Proceedings SampTA-2001, May 2001, Orlando, Florida

Stable interpolation and error correction: concatenated real-codes with an interleaver

Paulo J. S. G. Ferreira*
Dep. Electrónica e Telecomunicações / IEETA
Universidade de Aveiro
3810-193 Aveiro Portugal
pjf@det.ua.pt www.ieeta.pt/˜pjf

Abstract

We propose a preprocessing technique which allows stable interpolation and error correction. It can be regarded as a parallel concatenated (real) code with an interleaver. We discuss the relative performance and stability of the technique, when compared to the single-channel case (band-limited interpolation, extrapolation, and error-correction). We give examples that show that the condition numbers of the linear operators involved can be smaller by many orders of magnitude, when compared to the usual band-limited reconstruction. The stability problem can be formulated using the language of frames. Algorithms for recovering unknown samples (erasures) and for correcting errors at unknown locations are briefly discussed and demonstrated.

1 Introduction

This paper addresses a number of closely related issues that can be viewed from multiple angles. In the language of sampling and interpolation, we are interested in detecting and even correcting errors in band-limited sampled data. Using the language of error control coding, we are interested in channel coding, and in particular in finding stable real codes. And in the language of frames, we wish to design redundant signal representations, and the corresponding reconstruction algorithms, while controlling the ratio of the frame bounds.

The whole paper will be concerned with the two-channel system depicted in Fig. 1. The signal a is coded and the result is the signal x, of N samples. The permuted signal Pa is also coded, using a similar coder, leading to a second signal y, also of N samples. The system can be regarded as having two output signals, x and y, or just one output in a space of higher dimensionality, obtaining by combining or concatenating x and y.

We were driven to the study of such a structure for several reasons. First, interleaving is a common way of protecting against contiguous losses, a form of data corruption which is particularly difficult to handle (for low-pass signals, or other signals with a contiguous spectrum). In the context of interpolation, contiguous losses lead to extremely ill-posed problems. Band-limited extrapolation and superresolution are examples of problems made difficult (or interesting) by the contiguous distribution of the unknown segments of the signal (or of its Fourier transform).

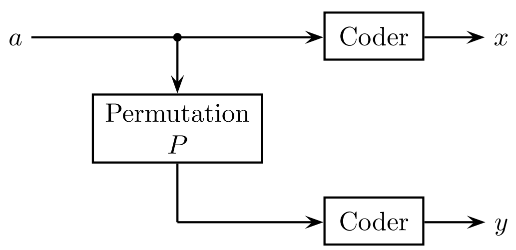

Figure 1: The signal a and its permuted version Pa are both coded, using coders similar to that depicted in Fig. 2, and the outputs of the coders are concatenated.

Interleaving the data maps contiguous losses to scattered losses, and improves the conditioning of the reconstruction problem in case of error bursts. Estimates for the condition numbers in terms of the minimum separation between errors are given in [1], for example.

However, the interleaver is a one-to-one mapping, and there will always exist error patterns that will again be mapped by the interleaver into contiguous, error patterns. This problem provided the motivation for using two data paths: one path in which the data are coded in the normal way, and another in which the data are interleaved before coding.

There is another reason for studying these structures, which has its roots in a very different context. We are indebted to Prof. J. F. Moura for bringing to our knowledge the striking similarity between the structure depicted in Fig. 1 and certain error control codes (concatenated codes with interleavers). Understanding these codes has been a priority among the coding theory community ever since their astonishing performance was reported by Berrou, Glavieux, and Thitimajshima in 1993. Such codes are capable of performances very close (a fraction of a dB) to the Shannon limit, at bit error rates of about 10^{-5}. For a re-

*Work partially supported by the FCT.

cent overview of this subject and coding theory in general see [2]. The framework that we are exploring is distinct (due to the nature of the underlying fields, $\mathbb{R}$ or $\mathbb{C}$ in our case, as opposed to the finite, Galois fields, commonly used in coding theory). But, due to the importance of the subject, it is worthwhile to study it from every possible angle. Hopefully, this may lead to cross-fertilization and further progress.

The building block of the two-channel system is the coder depicted in Fig. 2. It has been recognized as the equivalent of error control coding in the complex field by several researchers, and as a result a number of applications are known [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. It shows how the coded signal x is obtained from the data a. First, a is zero-padded to create a larger vector, with N samples (sometimes it can be convenient to work with a transform of a instead of a itself). The coded signal is the DFT of the zero-padded vector:

$$x = F \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

Although, to simplify the notation, we appended the zeros to a, it is possible to insert the zeros in such a way that x stays real. We will assume that $x \in \mathbb{R}^N$. It is precisely because x is confined to a lower dimensional subspace of $\mathbb{R}^N$ (to which we will call the code subspace) that error detection and correction becomes a possibility. In the examples below, $a \in \mathbb{R}^{2M+1}$, and $x \in \mathbb{R}^N$, obviously with $2M+1 < N$. One may identify with $2M+1$ the bandwidth of the data, and with $(2M+1)/N$ the oversampling ratio, or the code rate.

Consider the (band-limiting) operator B, which projects a vector of N samples on the code subspace. Hence, $x = Bx$ and $y = By$, and

$$F \begin{bmatrix} a \\ 0 \end{bmatrix} = F \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} F^{-1} F \begin{bmatrix} a \\ 0 \end{bmatrix}$$

shows that

$$B := F \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} F^{-1}.$$

Note that

$$y = F \begin{bmatrix} Pa \\ 0 \end{bmatrix} = F \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

Hence,

$$y = F \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} F^{-1} x =: Tx,$$

that is, $y = Tx$, with

$$T := F \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} F^{-1}.$$

which is the starting point for computing T using the FFT. On the other hand,

$$x = F \begin{bmatrix} a \\ 0 \end{bmatrix} = F \begin{bmatrix} P^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Pa \\ 0 \end{bmatrix}.$$

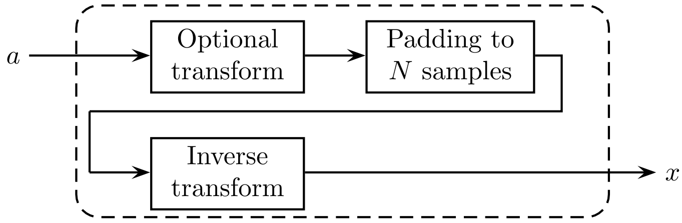

Figure 2: The single-channel coder, a reference against which the two-channel case (described in Fig. 1) is compared.

Hence,

$$x = F \begin{bmatrix} P^{-1} & 0 \\ 0 & 0 \end{bmatrix} F^{-1} y = T^H y.$$

We also note that

$$TT^H = T^H T = F \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} F^{-1} = B,$$

and

$$BT = TB = T.$$

2 Handling erasures and errors

After introducing the basic operators B and T, we are ready to formulate our first problem. A sampling matrix is a diagonal matrix of the form

$$D = \begin{bmatrix} d_0 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & d_n \end{bmatrix}, \qquad d_i \subset \{0, 1\}$$

It is convenient to exclude the trivial cases $D = I$ and $D = 0$ from consideration. Assume that instead of x and y we are given $x_0 = D_1 x$ and $y_0 = D_2 y$, where D_1 and D_2 are sampling matrices. In other words, we do not have access to x and y, but only to a subset of their samples. The matrices D_1 and D_2 can be thought of as the indicator matrices of these subsets. How can x and y, and consequently a, be recovered?

The simplest possibility is an alternating projection algorithm. At the end of step n, there will be two approximations to x and y, denoted by x_n and y_n. The next pair of approximations can be computed as follows:

$$\begin{aligned} x_{n+1} &= x_0 + (I - D_1) B x_n, \\ y_{n+1} &= T x_{n+1}, \\ y_{n+2} &= y_0 + (I - D_2) B y_{n+1}, \\ x_{n+2} &= T^H y_{n+2}. \end{aligned}$$

Any criteria for the convergence of alternating projections can now be applied [13, 14]. The analysis of the algorithm

$$x_{n+1} = x_0 + (I - D) B x_n$$

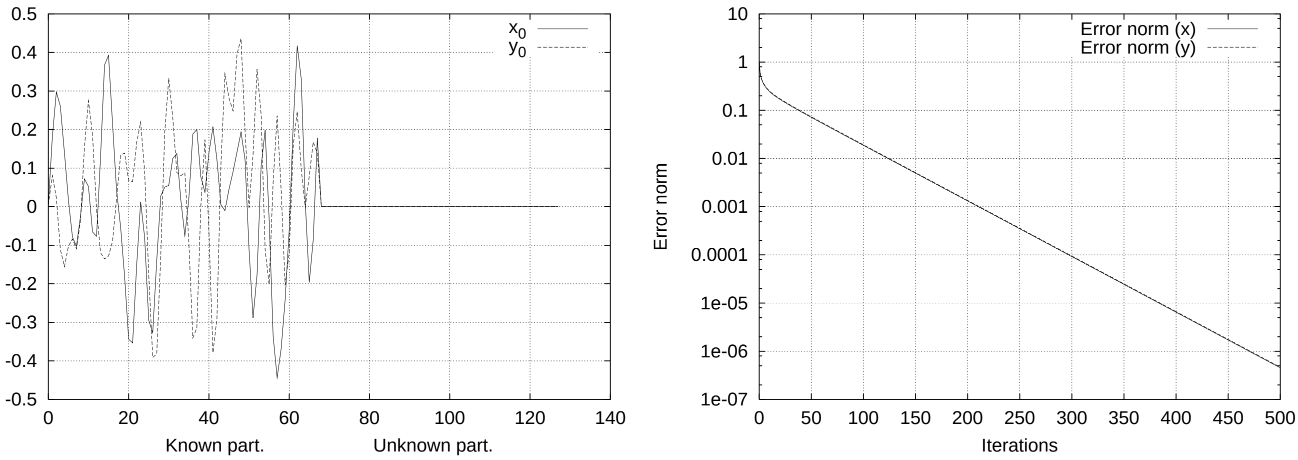

Figure 3: Left: the observed signals x_0 and y_0, consisting of a known segment of x and y, and an unknown segment, replaced by zeros. Right: the error evolution of the algorithm.

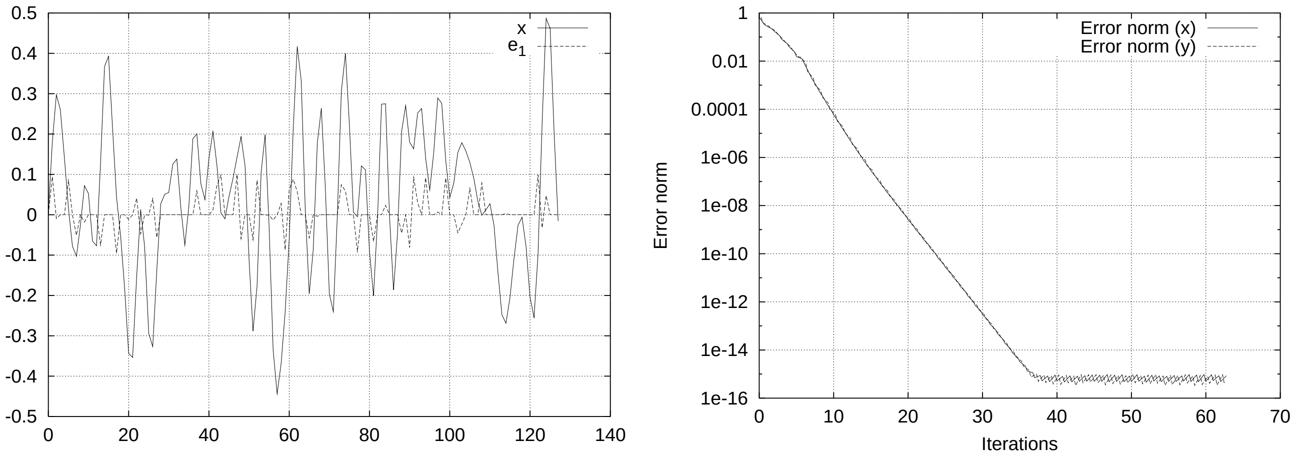

Figure 4: Left: the signal x and the error added to it e_1. Right: the error evolution of the error-correcting algorithm. Due to space limitations, the signal y and the error e_2 is not shown. The error signals e_1 and e_2 are unknown to the algorithm, and the task is to recover x and y, from which a follows. In this example, the total number of errors was 90.

which arises in the one-channel context was presented in detail in [15]. There is one obvious guiding principle: the number of known samples must be at least equal to the dimension of the subspace to which a belongs.

In practice, the algorithm performs remarkably well even for contiguous losses, a situation that leads to extremely poor conditioned problems in the one-channel case (and even in the multiple channel case, provided that the data are only subject to linear time-invariant filters). See Fig. 3.

The previous algorithm is for correcting erasures, that is, it can be applied only if D_1 and D_2 are known, which is true only if the positions of the errors are known (in both channels). A more challenging problem is that of error detection and correction: we are given two signals x_0 and y_0, which may differ from x and y at certain samples, and we wish to recover x, y, and a. We do not know how many errors occurred, nor their amplitudes or positions.

It turns out that the problem can be solved using an iterative algorithm, which also estimates D_1 and D_2. The idea is to take advantage of the fact that the (unknown) error signals $e_1 = x - x_0$ and $e_2 = y - y_0$ are sparse. This can be achieved using thresholding, as explained in [16]. The result of one such experiment is reported in Fig. 4.

3 Stability and frames

Let J be an index set (a nonempty ordered subset of $\{0, 1, \ldots, N-1\}$, containing no duplicates). The problem of correcting erasures at $\{0, 1, \ldots, N-1\} \setminus J$ can be rephrased (in this context) using the language of frames. It is interesting to do so, despite the finite-dimensional character of the problem.

We begin by considering the one-channel case, as illustrated in Fig. 2 . Let B_i denote the ith column of the matrix B. The question is: can $\{B_i\}_{i \in J}$ be a frame for the code subspace, $\mathcal{B} = \{x \in \mathbb{C}^N : x = Bx\}$, and what are the frame bounds? The answer is yes, provided of course that the cardinal of J is sufficiently large. The frame bounds can be readily related to a certain eigenproblem, by proceeding as follows. Define D as the indicator matrix associated with J, that is, D is diagonal and

$$D_{ii} = \begin{cases} 1, & i \in J, \\ 0, & i \notin J. \end{cases}$$

Then,

$$\sum_{i \in J} | \langle x, B_i \rangle |^2 = \|DBx\|^2,$$

and the frame bounds follow from the eigenvalues of BDB:

$$\lambda_{\mathsf{min}}(BDB) \le \frac{\|DBx\|^2}{\|x\|^2} = \frac{x^H BDBx}{\|x\|^2} \le \lambda_{\mathsf{max}}(BDB).$$

For a tutorial on this issue see [17]. An example, which shows how difficult contiguous losses can be, is illustrated in Fig. 7 and Fig. 5. The block size is $N = 256$, and $M = 10$. The known 60 data are distributed along three contiguous sets of 20 samples each (the set is outlined in Fig. 5). Despite the low oversampling ratio of $r = 21/256$, the condition number (ratio of the frame bounds) for the problem exceeds 10^{14}. Inspecting Fig. 7 we see that it is already difficult to tell apart the smallest nonzero eigenvalue from the noise floor — the theoretically "zero eigenvalues". Under such circumstances, there exist signals of the required bandwidth (belonging to the code subspace) whose energy is almost entirely contained in the samples x_i, $i \in J$. The same subspace contains signals with energy concentrated outside J, that is, in the samples x_i, $i \notin J$. The signals that maximize and minimize these energy concentrations are the eigenvectors depicted in Fig. 5.

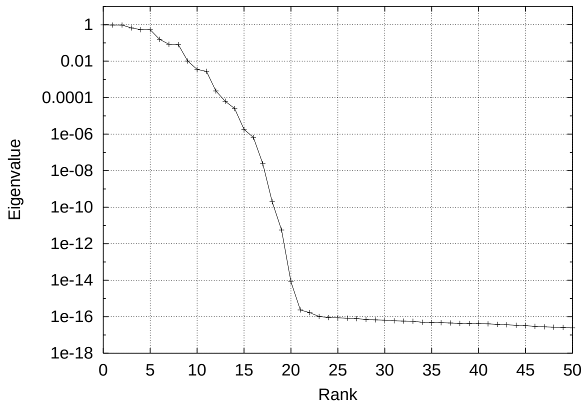

Figure 7: One-channel case: the 50 largest eigenvalues of BDB, corresponding to a block size of $N = 256$. The matrix B band-limits to $2M + 1$ nonzero harmonics, with $M = 10$. The 60 known samples were distributed along three equal intervals (as shown in Fig. 5). The condition number (ratio of the frame bounds) for this problem was $> 10^{14}$.

In the two-channel case, we have two sets J_1 and J_2, since the erasures now occur on both channels. The question now is: can $\{B_i\}_{i \in J_1} \cup \{T_i\}_{i \in J_2}$ be a frame for the code subspace, and what are the frame bounds? Naturally, the bounds have to be deduced from a different eigenproblem. First, define D_1 and D_2 as the indicator matrices associated with the sets J_1 and J_2, respectively. Now, because

$$\sum_{i \in J_1} | \langle x, B_i \rangle |^2 + \sum_{i \in J_2} | \langle x, T_i \rangle |^2 = \|D_1 Bx\|^2 + \|D_2 Tx\|^2,$$

the bounds follow from the eigenvalues of

$$C := BD_1B + T^H D_2 T,$$

which can be written as $C = A^H A$, with

$$A = \begin{bmatrix} D_1 B \\ D_2 T \end{bmatrix}.$$

It remains to design an experiment for the two-channel case which can be compared with the one-channel case, already described (Fig. 7 and Fig. 5). Recall that, in the one-channel case, $N = 256$, $M = 10$, and there were 60 erasures. Therefore, for the two-channel case we selected $N = 128$, $M = 10$, and 30 erasures per channel. The

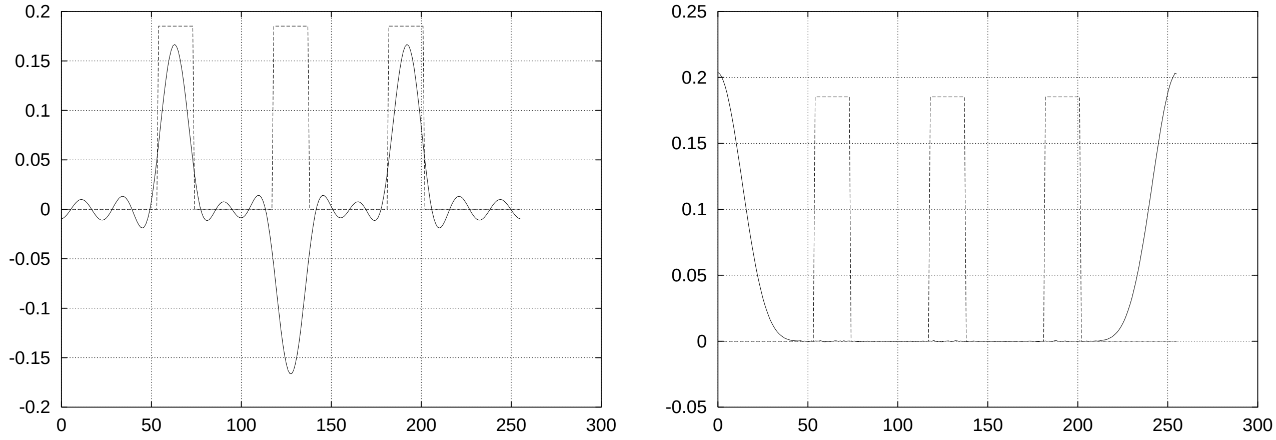

Figure 5: One-channel case: the signals of $N = 256$ samples, band-limited to $2M + 1$ nonzero harmonics, with $M = 10$ as in Fig. 7, which assume the maximum and minimum energy concentration in the set J and its complement.

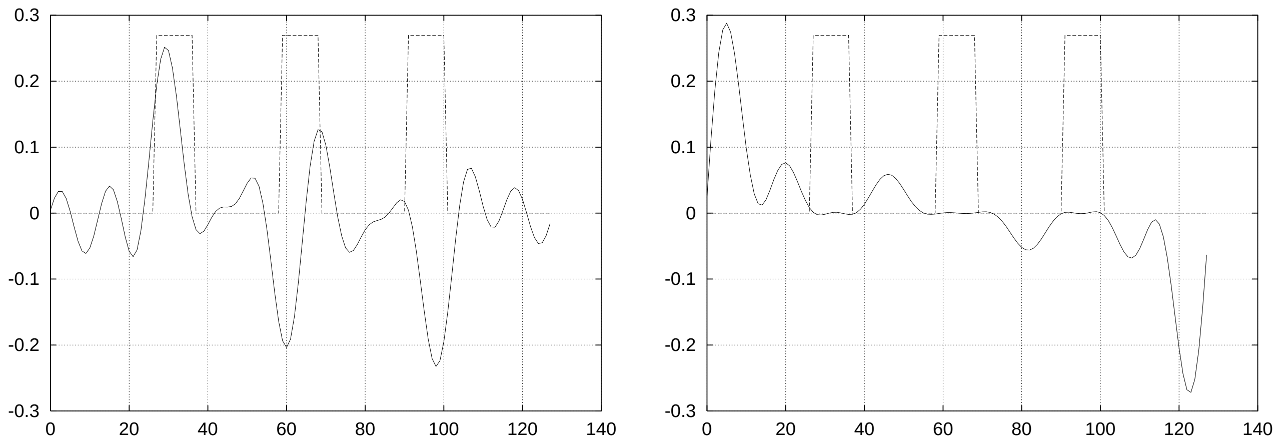

Figure 6: Two-channel case: the signals of $N = 128$ samples, band-limited to $2M + 1$ nonzero harmonics, with $M = 10$ as in Fig. 8, which correspond to the maximum and minimum energy concentration.

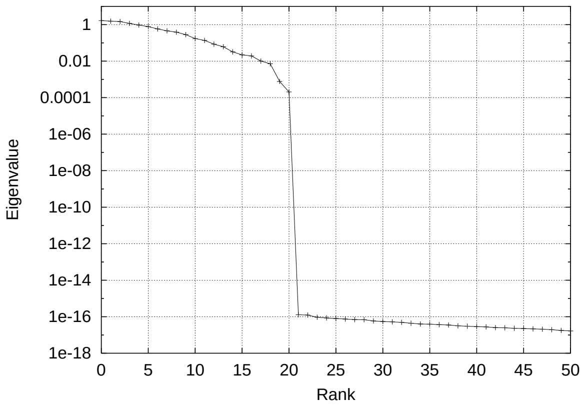

Figure 8: Two-channel case: the 50 largest eigenvalues of the matrix, for a total block of 256 samples ($N = 128$ per channel). The band-limit is $M = 10$, as in Fig. 7. The 60 known samples (30 per channel) were distributed along three equal intervals (as shown in Fig. 6). The condition number (ratio of the frame bounds) for this problem was $\approx 8\,000$, 10 orders of magnitude better than the comparable one-channel experiment.

results are presented in Fig. 8 and Fig. 6. The oversampling rate in each of the channels is now twice as high, and therefore each channel data has less redundancy. However, we do have two channels, and therefore the same overall degree of redundancy. The condition number for this problem is now close to 8 000, smaller by 10 orders of magnitude, and the smallest nonzero eigenvalue of C is still well above the noise level of the theoretically "zero eigenvalues" (see Fig. 8).

4 Conclusions

The preprocessing method proposed in this work maps a band-limited signal to another band-limited signal, in such a way that error detection and correction on the two signals becomes numerically much more tractable than when dealing with a single signal, even if the latter is twice as redundant.

We have seen that the condition numbers of the linear operators involved in the reconstruction can be many orders of magnitude smaller than the condition numbers of the operators that occur in the usual band-limited reconstruction problems.

In the language of frames, we are dealing with a frame $\{B_i\}_{i\in J_1} \cup \{T_i\}_{i\in J_2}$, and the ratio of the frame bounds can in this case be orders of magnitude smaller than that for frames $\{B_i\}_{i\in J}$, that is, frames formed by translating

$$B(k) = \frac{\sin(\pi(2M+1)k/N)}{N\sin(\pi k/N)}$$

to all $i \in J$. Algorithms for recovering unknown samples (erasures) and for correcting errors at unknown locations were briefly discussed and demonstrated, and show fast convergence even for contiguous errors.

References

[1] P. J. S. G. Ferreira. Stability issues in error control coding in the complex field, interpolation, and frame bounds. *IEEE Sig. Proc. Letters*, 7(3):57–59, Mar. 2000.

[2] S. Benedetto and E. Biglieri, editors. *Principles of Digital Transmission with Wireless Applications.* Kluwer Academic / Plenum Publishers, New York, 1999.

[3] R. E. Blahut. Transform techniques for error control codes. *IBM J. Res. Develop.*, 23(3):299–315, May 1979.

[4] J. K. Wolf. Redundancy, the discrete Fourier transform, and impulse noise cancellation. *IEEE Trans. Commun.*, 31(3):458–461, Mar. 1983.

[5] R. E. Blahut. Algebraic fields, signal processing, and error control. *Proc. IEEE*, 73(5):874–893, May 1985.

[6] T. G. Marshall, Jr. Coding of real-number sequences for error correction: A digital signal processing problem. *IEEE J. Select. Areas Commun.*, 2(2):381–391, Mar. 1984.

[7] T. G. Marshall, Jr. Image filtering with orthogonal projectors for burst error control. *IEEE Trans. Circuits Syst. Video Technol.*, 1(3):269–278, Sep. 1991.

[8] M. Nafie and F. M. Marvasti. Implementation of recovery of speech with missing samples on a DSP chip. *Electron. Letters*, 30(1):12–13, Jan. 1994.

[9] C. Wong, F. Marvasti, and W. G. Chambers. Implementation of recovery of speech with impulsive noise on a DSP chip. *Electron. Letters*, 31(17):1412–1413, Aug. 1995.

[10] F. Marvasti, M. Hasan, M. Echhart, and S. Talebi. Efficient algorithms for burst error recovery using FFT and other transform kernels. *IEEE Trans. Signal Processing*, 47(4):1065–1075, Apr. 1999.

[11] P. Marziliano and M. Vetterli. Reconstruction of irregularly sampled discrete-time bandlimited signals with unknown sampling locations. *IEEE Trans. Signal Processing*, 48(12):3462–3471, Dec. 2000.

[12] P. Marziliano. *Sampling Innovations.* PhD thesis, École Polytechnique Fédérale de Lausanne, Feb. 2001.

[13] D. C. Youla. Generalized image restoration by the method of alternating orthogonal projections. *IEEE Trans. Circuits Syst.*, 25(9):694–702, Sep. 1978.

[14] D. C. Youla and H. Webb. Image restoration by the method of convex projections: Part 1 — theory. *IEEE Trans. Med. Imag.*, 1(2):81–94, Oct. 1982.

[15] P. J. S. G. Ferreira. Interpolation and the discrete Papoulis-Gerchberg algorithm. *IEEE Trans. Signal Processing*, 42(10):2596–2606, Oct. 1994.

[16] J. M. N. Vieira and P. J. S. G. Ferreira. Interpolation, spectrum analysis, error-control coding, and fault-tolerant computing. In *Proceedings of ICASSP 97*, vol. III, p. 1831–1834, Munich, Germany, Apr. 1997.

[17] P. J. S. G. Ferreira. Mathematics for multimedia signal processing II — discrete finite frames and signal reconstruction. In J. S. Byrnes, editor, *Signal Processing for Multimedia*, p. 35–54, IOS Press, 1999. Presented at the NATO Advanced Study Institute "Signal Processing for Multimedia", Il Ciocco, Italy, July 1998.