Sorting Continuous-Time Signals and the Analog Median Filter

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***Abstract*—Continuous time signals can be meaningfully "sorted," or rearranged, leading to precise formulations of analog median filters and other variants of ranked order filters. The proposed interpretation leads to a broader view of the analog median and related filters and provides a framework for discussing the correct definitions in spaces such as L_1 or L_2 and certain issues regarding continuity and root signals.**

***Index Terms*—Continuous time filters, continuous time systems, median filters, nonlinear filters, nonlinear systems, sorting.**

I. Introduction

THE MEDIAN filter, ever since Tukey's work in the early 1970s, has been at the center of vigorous research in nonlinear filtering (see [1] and [2] for overviews). Many variants and generalizations of the basic running median have been proposed, leading to a rich class of nonlinear operators.

In this letter, we consider the problem of sorting continuous-time signals. The solution is given in terms of the distribution and rearrangement of the signal, and the analog median and other analog ranked order-based filters emerge as straightforward applications.

Our interest in that problem is due to the following factors. First, the new analytical tools that can be used in the continuous-time case might bring new insights to the analysis of these nonlinear filters. Second, the analog filters might lead to simpler and faster circuits and large power and area savings. Analog circuits able to select the kth largest element from a set of inputs, although not SISO analog median filters, show some of these advantages [3].

The lack of theoretical results on analog medians is noted in [4], where a nonlinear dynamical system that sorts n data is proposed. Nearly all the literature on ranked order filters concerns the discrete time case, hardly a surprising fact. There is no conceptual difficulty in sorting signal samples, but it is less obvious how to devise a similar operation for continuous-time signals. The exceptions are [5], which discusses the analog median filter and [6], which shows by means of a counterexample that some properties of the analog median, as defined in [5], are more subtle than they initially seem. The issues raised in [6], related to the nature and existence of root signals of the median filter, do not appear to have been addressed yet.

Our formulation, in terms of the distribution and rearrangement, brings out new insights and a more general framework within which to study median-type filters. The two nonequivalent definitions of the median filter in [5] turn out to be related to the left- or right-continuous inverses of the distribution function, sampled at $w/2$. Some of the issues raised in [6] will be clarified. The filter will behave as expected if the rearrangement of the windowed signal of length w is continuous at $w/2$.

The reinterpretation in terms of rearrangements allows a broader view of the analog median, leads to other ranked order filters, provides a framework for discussing the correct definitions for spaces such as L_1 or L_2, the connection between the analog and digital cases, the role of continuity, root signals, and the behavior under noise. This letter is a start toward that program.

Manuscript received June 7, 2000. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. G. Ramponi.

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Publisher Item Identifier S 1070-9908(00)08362-0.

II. Distribution Function

The distribution function M_f associated with the real measurable function f is defined by

$$M_f(y) := \text{meas}\{x\colon f(x) > y\}$$

where "meas S" denotes the Lebesgue measure of the set S, and restriction of x to the domain of f is tacitly assumed. The definition is meaningful even if $f\colon \mathbb{R}^N \longmapsto \mathbb{R}$, but we will restrict ourselves to functions $f : \mathbb{R} \longmapsto \mathbb{R}$ such that $M_f(y) \neq \infty$ almost everywhere.

The behavior of f at its points of discontinuity is irrelevant, in the sense that it leads to the same distribution function (f can be arbitrarily modified at any set of measure zero without changing M_f).

The mapping $f \longmapsto M_f$ is many-to-one (for example, M_f does not change if f is translated). Any two functions with the same distribution are called "equidistributed." The distribution function M_f is nonincreasing and right-continuous, because the measure is monotone and continuous from below. The alternate definition $M_f(y) := \text{meas}\{x\colon f(x) \geq y\}$ leads to a left-continuous distribution function.

III. Sorting Continuous-Time Signals

The (nonincreasing) rearrangement of a function is essentially the inverse function of its distribution function.

The motivation for this definition is simple. Consider splitting the domain of f in any finite number of nonoverlapping intervals and then permuting, translating, or time-reversing each "piece" of the function thus obtained, being careful not to overlap any two of them. The distribution function of the new function is still M_f due to the properties of the measure. Because M_f stays invariant under these operations and is nonincreasing, one might be tempted to take it as the rearrangement of f. However, this is clearly wrong. If $f(t)$ is interpreted as "voltage as a function of time," then $M_f(y)$ is "time as a function of voltage." Swapping axes, that is, considering M_f^{-1} instead of M_f, corrects the units and still leads to a nonincreasing function.

The difficulty is that M_f, in general, does not have an inverse in the strict sense (it may be constant over certain intervals). But we may define a generalized inverse that reduces to the inverse whenever it exists. Graphically and roughly speaking, we might draw a horizontal line of height x across the plot of M_f and search for the leftmost point y still satisfying $M_f(y) < x$, or the rightmost point y still satisfying $M_f(y) > x$.

Both possibilities lead to the usual inverse when M_f is continuous and decreasing. We chose to define the rearrangement $\overline{f}$ of f by

$$\overline{f}(x) := \inf_y \{M_f(y) < x\}$$

obtaining a left-continuous function. Defining $\overline{f}(x) := \inf_y \{M_f(y) \le x\}$ would lead to right-continuity instead.

To confirm that this is in fact the inverse of the distribution function M_f, properly defined in the case of discontinuities, check that $\overline{f}(M_f(y)) = y$, if $M_f(y)$ is a point of continuity of $\overline{f}$ and $M_f(\overline{f}(x)) = x$ if $\overline{f}(x)$ is a point of continuity of M_f. At points of discontinuity, $\overline{f}(M_f(y)) \ge y$, and $M_f(\overline{f}(x)) \le x$.

It is easy to see that

$$M_{\overline{f}}(y) := \text{meas}\{\overline{f}(x) > y\} = M_f(y)$$

that is, f and its rearrangement have the same distribution function. They are equidistributed, and the integrals of f and $\overline{f}$ are equal

$$\int f(x)\,dx = \int \overline{f}(x)\,dx. \tag{1}$$

It can be shown that if g and h are two nonincreasing functions equidistributed with a given f, then $g = h$ almost everywhere. Thus, the nonincreasing rearrangement is (essentially) unique. Requiring left- or right-continuity are two possible ways of removing the ambiguity.

The rearrangements of certain simple functions confirm that we are on the right track. For example, if $f: \mathbb{R}_0^+ \longmapsto \mathbb{R}$ is continuous and decreasing, then $M_f(y) = f^{-1}(y)$, and $\overline{f}(x) = f(x)$, as expected. If $f: \mathbb{R} \longmapsto \mathbb{R}$ is even, continuous, and decreasing in $\mathbb{R}^+$, then $M_f(y) = 2f^{-1}(y)$, and hence, $\overline{f}(x) = f(x/2)$. It can also be seen that the rearrangement of a continuous increasing function f: $[0, a] \longmapsto \mathbb{R}$ can be obtained by "reversing" the function $\overline{f}(x) = f(a - x)$.

When f is decreasing but discontinuous, the equality $f = \overline{f}$ may not hold at the points of discontinuity of f. A very simple discontinuous function and its rearrangement are shown in Fig. 1. If f is not left-continuous, $f = \overline{f}$ will fail to hold at the points of discontinuity of f.

The concept of rearrangement was introduced by Hardy and Littlewood [7]. The reader curious about the application of the Hardy and Littlewood maximal function to the theory of the Fourier series might consult [8].

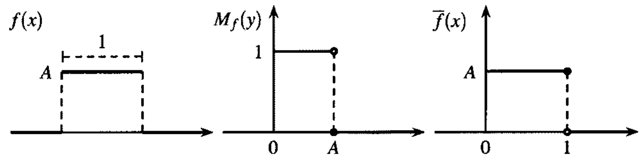

Fig. 1. An example of distribution and rearrangement. The function f can be arbitrarily translated or even replaced by the characteristic function of any set of measure one without affecting M_f and $\overline{f}$. The behavior or f at the discontinuities is not indicated because it has no effect upon M_f.

IV. Analog Median and Similar Filters

We denote by f_t the product of f by a rectangular window of length w, centered at t. The output of the analog median filter at t is, by definition, the value of the rearrangement of f_t at $w/2$

$$g(t) = \overline{f_t}(w/2).$$

If a function belongs to Lebesgue measurable functions spaces such as L_1 or L_2, it is meaningless to refer to its value at particular points. If f is piecewise continuous, the rearrangement may still be discontinuous at $w/2$, and the behavior of the filter will then depend on whether the rearrangement was defined as being right- or left-continuous.

To get other ranked order filters, replace the evaluation functional at $w/2$ by other functionals of $\overline{f_t}$

$$g(t) = F[\overline{f_t}].$$

A family of operators depending on one parameter a can be obtained using the functional

$$F_a[h] := \frac{1}{a}\int_{(w-a)/2}^{(w+a)/2} h(x)\,dx$$

where h: $[0, w] \longmapsto \mathbb{R}$, and $0 < a \le w$. The corresponding filter is defined by

$$g(t) = F_a[\overline{f_t}].$$

It becomes increasingly closer to the median filter as $a \to 0$, and to the (linear) moving average when $a \to w$ [see (1)]. Such definitions are adequate for spaces such as L_1 or L_2.

V. Relation with Previous Work

One of the definitions proposed in [5] for the analog median filter of width w can be recast as

$$g(t) = \sup\left\{r\text{: meas}\{\tau\text{: } f_t(\tau) \ge r\} \ge \frac{w}{2}\right\}$$

where g is the output of the filter, and f_t is the input multiplied by a rectangular window centered at t. The function

$$M(r) := \text{meas}\{\tau\text{: } f_t(\tau) \ge r\}$$

is the (left-continuous) distribution function of the windowed signal. The remaining terms determine one of its possible "reasonable" inverses

$$M^{-1}(x) := \sup_r \{M(r) \ge x\}$$

(see the discussion in Section III). The inverse, which is the rearrangement of the windowed signal, is then evaluated at $w/2$. The definitions in [5] thus correspond to sampling the left- or right-continuous inverses of the distribution function at $w/2$, and the constraints mentioned in Section IV apply.

VI. Fixed Points

The roots or fixed points of the median filter operator $\mathcal{M}$ are the signals that satisfy $f = \mathcal{M}f$. From Section III, left-continuous monotonic signals are roots. Without left-continuity, $f \neq \mathcal{M}f$ at the points of discontinuity of f.

The question raised in [6] concerns possibly nonmonotonic signals consisting of monotonic segments separated by constant segments of length $w/2$. Denote this set of signals by A. Consider the response of the median filter with $w = 2$ to the signal $f \in A$ of Fig. 1. When $t \in S$, S denoting the support of f, the output of the filter will be, by definition, $\overline{f}(w/2) = 1$. Thus, $f = \mathcal{M}f$ (except possibly at the points of discontinuity of f).

Consider now the response to $g = 1 - f$, $g \in A$. The rearrangements of g_t and f_t are equal for $t \in S$, and so the response of the median filter to g and f for these t will be identical. The distinct inputs f and g lead to the same output for $t \in S$, and consequently, one of them (in this case g) cannot be a root. Defining the rearrangement as a right-continuous function reverses the behavior.

We recall the observations made in [6]. The continuous elements of A are roots, but continuity is not necessary for a signal to be a root (step functions are roots). On the other hand, if the length ℓ of the constant segments is allowed to exceed $w/2$, the discontinuities cause no problem. But the continuous signals of A with $\ell = w/2$ are roots. Therefore, $\ell > w/2$ is not a necessary condition for a root either.

The presence of a discontinuity in the rearrangement at $w/2$ explains this behavior. When the length of the constant segments exceeds $w/2$, or when f_t is itself continuous, the rearrangement will be continuous at $w/2$, and there will be no difference between median filters based on right-continuous or left-continuous rearrangements. The rearrangement and its properties are crucial to the understanding of the analog ranked order filter.

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