particles *J. Phys. A: Math. Gen.* **37** 12067–76
[4] Berry M V 1994 Evanescent and real waves in quantum billiards and Gaussian beams *J. Phys. A: Math. Gen.* **27** L391–8
[5] Berry M 1994 Faster than Fourier *Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov* ed J S Anandan and J L Safko (Singapore: World Scientific) pp 55–65
[6] Qiao W 1996 A simple model of Aharonov–Berry's superoscillations *J. Phys. A: Math. Gen.* **29** 2257–8
[7] Ferreira P J S G and Kempf A 2006 Superoscillations: faster than the Nyquist rate *IEEE Trans. Signal Processing* **54** 3732–40
[8] Calder M S and Kempf A 2005 Analysis of superoscillatory wave functions *J. Math. Phys.* **46** 012101
[9] Higgins J R 1996 *Sampling Theory in Fourier and Signal Analysis: Foundations* (Oxford: Oxford University Press)
[10] Benedetto J J and Ferreira P J S G 2001 *Modern Sampling Theory: Mathematics and Applications* (Boston: Birkhäuser)
[11] Ferreira P J S G 1992 Incomplete sampling series and the recovery of missing samples from oversampled band-limited signals *IEEE Trans. Signal Processing* **40** 225–7