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Linear combinations of B-splines as generating functions for signal approximation ☆

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Abstract

The advantages of B-splines for signal representation are well known. This paper explores a fact that seems to be less well known, namely, the possibility of using linear combinations of B-splines to obtain representations that are more stable than the usual ones. We give the best possible Riesz bounds for these linear combinations and calculate their duals, in a generalized sampling context.
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1. Introduction

The selection of the interpolating or kernel function is a key step in obtaining adequate signal representations. The advantages and flexibility of B-splines are well known [1,2], and its use is compatible with the approximation of non-bandlimited signals.

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Practical implementations are possible up to a certain truncation error, which generally tends to be more of a concern when the kernel has infinite support and slow decay (such as the sinc function). B-splines have compact support, and as a result the value of a certain signal sample cannot influence the reconstruction outside a certain finite interval.

The kernel function will often lead to a Riesz sequence or a Riesz basis. Riesz bases are a generalization of orthonormal bases, and can be regarded as the result of applying a bounded invertible operator to the elements of an orthonormal basis. A Riesz sequence, on the other hand is an "incomplete basis," as it is merely a Riesz basis for the closed linear span of the set of functions under consideration.

The numerical stability of the representations is very important in practice, as it dictates the magnitude of the effect of dealing with imperfect data. The stability of Riesz bases and Riesz sequences can be measured by looking at the size of their Riesz bounds. Orthonormal bases are perfect from the viewpoint of numerical stability, and their Riesz bounds are both equal to unity. As the ratio of the upper to the lower bound increases, the numerical stability of the representation decreases. It is well known [3], that the tighter (closer to each other) these bounds are the less any small perturbations in the input data will be felt at the output.

This paper points out that the Riesz bounds associated with bases built using certain linear combinations of B-splines are better from the stability point of view than bases directly based on B-splines.

The notation used is standard: L_2 is the space of all quadratically Lebesgue integrable f, with norm $\|f\|_2 := (\int |f|^2)^{1/2}$ and inner product $\langle f, g\rangle := \int f g^*$. The space ℓ_2 is its discrete counterpart (quadratically summable sequences), with the usual norm and inner product. The Fourier transform $\hat{f}$ of f is

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{j}wt}\, dt,$$

implying an inverse given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{\mathrm{j}wt}\, dw.$$

Section 2 provides the necessary background and describes the kernels used, and Section 3 deals with the stability issues. The spaces V_φ generated by the kernels are discussed in Section 4, and the analysis filters are computed in Section 5.

2. The general sampling scheme

It is well known that substitution of the sinc kernel by other functions more adjusted to the specific problem considered may lead to better approximations [4–6]. B-splines are a possibility, and their advantages are well known [2]. Consider the signal space [7]

$$V_\varphi := \left\{ f(t) = \sum_{k\in\mathbb{Z}} c_k \varphi(t-k):\ c_k \in l_2 \right\},$$

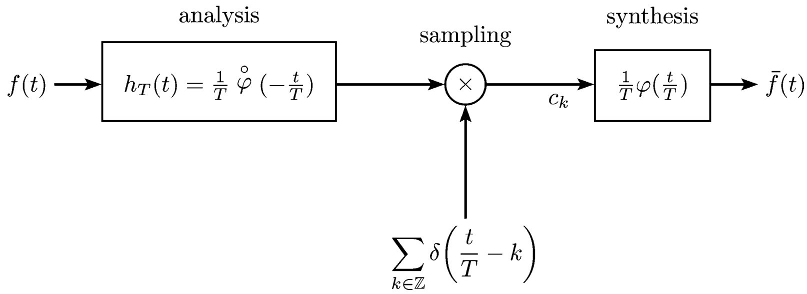

Fig. 1. Typical sampling scheme. Usually, the analysis and synthesis filters are low-pass, i.e., $h(t) = \varphi(t) = \mathrm{sinc}(t)$, for $T = 1$. In the general case they must satisfy the bi-orthogonality condition $\langle \varphi(t-k), \mathring{\varphi}(t-l)\rangle = \delta_{k-l}$.

where we have chosen the sampling step $T = 1$. When the function φ generates a Riesz basis, the coefficients c_k completely characterize the signal. When φ is the sinc function, the coefficients c_k are essentially the signal samples, but in the more general case they are given by the inner product between f and $\mathring{\varphi}_k$, i.e., $c_k = \langle f, \mathring{\varphi}_k\rangle$. See Fig. 1.

The family $\{\mathring{\varphi}_k\}$ is dual to $\{\varphi_k\}$ and its Fourier transform is given by [8]

$$\hat{\mathring{\varphi}}(w) = \frac{\hat{\varphi}(w)}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(w + 2\pi k)|^2}. \tag{1}$$

In the least squares sense this means that

$$P_{V_\varphi} f = \sum_{k\in\mathbb{Z}} \langle f, \mathring{\varphi}_k\rangle \varphi_k,$$

where $P_{V_\varphi} f$ denotes the projection of f in V_φ.

Butzer et al. [9] studied interpolation and approximation problems in which the sinc kernel is replaced by certain interpolation functions φ_i with compact support, among which the following will be relevant for us:

$$\begin{aligned}
\varphi_1(t) &= M_2(t),\\
\varphi_2(t) &= 4M_3(t) - 3M_4(t),\\
\varphi_6(t) &= M_4(t) + \frac{1}{3}M_2(t) - \frac{1}{6}\big(M_2(t+1) + M_2(t-1)\big),\\
\varphi_7(t) &= 4M_4(t) + \frac{1}{2}\big(M_4(t+1) + M_4(t-1)\big) - 2\big(M_5(t+1/2) + M_5(t-1/2)\big).
\end{aligned}$$

These interpolating kernels are linear combinations of the B-splines defined by

$$M_n(t) = \begin{cases} \sum_{j=0}^{\lceil n/2-|t|\rceil} \frac{(-1)^j n(n/2-|t|-j)^{n-1}}{j!(n-j)!}, & |t| \leqslant n/2,\\ 0, & |t| > n/2, \end{cases}$$

where $\lceil x\rceil$ denotes the largest integer less than or equal to x. They can also be represented in terms of their Fourier transforms

$$\hat{M}_n(w) = \left[\frac{\sin w/2}{w/2}\right]^n, \quad w \in \mathbb{R}.$$

Plots are shown in Fig. 2.

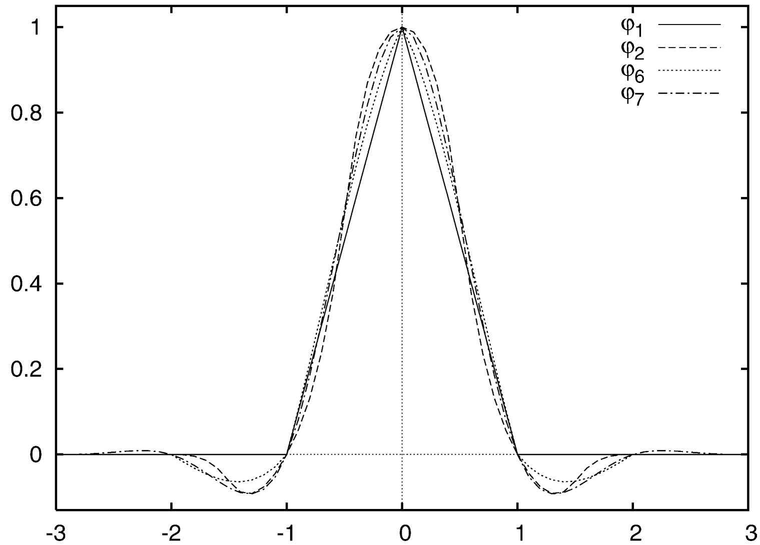

Fig. 2. Interpolation kernels φ_1, φ_2, φ_6, and φ_7.

Although Butzer et al. have discussed a wider class of functions, here we use only those with the interpolation property (see Ref. [9] for details).

3. Stability and Riesz bounds

The set of integer translations of a B-spline generates a Riesz basis. In fact, the following result holds [10].

Theorem 1. *For any function* $\phi \in L^2(\mathbb{R})$ *and constants* $0 < \alpha \leqslant \beta < \infty$ *the following two statements are equivalent*:

(i) $\{\phi(\cdot - k)\colon\ k \in \mathbb{Z}\}$ *satisfies the Riesz condition with Riesz bounds* α *and* β; *that is, for any* $\{c_k\} \in l^2$,

$$\alpha \sum_{k\in\mathbb{Z}} |c_k|^2 \leqslant \left\| \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k) \right\|_2^2 \leqslant \beta \sum_{k\in\mathbb{Z}} |c_k|^2.$$

(ii) *The Fourier transform* $\hat{\phi}$ *of* ϕ *satisfies*

$$\alpha \leqslant \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(w + 2\pi k)\right|^2 \leqslant \beta,$$

almost everywhere.

The Riesz bounds corresponding to the B-splines and the linear combinations of B-splines used by Butzer et al. can now be found. Setting $w = 2x$ and $\phi(w) = M_n(w)$ we see that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\hat{M}_n(2x+2\pi k)\right|^2 &= \sum_{k\in\mathbb{Z}}\left|\frac{\sin[(2x+2\pi k)/2]}{(2x+2\pi k)/2}\right|^{2n} = \sum_{k\in\mathbb{Z}}\frac{\sin^{2n}(x+\pi k)}{(x+\pi k)^{2n}}\\ &= \sin^{2n}x\sum_{k\in\mathbb{Z}}\frac{1}{(x+\pi k)^{2n}},\end{aligned}$$

where we have used $\sin^{2n}(x+\pi k)=\sin^{2n}x$. It is known [11] that

$$\sum_{k\in\mathbb{Z}}\frac{1}{(x+\pi k)^{2n}} = -\frac{1}{(2n-1)!}\frac{d^{2n-1}}{dx^{2n-1}}\cot x,$$

which immediately yields

$$\sum_{k\in\mathbb{Z}}\left|\hat{M}_n(2x+2\pi k)\right|^2 = \frac{-\sin^{2n}x}{(2n-1)!}\frac{d^{2n-1}}{dx^{2n-1}}\cot x. \tag{2}$$

Now, with the help of (2), we find for M_1

$$\tilde{f}_1(x) = \sum_{k\in\mathbb{Z}}\left|\hat{M}_1(2x+2\pi k)\right|^2 = \frac{-\sin^2 x}{(2-1)!}\frac{d^{2-1}}{dx^{2-1}}\cot x = 1, \tag{3}$$

that is, $\alpha=\beta=1$.

For M_2 we have

$$\tilde{f}_2(x) = \sum_{k\in\mathbb{Z}}\left|\hat{M}_2(2x+2\pi k)\right|^2 = \frac{-\sin^4 x}{3!}\frac{d^3}{dx^3}\cot x = \frac{1}{3}\left(1+2\cos^2 x\right). \tag{4}$$

This equation has its minimum $1/3$ at $x=\pi/2$ (or $w=\pi$), and its maximum 1 at $x=w=0$, i.e., $\alpha=1/3$ and $\beta=1$.

In the same way,

$$\begin{aligned}\tilde{f}_3(x) &= \sum_{k\in\mathbb{Z}}\left|\hat{M}_3(2x+2\pi k)\right|^2 = \frac{-\sin^6 x}{5!}\frac{d^5}{dx^5}\cot x\\ &= \frac{1}{15}\left(2+11\cos^2 x+2\cos^4 x\right).\end{aligned} \tag{5}$$

The minimum is $\alpha=2/15$ and the maximum is $\beta=1$.

As for M_4, we see that

$$\begin{aligned}\tilde{f}_4(x) &= \sum_{k\in\mathbb{Z}}\left|\hat{M}_4(2x+2\pi k)\right|^2 = \frac{-\sin^8 x}{7!}\frac{d^7}{dx^7}\cot x\\ &= \frac{1}{315}\left(17+180\cos^2 x+114\cos^4 x+4\cos^6 x\right),\end{aligned} \tag{6}$$

yielding $\alpha=17/315$ and $\beta=1$.

Finally, for M_5 we have

$$\tilde{f}_5(x) = \sum_{k\in\mathbb{Z}} \left|\hat{M}_5(2x+2\pi k)\right|^2 = \frac{-\sin^{10} x}{9!}\frac{d^{10}}{dx^{10}}\cot x$$
$$= \frac{1}{2835}\left\{62 + 1072\cos^2 x + 1452\cos^4 x + 247\cos^6 x + 7\cos^8 x\right\}, \tag{7}$$

showing that $\alpha = 62/2835$ and $\beta = 1$.

Figure 3 shows the plots of $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$, $\tilde{f}_4$, $\tilde{f}_5$ defined by (3), (4), (5), (6), and (7), respectively. Table 1 presents a summary of these bounds.

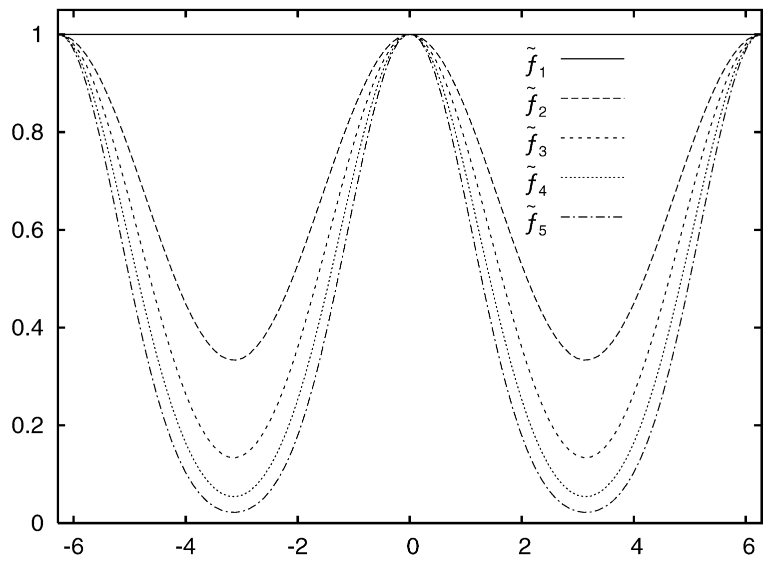

Fig. 3. Plot of the periodic functions $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$, $\tilde{f}_4$, and $\tilde{f}_5$, associated to the Riesz bases generated by the translations sets of M_1, M_2, M_3, M_4, and M_5.

Table 1
The Riesz bounds and supports of the kernels considered, and those of the B-splines for comparison

Kernel	Support	Lower bound (α)	Upper bound (β)
$M_1(t)$	$[-\frac{1}{2}, \frac{1}{2}]$	1	1
$M_2(t)$	$[-1, 1]$	$1/3 \approx 0.333$	1
$M_3(t)$	$[-\frac{3}{2}, \frac{3}{2}]$	$2/15 \approx 0.133$	1
$M_4(t)$	$[-2, 2]$	$17/315 \approx 0.054$	1
$M_5(t)$	$[-\frac{5}{2}, \frac{5}{2}]$	$62/2835 \approx 0.0022$	1
$\varphi_1(t)$	$[-1, 1]$	$1/3 \approx 0.333$	1
$\varphi_2(t)$	$[-2, 2]$	$41/70 \approx 0.586$	1
$\varphi_6(t)$	$[-2, 2]$	$359/945 \approx 0.380$	1
$\varphi_7(t)$	$[-3, 3]$	$17/35 \approx 0.486$	1

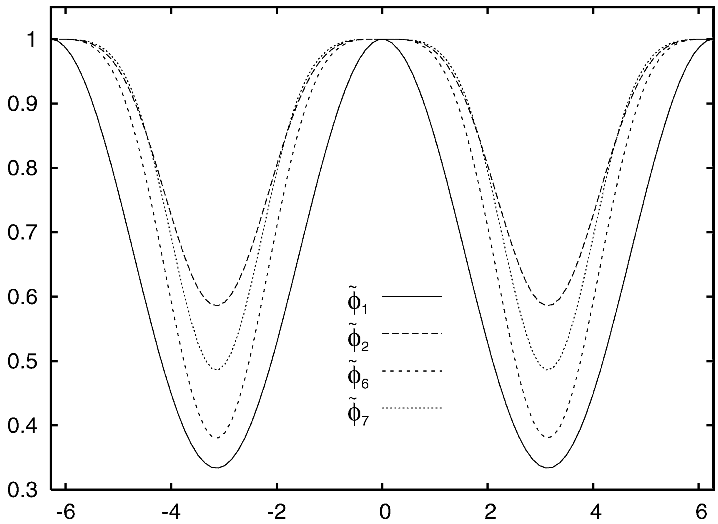

Fig. 4. Plot of the periodic functions $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_6$, and $\tilde{\phi}_7$, associated to the Riesz bases generated by the φ_1, φ_2, φ_6, and φ_7.

To investigate the bounds associated with φ_1, φ_2, φ_6, and φ_7 we observe that, for odd integers p and q,

$$\sum_{k\in\mathbb{Z}} \frac{\sin^p(x+\pi k)}{(x+\pi k)^p} = \sin^p x \left\{ \sum_{k\in\mathbb{Z}} \frac{1}{(x+2k\pi)^p} - \sum_{k\in\mathbb{Z}} \frac{1}{[x+(2k+1)\pi]^p} \right\}, \tag{8}$$

and

$$\sum_{k\in\mathbb{Z}} \frac{\cos^q(x+\pi k)\sin^p(x+\pi k)}{(x+\pi k)^p} = \cos^q x \sin^p x \left\{ \sum_{k\in\mathbb{Z}} \frac{1}{(x+2k\pi)^p} + \sum_{k\in\mathbb{Z}} \frac{1}{[x+(2k+1)\pi]^p} \right\}. \tag{9}$$

The case φ_1 has been solved, since $\varphi_1 = M_2$. Hence,

$$\tilde{\phi}_1(x) = \frac{1}{3}\big(1 + 2\cos^2 x\big), \tag{10}$$

i.e., $\alpha = 1/3$ and $\beta = 1$.

With the help of (8) and (9) we get

$$\tilde{\phi}_2(x) = \frac{41}{70} + \frac{191}{210}\cos^2 x - \frac{121}{210}\cos^4 x + \frac{17}{210}\cos^6 x, \tag{11}$$

$$\tilde{\phi}_6(x) = \frac{359}{945} + \frac{48}{35}\cos^2 x - \frac{278}{315}\cos^4 x + \frac{124}{945}\cos^6 x, \tag{12}$$

$$\tilde{\phi}_7(x) = \frac{17}{35} + \frac{4016}{2835}\cos^2 x - \frac{566}{405}\cos^4 x + \frac{116}{189}\cos^6 x - \frac{368}{2835}\cos^8 x + \frac{32}{2835}\cos^{10} x. \tag{13}$$

Figure 4 shows the plots of $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_6$, and $\tilde{\phi}_7$ defined by (10), (11), (12), and (13), respectively. Note that the largest variation, $1-\alpha_1 \approx 0.667$, occurs for $\tilde{\phi}_1$ and the smallest $1-\alpha_2 \approx 0.414$ for $\tilde{\phi}_2$.

Table 1 summarizes the bounds, and shows that the linear combinations of B-splines φ_1, φ_2, φ_6, and φ_7, lead to more stable representations than the B-splines M_1, M_2, M_3, M_4, and M_5 for equivalent polynomial spaces. Compare, for example, the bounds associated with M_5 and φ_7, both of the 4th degree.

4. The spaces V_φ

Suppose that the spaces V_a and V_b are generated by φ_a and φ_b, respectively. It is known [3] that when the Riesz bounds of V_a are tighter than those of V_b, the representations produced in V_a are numerically more stable. This statement is clear when φ_a and φ_b generate the same space, i.e., $V_a = V_b$. In the more general case $V_a \neq V_b$ the comparison is less straightforward. This will now be discussed.

We note that already in [7] the authors considered the spaces generated by integer translations of a generating function φ_a, and linear combinations of the translations of the same generating function (what the authors call *equivalent basis functions*).

Generally, when the sampling set T approaches zero, the approximation error $\|f - P_{V_T} f\|$ decreases. The Strang–Fix conditions [12] relate the approximation power of the representation to the spectral characteristics of the generating function:

$$\|f - P_{V_T} f\| \leqslant C_L T^L \|f^{(L)}\|, \quad \forall f \in W_2^L,$$

where C_L is a known constant and

$$\|f^{(L)}\| = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} w^{2L} \left|\hat{f}(w)\right|^2 \right)^{1/2}.$$

Here, W_2^L denotes the space of functions that are L times differentiable in the L_2 or finite-energy sense, and $P_{V_T} f$, as before, the orthogonal projection of f into V_T. The error will decay like $O(T^L)$, where the order of approximation is $L = n + 1$, and n the polynomial degree. Spline interpolation gives an identical error of approximation but a larger C_L constant [13,14].

In our case, the spaces $V_\varphi \subset L_2$ formed from the M_i are all disjoint (the constant function is not in L_2). Hence, they do not intersect and are not a subset of each other (for example, we cannot say that $V_{M_1} \subset V_{M_2}$ or *vice-versa*).

For φ_1 we have that $\varphi_1(t) = M_1(t)$, that is, the spaces generated from the translations set of φ_1 and M_1 are equivalent, i.e., $V_{\varphi_1} \equiv V_{M_1}$.

Now, suppose that $\psi(t) = \alpha M_3(t) + \beta M_4(t)$, for some fixed reals α and β. Suppose that for these values of α and β the Riesz bounds are better for V_ψ than for V_{M_3} and V_{M_4}. Consider the projection of a signal $f \in L_2$ in V_{M_3}, V_{M_4}, and V_ψ. Controlling T leads to approximations as good as necessary, but the representation in V_ψ will remain the most stable. This is also the case when $\alpha = 4$ and $\beta = -3$, that is, the approximation of a signal $f \in L_2$ will be numerically better in V_{φ_2} than in V_{M_3} or V_{M_4}.

A similar argument can be applied to the representations produced by the projections into V_{φ_6}, which are found to be more stable than the ones in V_{M_2} and V_{M_4}. The same is true for V_{φ_7}, in this case with respect to V_{M_4} and V_{M_5}.

5. The dual bases

To complete this study, and in reference to Fig. 1, it remains to describe the dual functions of the interpolation kernels φ_1, φ_2, φ_6, and φ_7.

For the B-splines, and according to (1), we find

$$\begin{aligned}
\hat{\mathring{M}}_1(w) &= \frac{2\sin\frac{w}{2}}{w},\\
\hat{\mathring{M}}_2(w) &= \frac{12(1-\cos^2\frac{w}{2})}{w^2(1+2\cos^2\frac{w}{2})},\\
\hat{\mathring{M}}_3(w) &= \frac{120\sin\frac{w}{2}(1-\cos^2\frac{w}{2})}{w^3(2+11\cos^2\frac{w}{2}+2\cos^4\frac{w}{2})},\\
\hat{\mathring{M}}_4(w) &= \frac{5040(1-2\cos^2\frac{w}{2}+\cos^4\frac{w}{2})}{w^4(17+180\cos^2\frac{w}{2}+114\cos^4\frac{w}{2}+4\cos^6\frac{w}{2})},\\
\hat{\mathring{M}}_5(w) &= \frac{90720\sin\frac{w}{2}(1-2\cos^2\frac{w}{2}+\cos^4\frac{w}{2})}{w^5(62+1072\cos^2\frac{w}{2}+1452\cos^4\frac{w}{2}+247\cos^6\frac{w}{2}+7\cos^8\frac{w}{2})}.
\end{aligned} \tag{14}$$

See Fig. 5 for plots. As for the interpolation kernels φ_1, φ_2, φ_6, and φ_7 we find

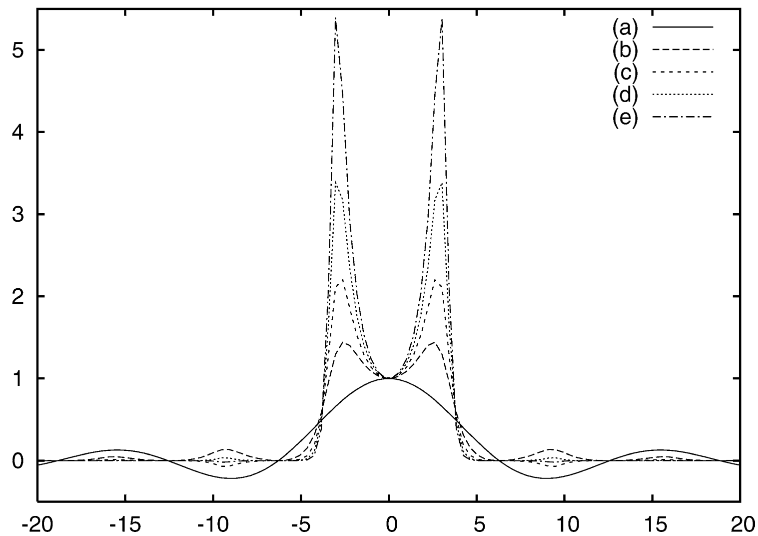

Fig. 5. The Fourier transforms of the dual functions of the B-splines (a) M_1, (b) M_2, (c) M_3, (d) M_4, and (e) M_5.

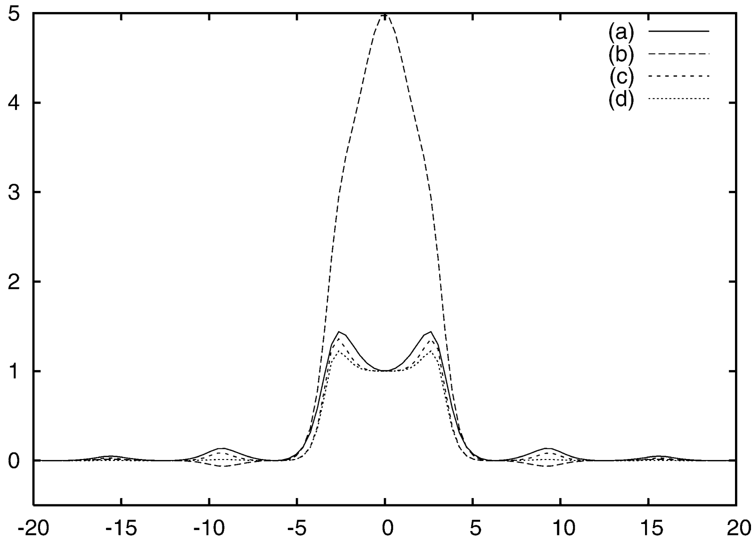

Fig. 6. The Fourier transforms of the dual functions of the interpolation kernels (a) φ_1, (b) φ_2, (c) φ_6, and (d) φ_7.

$$
\begin{aligned}
\hat{\mathring{\varphi}}_1(w) &= \frac{12(1-\cos^2\frac{w}{2})}{w^2(1+2\cos^2\frac{w}{2})},\\
\hat{\mathring{\varphi}}_2(w) &= \frac{-3360\big(-2w\sin\frac{w}{2}+2w\sin\frac{w}{2}\cos^2\frac{w}{2}-1+2\cos^2\frac{w}{2}-\cos^4\frac{w}{2}\big)}{w^4\big(123+191\cos^2\frac{w}{2}-121\cos^4\frac{w}{2}+17\cos^6\frac{w}{2}\big)},\\
\hat{\mathring{\varphi}}_6(w) &= \frac{5040\big(3+\frac{w^2}{2}-6\cos^2\frac{w}{2}-w^2\cos^2\frac{w}{2}+3\cos^4\frac{w}{2}+\frac{w^2}{2}\cos^4\frac{w}{2}\big)}{w^4\big(359+1296\cos^2\frac{w}{2}-834\cos^4\frac{w}{2}+124\cos^6\frac{w}{2}\big)},\\
\hat{\mathring{\varphi}}_7(w) &= 90720\sin^4\frac{w}{2}\left(\frac{3w}{2}+w\cos^2\frac{w}{2}-4\cos\frac{w}{2}\sin\frac{w}{2}\right)\\
&\quad\times\Bigg[w^5\bigg(1377+4016\cos^2\frac{w}{2}-3962\cos^4\frac{w}{2}+1740\cos^6\frac{w}{2}\\
&\qquad-368\cos^8\frac{w}{2}+32\cos^{10}\frac{w}{2}\bigg)\Bigg]^{-1}.
\end{aligned}
\tag{15}
$$

See Fig. 6. The expressions (14) and (15) define the frequency response of each analysis filter, or pre-filter, completing the framework of Fig. 1.

6. Conclusions

For all practical purposes, signal representation procedures must be stable under small perturbations in the input data, and sometimes it is also desirable that the support of the interpolating kernel be as small as possible. This limits the truncation error, since only a few coefficients contribute to the representation at any given point. The kernels that have been discussed meet both conditions. In the context of the sampling scheme described in [7], certain linear combinations of B-splines, already studied in a different context [9], lead to representations with tighter Riesz bounds than those generated by B-splines alone. Numerically, they therefore lead to more stable representations.

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