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TWO FAST EXTRAPOLATION / SUPERRESOLUTION ALGORITHMS

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ABSTRACT

In this paper we propose two fast algorithms for the extrapolation of band-limited images (or, equivalently, for spectrum extrapolation of finite support objects). The algorithms are noniterative and based on the Cholesky decomposition of certain Hermitian positive-definite matrices. One of the algorithms is based on a image domain formulation of the problem, whereas the other is based on the dual frequency domain formulation. Both formulations lead to a wide range of possible algorithms, but the proposed techniques seem very adequate to the problem under consideration. It is shown that there are problems for which the algorithms that spring from one formulation will outperform those that arise from the other. Some of the issues that should be taken in consideration when designing algorithms for specific problems are discussed. The connection between these reconstruction algorithms and error control coding in the complex field is briefly mentioned.

1. INTRODUCTION

This paper addresses the discrete-discrete extrapolation / superresolution problem. Iterative reconstruction algorithms based on alternating projections [1] have been successfully applied in this context. One advantage of these methods is the possibility of incorporating other possibly nonlinear constraints in the reconstruction process, leading to iterations of the POCS type [2]. A simple nonlinear method, based upon spectral analysis, is compared with an iterative method in [3]. Faster methods are of interest since the convergence rate of the iterative methods often reaches very low values. This was the motivation for the method proposed in [4], which however should be used with care because of its numerical sensitivity to out-of-band noise. Iterations based on the preconditioned conjugate gradient method [5] can also be used.

In this paper we concentrate on two new classes of methods for the solution of the extrapolation problem. We refer to these formulations as image-domain and frequency-domain minimum dimension methods. The image-domain formulation leads to linear equations for the unknown image pixels, as opposed to the frequency-domain formulation, which leads to equations for the unknown image DFT samples. As a result, solving the image domain equations yields the unknown pixels of the image itself, without any further processing, whereas solving the frequency domain equations leads to DFT coefficients, and an additional Fourier inversion is required to obtain the image.

A variety of iterative and noniterative reconstruction algorithms spring from these formulations. The matrices of the underlying linear equations, either in the image or frequency domain case, have minimum dimension because the number of equations is already equal to the number of unknown quantitities, in the image or frequency domains. In both cases, the matrices are Hermitian and positive-definite.

We will explore one noniterative method, based on Cholesky decomposition of the matrices that occur in the image domain and frequency domain formulations. We will identify the problems for which the algorithms that emerge from one of these two formulations outperform the corresponding algorithms that spring from the other. In particular, we show that there is no "fastest" method overall. It will be seen that the relative performance depends on the details of the problem, and mainly on the size of the area that has to be extrapolated, as compared to the size of the support of the Fourier transform of the image.

The approach proposed in this paper shares one common characteristic with the algorithm proposed for 2-D extrapolation in [6], namely, the fact that in both methods the image is processed row by row (or column by column), reducing the 2-D problem to a number of identical 1-D problems. There is, however, one major difference: we provide both image-domain and frequency-domain equations, from which a variety of specific algorithms can be extracted. It will be shown that any one of the two formulations may clearly outperform the other for specific problems. Being aware of this fact may bring important computational advantages: there is no "best" method overall.

2. RECONSTRUCTION, ERASURES, AND ERROR CONTROL CODING

Although natural images often have a spectrum which is far from being flat, they are also not exactly band-limited. However, if the image is band-limited by zeroing those frequency components that correspond to a negligible energy content, the effect can be visually imperceptible. The filtering process that leads to the new image can be regarded as "channel coding" in the complex field, using the DFT instead of the Fourier transform in a Galois field. Although error control codes are usually formulated in finite fields, they can certainly be meaningfully formulated in other fields. One possible benefit is the greater freedom in picking block lengths, the price to pay is round-off error. See [7] for a discussion of the stability of the decoding or reconstruction problem. From this point of view, the reconstruction algorithms discussed in this paper can be interpreted as decoding methods for erasure channels, in the complex field.

3. THE BASIC EQUATIONS

We assume that the rows (or columns) of the image are vectors of N samples, with $2M+1$ nonzero DFT harmonics, and that only $N-n$ pixels are known. The remaining pixels will be interpolated or extrapolated. The set of known pixels is S_i, and $\bar{S}_i$ is its complement. In the frequency domain, the set of known (vanishing) DFT pixels is denoted by S_f, and its complement by $\bar{S}_f$. We express the row (or column) x and its DFT X as $x = u + v$, $X = G + H$, where v and H contain only known data.

Below, we will need the following diagonal matrix,

$$D^g_{ii} = \begin{cases} 0, & i \in S_f \\ 1, & i \notin S_f \end{cases}$$

which will occur in the matrix B that arises in the space-domain formulation, and a second diagonal matrix

$$D^u_{ii} = \begin{cases} 0, & i \in S_i \\ 1, & i \notin S_i \end{cases}$$

which will occur in the matrix T, in reference to the frequency-domain formulation.

The equations which are the starting point of the algorithms are the following.

Proposition 1 *The image row (or column) $x \in \mathbb{C}^N$ band-limited to $2M+1$ harmonics satisfies $x = Bx + h$, and its DFT satisfies $X = TX + V$. The matrices B and T are circulant projections, with $B = F^{-1}D^gF$ and $T = FD^uF^{-1}$, and the matrices D^g and D^u are defined above. Explicitly,*

$$\begin{aligned} B_{ab} &= \frac{1}{N}\sum_{p\notin S_f} e^{\mathrm{i}\frac{2\pi}{N}(a-b)p}, \\ T_{ab} &= \frac{1}{N}\sum_{p\notin S_i} e^{-\mathrm{i}\frac{2\pi}{N}(a-b)p}. \end{aligned}$$

The equations $x = Bx + h$ and $X = TX + V$ cannot be solved for x or X, because the matrices $I-B$ and $I-T$ are singular. The following propositions solve that problem.

Proposition 2 *Let the number of unknown pixels in the row (or column) x be denoted by n (the cardinal of $\bar{S}_i$). Let P be the $n \times n$ principal submatrix of B, obtained by deleting from B all rows and columns indexed by elements of S_i. Denote by $c \in \mathbb{C}^n$ the vector with elements*

$$c_i = \sum_{j\in S_i} B_{ij}v_j + h_i \quad (i \notin S_i).$$

Then, the vector $y \in \mathbb{C}^n$ formed by the n image domain unknowns x_i $(i \notin S_i)$ satisfies

$$y = Py + c. \tag{1}$$

Proposition 3 *Let the number of unknown DFT coefficients (the cardinal of $\bar{S}_f$) be denoted by r. Let Q be the $r \times r$ principal submatrix of T, obtained by deleting from T all rows and columns indexed by elements of S_f. Denote by $d \in \mathbb{C}^r$ the vector with elements*

$$d_i = \sum_{j\in S_f} T_{ij}H_j + V_i \quad (i \notin S_f).$$

Then, the vector $z \in \mathbb{C}^r$ formed by the r unknown DFT harmonics X_i $(i \notin S_f)$ satisfies

$$z = Qz + d. \tag{2}$$

The results in [8] can be used to show that the matrices $I-P$ and $I-Q$ are nonsingular. The equations (1) and (2) are the key equations from which the two fast algorithms (image and frequency domain, respectively) will emerge.

4. THE ALGORITHMS

The solution of the image extrapolation problem can be based upon (1) or (2). Selecting as a starting point, say, (1), we propose to solve the equations row by row (or column by column) by Cholesky decomposition of the relevant matrix (Algorithm 1).

```
Compute the Hermitian positive-definite matrix I − P
Compute its Cholesky factorization I − P = GG^H
for all columns with unknown pixels do
   Compute the vector c in y = Py + c
   Solve (I − P)y = c for y by backsubstitution.
end for
```

Algorithm 1: Image domain formulation of the extrapolation algorithm.

The algorithm for the frequency domain equations is identical and it is omitted. Note, however, that it solves distinct equations, of distinct size. Also, it requires complex arithmetic since it manipulates complex quantities (the DFT of the image). A reconstruction example is given in Fig. 1 and Fig. 2.

The image with errors Reconstruction

Fig. 1. Reconstruction of an artifically band-limited image with the minimum dimension method in the image domain (this can be interpreted as error control coding in the complex field). The reconstruction was performed column by column using Cholesky decomposition and backsubstitution. Since the geometry of the problem is constant across all columns with errors, the Cholesky decomposition has to be computed only once. This can be interpreted as error correction in an erasure channel, in the complex field.

5. DISCUSSION AND COMPARISONS

Why do we claim that these algorithms are efficient? First of all, when compared to alternating projection methods, they manipulate matrices and vectors of much smaller size. In addition to this, the time-consuming step is the Cholesky decomposition, which needs to be performed once only, since the extrapolation of each row (or column) can and should be performed using the same matrix (the matrices P and Q depend only on the geometry of the problem, which is invariant across all rows or columns).

Let us assume that the image is $N \times N$, and that its DFT has $2M+1 < N$ nonzero harmonics in each direction. Assume further that the image segment to restore is square of size $n \times n$. None of this restrictions is essential, but it simplifies the presentation and comparison.

The alternating projections method requires one filtering and resampling operation on images of size $N \times N$, for each iteration performed. The solutions that we propose require the following steps.

For the image domain formulation, and assuming that the image will be processed by columns, we need first of all to compute the Cholesky factor of the matrix $I - P$, of size n. This step takes $O(n^3)$ flops. We then need to form the vector c, for each of the n columns with unknown data, and then solve each of these equations by backsubstitution. These steps take $O(n^2)$ flops each.

In the frequency domain case, we need to compute the Cholesky factor of $I - Q$, of size $2M + 1$, which takes $O[(2M + 1)^3]$ flops. We then need to form the vector d, for each of the $2M + 1$ columns with unknown data, and then solve each of these equations by backsubstitution in $O[(2M + 1)^2]$ flops.

In both cases, the result is a method that requires $O(p^3)$ flops, p being the size of the problem in the image or frequency domain, depending on the formulation selected, and assuming that the Cholesky decomposition, which dominates the CPU time, has to be performed once.

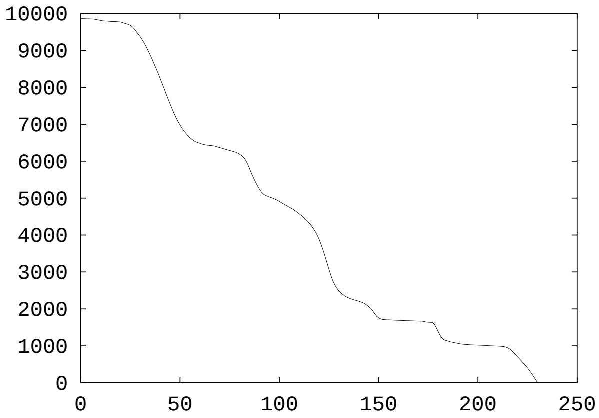

Fig. 2. The evolution of the quadratic error after reconstructing each column with errors in Fig. 1.

It is intuitively obvious that it is better to select the image domain formulation when the number of unknown pixels is small compared to the bandwidth of the image (wideband images, and a small area to extrapolate). The opposite may be true when the area to extrapolate is large and the bandwidth very narrow.

Refer to Fig. 3 and Fig. 4 for examples that show the difference that may exist between Cholesky decomposition of the image domain and frequency domain equations.

The horizontal lines in the figures correspond to the time needed for the Cholesky factorization, the remaining (and by comparison negligible) time corresponds to the solution of the equations by backsubstitution.

Note that in the first case (Fig. 3) the size of the image domain matrices is much smaller ($n = 150$ against $2M + 1 = 521$), whereas in the second case (Fig. 4) the opposite

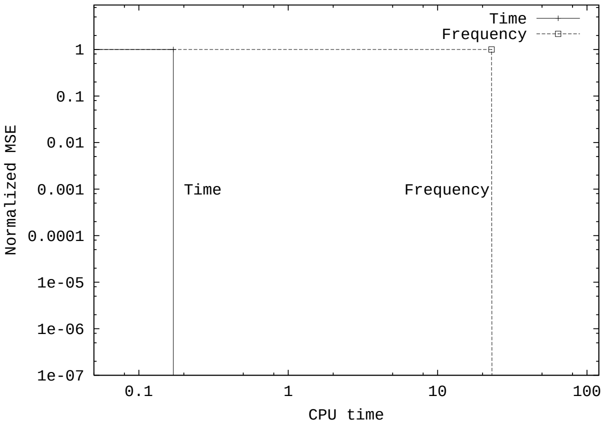

Fig. 3. First example: Cholesky factorization of the minimum dimension image-domain and frequency-domain equations. The initial and final mean-square errors are plotted *versus* time. In this case, $n = 150$ and $2M + 1 = 521$.

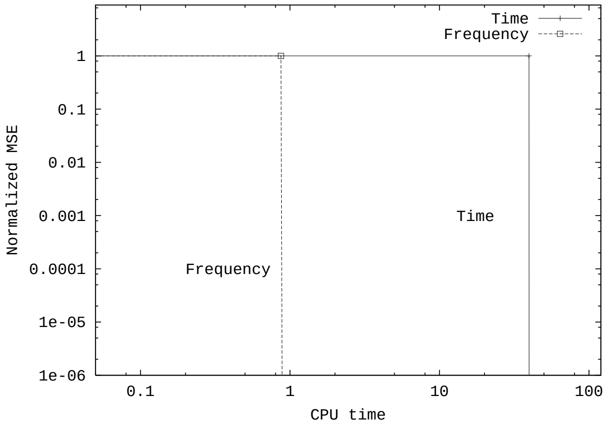

Fig. 4. Second example: Cholesky factorization of the minimum dimension image-domain and frequency-domain equations. The initial and final mean-square errors are plotted *versus* time. In this case, $n = 700$ and $2M + 1 = 201$.

is true ($n = 700$ against $2M+1 = 201$). Clearly, in the first case one should consider the image domain formulation, but in the second case the frequency domain formulation will lead to much faster algorithms.

When picking the algorithm for a certain reconstructing problem there are many other factors that should be taken in consideration. The domain (space or frequency), the size of the problem, the structure of the matrices, the nature of the arithmetic (real or complex), the method (iterative, non-iterative, or semi-iterative), and the numerical stability should all be considered, among other. Asymptotically faster methods can be slower if the size of the problem is smaller than the break-even point, faster methods can be much less stable than slower methods, and the domain of formulation (space or frequency) can have a drastic impact on the computational demands. We will attempt to report on some of these issues in future works.

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