Stability Issues in Error Control Coding in the Complex Field, Interpolation, and Frame Bounds

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***Abstract*—We give bounds for the eigenvalues and condition number of a matrix, with applications to error control coding in the complex field, spectrum analysis, the missing data problem, interpolation, and the determination of discrete finite frame bounds.**

***Index Terms*—Condition number, eigenvalues, error control coding, frame bounds, frames, interpolation, numerical stability, real codes.**

I. Statement of the Problem

All SIGNALS considered here are vectors $x \in \mathbb{R}^N$, with components or samples denoted by $x(i)$. Periodic extension $x(i) = x(N+i)$ is tacitly assumed. Consider the function

$$\hat{s}_m(\xi) \triangleq \frac{\sin \pi(2m+1)\xi/N}{N \sin \pi\xi/N} \tag{1}$$

a set of n distinct integers $\{i_k\}_{k=1}^n$, $n < N$, and the matrices

$$S = [S_{ab}]_{a,\,b=1}^n \triangleq [\hat{s}_m(i_a - i_b)]_{a,\,b=1}^n \tag{2}$$

and $A = I - S$. The problem addressed is the following.

Problem A: To obtain bounds for the eigenvalues $\lambda(A)$ and (spectral) condition number $\kappa(A)$, under the hypothesis that the $\{i_k\}_{k=1}^n$ satisfy

$$|i_a - i_b| \geq d|a-b|$$

for some integer $d > 1$ (subtractions mod N, and $1 \leq a, b \leq n$ with $a \neq b$). It will also be assumed that there is a maximum separation of $N/2$, that is, $|i_a - i_b| \leq N/2$.

The mod N subtractions are a natural consequence of the periodicity of the signals.

II. Motivation and Applications

The matrix $A = I - S$ occurs in several problems, some of which are mentioned below. Understanding its conditioning is important in practice, when instead of solving $Ax = b$, one is forced to solve

$$A(x + \delta x) = b + \delta b$$

the relative error satisfying [1]

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\,\frac{\|\delta b\|}{\|b\|}.$$

The condition number $\kappa(A)$ is therefore crucial for the estimation of the relative error.

Note that S is a $n \times n$ submatrix of the ideal low-pass $N \times N$ matrix B that preserves $2m+1$ harmonics out of N (B has $2m+1$ eigenvalues $\lambda = 1$, and $N - (2m+1)$ eigenvalues $\lambda = 0$). Therefore, by the interlacing inequalities [1], one has $0 \leq \lambda(S) \leq 1$.

When $n \leq N - (2m+1)$, A becomes nonsingular, independently of the i_k [2], but its conditioning ranges from excellent to extremely poor and critically depends on the i_k. Experimentally, it can be verified that increasing the separation between the i_k improves the conditioning. Studying the problem formulated above leads to a better understanding of this behavior.

When the i_k are consecutive integers mod N (and so $d = 1$), the asymptotic equivalence of S to the prolate matrix can be combined with results that go back to Slepian [3] and Grenander and Szegö [4] to quantify its ill-conditioning. The prolate matrix has even been proposed as a good test matrix for numerical algorithms [5]. Our interest in the case $d > 1$ is due to the following factors.

- Consider the n signals $h_k(j) \triangleq \hat{s}_m(j - i_k)$, $1 \leq k \leq n$. Can any $x \in \mathbb{R}^N$ band-limited to $2m+1$ harmonics be written as a linear combination of the n translates h_k? Do they form a discrete finite frame for this subspace of band-limited signals? What are the frame bounds? Since $\langle x(\cdot),\, \hat{s}_m(\cdot - i_k)\rangle = x(i_k)$, the h_k form a frame with bounds α and β if

  $$\alpha\|x\|^2 \leq \sum_{k=1}^n |x(i_k)|^2 \leq \beta\|x\|^2.$$

  But

  $$\sum_{k=1}^n |x(i_k)|^2 = \|DBx\|^2$$

  where $D_{i_k,\,i_k} = 1$, otherwise $D_{ij} = 0$. Thus, α and β can be found from the singular values of DB or equivalently, the eigenvalues of BDB or DBD. The observation that $(DBD)_{ab}$ is either zero or $S_{ab} = B(i_a - i_b)$ leads back to Problem A.
- Equations involving A or closely related matrices occur in the estimation of missing samples, interleaving factors, and relaxation constants for iterative reconstruction

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methods, as described in [2] and [6]. Our results generalize those in [6] and [7], where the i_k are multiples of the same integer.

- In a cyclic block code implemented with the FFT, the DFT of a message of $2m+1$ samples is padded with $N-(2m+1)$ samples, leading to a low-pass signal of N samples. Theoretically, up to $n = N-(2m+1)$ erasures at $\{i_k\}_{k=1}^n$ can be corrected (in the time or frequency domain [8]). The matrix of the time domain equations is exactly A. The problem may be ill-conditioned, even for very small blocks. The present results show how the condition number varies as a function of the $\{i_k\}_{k=1}^n$ and therefore, the extent to which the solution is affected by errors in the data as a function of the error pattern n, m, and N.
- The difficulty of estimating the parameters of a harmonic signal with frequencies proportional to i_k from a subset of its samples increases as the minimum separation between the frequencies decreases. Our results quantify the numerical difficulty as a function of the minimum frequency separation. For a discussion of the connections between this problem, algorithm-based fault-tolerant computing, and other topics, see [9].
- BCH or Reed–Solomon codes over the complex field [10] have been claimed as advantageous for wireless channels [11]. However, the stability issue has not been studied. Such study can be based on the results given here.

III. Results

The Geršgorin discs associated with the matrix S are the sets

$$D_i \triangleq \{z \in \mathbb{C}: |z - S_{ii}| \le R_i(S)\}$$

where

$$S_{ii} = \hat{s}_m(0) = \frac{2m+1}{N}, \qquad R_i(S) \triangleq \sum_{\substack{j=0\\ j\neq i}}^{n-1} |S_{ij}|.$$

These discs are not useful, because S is not diagonally dominant. The numbers $\hat{s}_m(k)$ do not decrease sufficiently fast with k (there are abrupt changes in its IDFT $s_m(k)$, which is of rectangular shape). To solve this difficulty, consider

$$\hat{s}_m^\ell(\xi) \triangleq \frac{\sin \dfrac{\pi\ell\xi}{N} \sin \dfrac{\pi(2m+\ell)\xi}{N}}{N\ell \sin^2 \dfrac{\pi\xi}{N}}$$

where m and ℓ are given positive integers (such that $2(m+\ell) \le N$). The $\hat{s}_m^\ell(i)$ ($i = 0, 1, \cdots, N-1$) are proportional to the DFT of the discrete even-trapezoidal signal $s_m^\ell(i)$ that vanishes for $|i| \ge m+\ell$, satisfies $s(i) = 1$ for $|i| \le m$, and decreases linearly from unity at $i = m$ down to zero at $i = m+\ell$. Note that $s_m^\ell(i)$ has rectangular shape for $\ell = 1$, that is, $\hat{s}_m^1(x)$ reduces to (1).

We will need the following two additional signals:

$$s^+(i) \triangleq s_m^\ell(i) \tag{3}$$

and

$$s^-(i) \triangleq s_{m-\ell+1}^\ell(i) \tag{4}$$

for some $\ell > 1$, which satisfy

$$0 \le s^-(k) \le s_m(k) \le s^+(k). \tag{5}$$

We associate with s^- and s^+, the matrices

$$S^+ = \left[S_{ab}^+\right]_{a,\,b=1}^n \triangleq \left[\hat{s}^+(i_a - i_b)\right]_{a,\,b=1}^n$$
$$S^- = \left[S_{ab}^-\right]_{a,\,b=1}^n \triangleq \left[\hat{s}^-(i_a - i_b)\right]_{a,\,b=1}^n.$$

The matrices S, S^+, and S^- are symmetric, and their extreme eigenvalues are the extreme values of the associated quadratic forms over the unit ball. Equation (5) shows that they are nonnegative definite and that they satisfy

$$0 \le S^- \le S \le S^+ \tag{6}$$

whereas their eigenvalues satisfy

$$\begin{aligned}\lambda_{\max}(S^-) &\le \lambda_{\max}(S) \le \lambda_{\max}(S^+),\\ \lambda_{\min}(S^-) &\le \lambda_{\min}(S) \le \lambda_{\min}(S^+).\end{aligned} \tag{7}$$

The condition number of $I - S$ is given by

$$\kappa(I-S) \triangleq \frac{\lambda_{\max}(I-S)}{\lambda_{\min}(I-S)} = \frac{1-\lambda_{\min}(S)}{1-\lambda_{\max}(S)}.$$

The bounds

$$\lambda_{\max}(S) \le \lambda_{\max}(S^+) \le \hat{s}^+(0) + \max_i R_i(S^+) \tag{8}$$

$$\lambda_{\min}(S) \ge \lambda_{\min}(S^-) \ge \hat{s}^-(0) - \max_i R_i(S^-) \tag{9}$$

lead to

$$\begin{aligned}\kappa(I-S) &\le \frac{1-\lambda_{\min}(S^-)}{1-\lambda_{\max}(S^+)}\\ &\le \frac{1-\hat{s}^-(0)+\max_i R_i(S^-)}{1-\hat{s}^+(0)-\max_i R_i(S^+)}.\end{aligned} \tag{10}$$

We will need the following upper bound on the radius of the Geršgorin disc of the matrix that corresponds to $\hat{s}_m^\ell$

$$\begin{aligned}\sum_{a\neq b} |\hat{s}_m^\ell(i_a - i_b)| &\le \sum_{a\neq b} \frac{1}{N\ell \sin^2 \dfrac{\pi(i_a-i_b)}{N}}\\ &\le \sum_{a\neq b} \frac{1}{N\ell\left[\dfrac{2(i_a-i_b)}{N}\right]^2}\\ &\le \sum_{a\neq b} \frac{1}{N\ell\left[\dfrac{2d(a-b)}{N}\right]^2}\\ &\le \frac{N}{4\ell d^2} \sum_{a\neq b} \frac{1}{(a-b)^2}\\ &\le \frac{2N}{4\ell d^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \le \frac{\pi^2 N}{12\ell d^2}.\end{aligned}$$

Noting that

$$\hat{s}_m^\ell(0) = \frac{2m+\ell}{N}$$

combining (3) with (8), and (4) with (9), leads to

$$\lambda_{\max}(S) \leq \frac{2m+\ell}{N} + \frac{\pi^2 N}{12\ell d^2},$$
$$\lambda_{\min}(S) \geq \frac{2m-\ell+2}{N} - \frac{\pi^2 N}{12\ell d^2}.$$

The upper bound is minimized (and the lower bound maximized) when

$$\ell_0 = \frac{\pi N}{\sqrt{12} d}.$$

At least one of the integers closest to ℓ_0 will yield the tightest bound. The exact value of ℓ_0 leads to the approximation

$$\frac{2m+1}{N} \pm \left(\frac{2\pi}{\sqrt{12}d} - \frac{1}{N}\right) \approx \frac{2m+1}{N} \pm \left(\frac{1.81}{d} - \frac{1}{N}\right)$$

for the interval containing the eigenvalues.

The bound for the condition number is independent of the matrix order and follows from (10). It confirms that increasing d leads to more stable problems. The result can be applied as described in Section II.

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