ON AUTOMATIC REES MATRIX SEMIGROUPS

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ABSTRACT

We consider a Rees matrix semigroup $S = \mathcal{M}[U; I, J; P]$ over a semigroup U, with I and J finite index sets, and relate the automaticity of S with the automaticity of U. We prove that if U is an automatic semigroup and S is finitely generated then S is an automatic semigroup. If S is an automatic semigroup and there is an entry p in the matrix P such that $pU^1 = U$ then U is automatic. We also prove that if S is a prefix-automatic semigroup, then U is a prefix-automatic semigroup.

1 INTRODUCTION AND DEFINITIONS

We consider automatic semigroups as defined in [**?**]. We are interested in the question of whether automaticity of semigroups is preserved by various semigroup constructions. Some semigroups can be described as Rees matrix semigroups over semigroups. In this work we start with an automatic semigroup U, and prove that a Rees matrix semigroup $S = \mathcal{M}[U; I, J; P]$ over U is automatic whenever it is finitely generated. This implies that if a semigroup is finitely generated and can be described as a Rees matrix semigroup over an automatic semigroup then it is automatic. We observe that, by the Main Theorem of [**?**], S is finitely generated if and only if both I and J are finite sets, U is finitely generated and the set $U \backslash V$ is finite, where V is the ideal of U generated by the entries in the matrix P. We also consider the converse problem: does the automaticity of S imply that of U? We prove that this is the case when S is prefix-automatic or when there is an element p in the matrix P such that $pU^1 = U$. Finally, we prove the analogous results for Rees matrix semigroups with zero.

We start by introducing the definitions we require. The Rees matrix semigroup $S = \mathcal{M}[U; I, J; P]$ over the semigroup U, with $P = (p_{ji})_{j \in J, i \in I}$ a matrix with entries in U, is the semigroup with the support set $I \times U \times J$ and multiplication defined by $(l_1, s_1, r_1)(l_2, s_2, r_2) = (l_1, s_1 p_{r_1 l_2} s_2, r_2)$ where $(l_1, s_1, r_1), (l_2, s_2, r_2) \in I \times U \times J$. We say that U is the *base semigroup* of the Rees matrix semigroup S.

If A is a finite set, we denote by A^+ the free semigroup generated by A consisting of non empty words over A under the concatenation, and by A^* the free monoid generated by A consisting of A^+ together with the empty word ϵ. Let S be a semigroup and $\psi : A \to S$ a mapping. We say that A is a *finite generating set for S with respect to ψ* if the unique extension of ψ to a semigroup homomorphism $\psi : A^+ \to S$ is surjective. For $u, v \in A^+$ we write $u \equiv v$ to mean that u and v are equal as words and $u = v$ to mean that u and v represent the same element in the semigroup i.e. that $u\psi = v\psi$. We say that a subset L of A^+ is *regular* if there is a finite state automaton accepting L. To be able to deal with automata that accept pairs of words and to define automatic semigroups we need to define the set $A(2, \$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) \backslash \{(\$, \$)\}$ where \$ is a symbol not in A (called the *padding symbol*) and the function $\delta_A : A^* \times A^* \to A(2, \$)^*$ defined by

$$(a_1 \ldots a_m, b_1 \ldots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n \\ (a_1, b_1) \ldots (a_m, b_m) & \text{if } 0 < m = n \\ (a_1, b_1) \ldots (a_m, b_m)(\$, b_{m+1}) \ldots (\$, b_n) & \text{if } 0 \leq m < n \\ (a_1, b_1) \ldots (a_n, b_n)(a_{n+1}, \$) \ldots (a_m, \$) & \text{if } m > n \geq 0. \end{cases}$$

Let S be a semigroup and A a finite generating set for S with respect to $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S (with respect to ψ)* if

- L is a regular subset of A^+ and $L\psi = S$,
- $L_= = \{(\alpha,\beta) : \alpha,\beta \in L, \alpha = \beta\}\delta_A$ is regular in $A(2,\$)^+$, and
- $L_a = \{(\alpha,\beta) : \alpha,\beta \in L, \alpha a = \beta\}\delta_A$ is regular in $A(2,\$)^+$ for each $a \in A$.

We say that a semigroup is *automatic* if it has an automatic structure. If (A, L) is an automatic structure for a semigroup S then there is an automatic structure (A, K) such that each element of S has a unique representative in K (see [**?**, Proposition 5.4]); we say that (A, K) is an *automatic structure with uniqueness*. We say that a semigroup is *prefix-automatic* or *p-automatic* if it has an automatic structure (A, L) such that the set

$$L'_= = \{(w_1, w_2)\delta_A : w_1 \in L, w_2 \in \mathrm{Pref}(L), w_1 = w_2\}$$

is also regular, where

$$\mathrm{Pref}(L) = \{w \in A^+ : ww' \in L \text{ for some } w' \in A^*\}.$$

For more details on automatic semigroups the reader is referred to [**?**] (introduction), [**?**] (geometric aspects and p-automaticity), [**?**], [**?**], [**?**] (computational and decidability aspects) and [**?**], [**?**] (other constructions).

2 GENERALIZED SEQUENTIAL MACHINES

It is known that the fellow-traveler property, which characterizes automatic groups, does not characterize automatic semigroups. So we have to use directly the definition and work with regular languages instead of the Cayley graph to prove that a semigroup is automatic. Since we are working with semigroup constructions, we usually have to construct automatic structures from known automatic structures. For that purpose we use the concept of a generalized sequential machine.

A *generalized sequential machine* (*gsm* for short) is a six-tuple $\mathcal{A} = (Q, A, B, \mu, q_0, T)$ where Q, A and B are finite sets, (called the *states*, the *input alphabet* and the *output alphabet* respectively), μ is a (partial) function from $Q \times A$ to finite subsets of $Q \times B^+$, $q_0 \in Q$ is the *initial state* and $T \subseteq Q$ is the *set of* *terminal states*. The inclusion $(q', u) \in (q, a)\mu$ corresponds to the following situation: if $\mathcal{A}$ is in state q and reads input a, then it can move into state q' and output u.

We can interpret $\mathcal{A}$ as a directed labelled graph with vertices Q, and an edge $q \xrightarrow{(a,u)} q'$ for every pair $(q', u) \in (q, a)\mu$. For a path

$$\pi \ : \ q_1 \xrightarrow{(a_1,u_1)} q_2 \xrightarrow{(a_2,u_2)} q_3 \ldots \xrightarrow{(a_n,u_n)} q_{n+1}$$

we define

$$\Phi(\pi) = a_1 a_2 \ldots a_n, \ \Sigma(\pi) = u_1 u_2 \ldots u_n.$$

For $q, q' \in Q$, $u \in A^+$ and $v \in B^+$ we write $q \xrightarrow{(u,v)}_+ q'$ to mean that there exists a path π from q to q' such that $\Phi(\pi) \equiv u$ and $\Sigma(\pi) \equiv v$, and we say that (u, v) is the *label* of the path. We say that a path is *successful* if it has the form $q \xrightarrow{(u,v)}_+ t$ with $t \in T$.

The gsm $\mathcal{A}$ induces a mapping $\eta_{\mathcal{A}} : \mathcal{P}(A^+) \longrightarrow \mathcal{P}(B^+)$ from subsets of A^+ into subsets of B^+ defined by

$$X\eta_{\mathcal{A}} = \{v \in B^+ \ : \ (\exists u \in X)(\exists t \in T)(q_0 \xrightarrow{(u,v)}_+ t)\}.$$

It is well known that if X is regular then so is $X\eta_{\mathcal{A}}$; see [**?**]. Similarly, $\mathcal{A}$ induces a mapping $\zeta_{\mathcal{A}} : \mathcal{P}(A^+ \times A^+) \longrightarrow \mathcal{P}(B^+ \times B^+)$ defined by

$$Y\zeta_{\mathcal{A}} = \{(w, z) \in B^+ \times B^+ \ : \ (\exists (u, v) \in Y)(w \in u\eta_{\mathcal{A}} \ \& \ z \in v\eta_{\mathcal{A}})\}.$$

The next lemmas asserts that, under certain conditions, this mapping also preserves regularity.

Lemma 2.1 *Let $\mathcal{A} = (Q, A, B, \mu, q_0, T)$ be a gsm, and let $\pi_A : (A^* \times A^*)\delta_A \longrightarrow A^* \times A^*$ be the inverse of δ_A. Suppose that there is a constant C such that for any two paths α_1, α_2 in $\mathcal{A}$, we have*

$$|\Phi(\alpha_1)| = |\Phi(\alpha_2)| \Longrightarrow ||\Sigma(\alpha_1)| - |\Sigma(\alpha_2)|| \leq C. \qquad (1)$$

If $M \subseteq (A^+ \times A^+)\delta_A$ is a regular language in $A(2, \$)^+$ then $N = M\pi_A\zeta_{\mathcal{A}}\delta_B$ is a regular language in $B(2, \$)^+$.

PROOF. To prove that N is regular we will define a gsm $\mathcal{B}$ such that $M\eta_{\mathcal{B}} = N$. First we define three functions with domain $B^* \times B^*$ that will be used in the definition of $\mathcal{B}$:

$$(a_1 \dots a_k, b_1 \dots b_l)\lambda = \begin{cases} a_{l+1} \dots a_k \text{ if } k > l \\ \epsilon \qquad\qquad \text{otherwise,} \end{cases}$$

$$(a_1 \dots a_k, b_1 \dots b_l)\rho = \begin{cases} b_{k+1} \dots b_l \text{ if } l > k \\ \epsilon \qquad\qquad \text{otherwise,} \end{cases}$$

$$(a_1 \dots a_k, b_1 \dots b_l)\kappa = (a_1, b_1) \dots (a_s, b_s), \;\; s = \min(k, l).$$

We now let $\mathcal{B} = (R, A(2,\$), B(2,\$), \nu, r_0, Z)$, where

$$\begin{aligned} R &= Q \times Q \times W \times W, \quad & W &= (\textstyle\bigcup_{k=0}^{C} B^k) \cup \{\$\}, \\ r_0 &= (q_0, q_0, \epsilon, \epsilon), & Z &= T \times T \times \{(\epsilon, \$), (\$, \epsilon), (\$, \$)\}. \end{aligned}$$

In order to define the transition ν, we first extend the transition μ to allow input \$:

$$(q, a)\bar{\mu} = \begin{cases} (q, a)\mu \;\text{ if } a \in A \\ \{(q, \epsilon)\} \text{ if } a = \$. \end{cases}$$

Now the transition ν is defined by

$$((q, q', w, w'), (a, a'))\nu = \bigcup_{\substack{(q_1, u) \in (q, a)\bar{\mu} \\ (q_1', u') \in (q', a')\bar{\mu}}} S_{(q_1, u, q_1', u', w, w')} \;\; (w, w' \in W \backslash \{\$\}),$$

$$((q, q', \$, \epsilon), (\$, a'))\nu = \bigcup_{(q_1', u') \in (q', a')\mu} \{((q, q_1', \$, \epsilon), (\epsilon, u')\delta_B)\} \;\; (a' \neq \$),$$

$$((q, q', \epsilon, \$), (a, \$))\nu = \bigcup_{(q_1, u) \in (q, a)\mu} \{((q_1, q', \epsilon, \$), (u, \epsilon)\delta_B)\} \;\; (a \neq \$),$$

where $q, q' \in Q$, $a, a' \in A \cup \{\$\}$, and

$$\begin{aligned} &S_{(q_1, u, q_1', u', w, w')} \\ &= \begin{cases} \{((q_1, q_1', (wu, w'u')\lambda, (wu, w'u')\rho), (wu, w'u')\kappa), \\ \qquad\qquad ((q_1, q_1', \$, \$), (wu, w'u')\delta_B)\} & \text{if } |wu|, |w'u'| > 0 \\ \{((q_1, q_1', \$, \epsilon), (\epsilon, w'u')\delta_B)\} & \text{if } 0 = |wu| < |w'u'| \\ \{((q_1, q_1', \epsilon, \$), (wu, \epsilon)\delta_B)\} & \text{if } 0 = |w'u'| < |wu| \end{cases} \end{aligned}$$

provided $||wu| - |w'u'|| \leq C$, and $S_{(q_1, u, q_1', u', w, w')} = \emptyset$ otherwise.

We now prove that $N \subseteq M\eta_{\mathcal{B}}$. Let $(v, v')\delta_B \in N$. By definition of N there is $(u, u')\delta_A \in M$ such that $(v, v') \in \{(u, u')\}\zeta_{\mathcal{A}}$. So $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$. This means that in $\mathcal{A}$ there are paths of the form

$$\begin{aligned} q_{i-1} &\xrightarrow{(a_i,w_i)} q_i \ (i = 1, \dots, m,\ a_i \in A,\ w_i \in B^+) \\ q'_{i-1} &\xrightarrow{(a'_i,w'_i)} q'_i \ (i = 1, \dots, n,\ a'_i \in A,\ w'_i \in B^+), \end{aligned}$$

with

$$u = a_1 \dots a_m,\ v = w_1 \dots w_m,\ u' = a'_1 \dots a'_n,\ v' = w'_1 \dots w'_n,\ q'_0 = q_0,\ q_m, q'_n \in T.$$

We now show that there is a successful path in $\mathcal{B}$ of the form

$$(q_{i-1}, q'_{i-1}, z_{i-1}, z'_{i-1}) \xrightarrow{((a_i,a'_i),\omega_i)} (q_i, q'_i, z_i, z'_i)\ (i = 1, \dots, p),$$

where $p = \max(m, n)$ and $q_m = q_{m+1} = \dots = q_p$, $q'_n = q'_{n+1} = \dots = q'_p$, $a_{m+1} = \dots = a_p = a'_{n+1} = \dots = a'_p = \$$, such that the output $\omega_1 \dots \omega_p$ is equal to $(v, v')\delta_B$. To begin with we follow a path visiting states from the set $Q \times Q \times (W\backslash\{\$\}) \times (W\backslash\{\$\})$ as long as $i < p$ and

$$|z_{i-1}w_i|, |z'_{i-1}w'_i| > 0. \tag{2}$$

The output ω_i in these transitions is the longest prefix of $(z_{i-1}w_i, z'_{i-1}w'_i)\delta_B$ that belongs to $(B \times B)^+$, z_i is equal to the remaining letters in $z_{i-1}w_i$ if $|z_{i-1}w_i| > |z'_{i-1}w'_i|$ (otherwise it is ϵ) and z'_i is equal to the remaining letters of $z'_{i-1}w'_i$ if $|z'_{i-1}w'_i| > |z_{i-1}w_i|$ (otherwise it is ϵ). We note that $|z_i|, |z'_i| \le C$ because of assumption (**??**). So after transition i, the complete output produced is the longest prefix of $(w_1 \dots w_i, w'_1 \dots w'_i)\delta_B$ that belongs to $(B \times B)^+$. If for $i = p$ condition (**??**) still holds then we set $z_p = z'_p = \$$ and $\omega_p = (z_{p-1}w_p, z'_{p-1}w'_p)\delta_B$, i.e. the output in this last transition is the remainder of $(v, v')\delta_B$. The machine ends in the terminal state $(q_p, q'_p, \$, \$)$ and the complete output is $(v, v')\delta_B$. Suppose now that condition (**??**) does not hold for some $i \in \{2, \dots, p\}$, and let j be the smallest such. Suppose that $0 = |z_{j-1}w_j| < |z'_{j-1}w'_j|$ (the other case is similar). Then $z_{j-1} = \epsilon$ and, since $w_1, \dots, w_m \in B^+$, we must have $j > m$, $a_j = \$$ and $w_j = \epsilon$. So the complete output produced until transition $j - 1$ is the longest prefix of $(v, v')\delta_B$ that belongs to $(B \times B)^+$. By definition of ν, we must have $z_j = \$$, $z'_j = \epsilon$ and the output of transition $j - 1$ is $(\epsilon, z'_{j-1}w'_j)\delta_B$.

Now we follow a path visiting states from $Q \times Q \times \{\$\} \times \{\epsilon\}$ and output the remainder $(\epsilon, w'_{j+1} \dots w'_n)\delta_B$ of the word $(v, v')\delta_B$ ending in the terminal state $(q_p, q'_p, \$, \epsilon)$. Again the complete output is the word $(v, v')\delta_B$. We conclude that $(v, v')\delta_B \in \{(u, u')\}\eta_{\mathcal{B}} \subseteq M\eta_{\mathcal{B}}$.

We now show the converse inclusion $M\eta_{\mathcal{B}} \subseteq N$. First we note that in a path in $\mathcal{B}$, each transition outputs a word in $(B^+ \times B^+)\delta_B$. Moreover if a word of the form $(w, w')\delta_B$, with $|w| < |w'|$, is output in a transition, then either the machine enters a state in $Q \times Q \times \{\$\} \times \{\$\}$ and stops or it enters the states of $Q \times Q \times \{\$\} \times \{\epsilon\}$, from where it cannot move out, and where the transitions output words of the form $(\epsilon, w)\delta_B$. So the complete output of a path is always a word in $(B^+ \times B^+)\delta_B$, noting that a similar argument applies if the first output containing a \$ is a word of the form $(w, w')\delta_B$, with $|w| > |w'|$.

Now let $(v, v')\delta_B \in M\eta_{\mathcal{B}}$. So there is a successful path in $\mathcal{B}$ with label $((u, u')\delta_A, (v, v')\delta_B)$ with $(u, u')\delta_A \in M$. We will prove that $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$ to conclude that

$$(v, v')\delta_B \in \{(u, u')\delta_A\}\pi_A\zeta_{\mathcal{A}}\delta_B \subseteq M\pi_A\zeta_{\mathcal{A}}\delta_B = N,$$

as required. Let

$$u = a_1 \dots a_m, \ u' = a'_1 \dots a'_n, \ p = \max(m, n).$$

By definition of ν a successful path in $\mathcal{B}$ labeled by $((u, u')\delta_A, (v, v')\delta_B)$ has the form

$$(q_{i-1}, q'_{i-1}, w_{i-1}, w'_{i-1}) \xrightarrow{((a_i, a'_i), \omega_i)} (q_i, q'_i, w_i, w'_i) \ (i = 1, \dots, p),$$

where $q_0 = q'_0$, $q_p, q'_p \in T$, $a_{m+1} = \dots = a_p = a'_{n+1} = \dots = a'_p = \$$, $q_m = \dots = q_p$, $q'_m = \dots = q'_p$, and $\omega_1 \dots \omega_p = (v, v')\delta_B$. This yields successful paths

$$\begin{aligned} q_{i-1} &\xrightarrow{(a_i, z_i)} q_i \ (i = 1, \dots, m) \\ q'_{i-1} &\xrightarrow{(a'_i, z'_i)} q'_i \ (i = 1, \dots, n) \end{aligned}$$

in $\mathcal{A}$, with $v = z_1 \dots z_m$ and $v' = z'_1 \dots z'_n$. So $v \in u\eta_{\mathcal{A}}$ and $v' \in u'\eta_{\mathcal{A}}$, completing the proof of the lemma. ■

3 AUTOMATICITY OF A REES MATRIX SEMIGROUP

We start this section by stating our first main result.

Theorem 3.1 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If U is an automatic semigroup and if S is finitely generated then S is automatic.*

Let V be the ideal of U generated by the entries of the matrix P i.e. $V = \{sp_{ji}s' : s, s' \in U^1, i \in I, j \in J\}$, where U^1 is the monoid obtained by adding an identity to U regardless of whether or not U already has an identity. ¿From the Main Theorem of [**?**] we know that $S = \mathcal{M}[U; I, J; P]$ is finitely generated if and only if U is finitely generated and I, J and $U\backslash V$ are finite. So the previous theorem has the following equivalent formulation:

Theorem 3.2 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup, where I, J are finite sets and $U\backslash V$ is finite, where V is the ideal of U generated by the entries of the matrix P. If U is an automatic semigroup then S is automatic.*

Proof. Since U is automatic and $U\backslash V$ is finite, by [**?**, Theorem 1.1], V is an automatic semigroup. Let (B, K) be an automatic structure with uniqueness for V, where $B = \{b_1, \dots, b_n\}$ is a set of semigroup generators for V. Since V is the ideal of U generated by the entries in the matrix P we can write each b_h ($h \in N = \{1, \dots, n\}$) as $b_h = s_h p_{\rho_h \lambda_h} s'_h$ where $s_h, s'_h \in U^1, \rho_h \in J, \lambda_h \in I$. Let $S_1 = \mathcal{M}[U^1; I, J; P]$. Given $(l, s, r) \in I \times V \times J$ we can write $s = b_{\alpha_1} \dots b_{\alpha_h}$ where $b_{\alpha_1} \dots b_{\alpha_h}$ is a word in K. So we can write

$$(l, s, r) = (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1} s_{\alpha_2}, \rho_{\alpha_2}) \dots (\lambda_{\alpha_h}, s'_{\alpha_h}, r).$$

We note that the elements in the above sequence are elements of S_1 but some of them can be outside S. Since $U^1\backslash V$ is finite and non empty we can write $U^1\backslash V = \{x_1, \dots, x_m\}$ with $m \geq 1$. We define a set $A = C \cup D$ of semigroup generators for S_1 by

$$C = \{c_{li} : l \in I, i \in N\} \cup \{d_{ij} : i, j \in N\} \cup \{e_{jr} : j \in N, r \in J\},$$
$$D = \{f_{lhr} : l \in I, h \in \{1, \dots, m\}, r \in J\}$$

with

$$\psi : A^+ \to S_1, \ c_{li} \mapsto (l, s_i, \rho_i), \ \ d_{ij} \mapsto (\lambda_i, s_i' s_j, \rho_j),$$
$$e_{jr} \mapsto (\lambda_j, s_j', r), f_{lhr} \mapsto (l, x_h, r).$$

We can define a language $L = L_1 \cup D$ to represent the elements of S_1 with

$$L_1 = \{c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} : b_{\alpha_1} \dots b_{\alpha_h} \in K, h \geq 1, l \in I, r \in J\}.$$

We are going to prove that (A, L) is an automatic structure for S_1. First we need to prove that L is a regular language. To this end let

$$L^{(l,r)} = L \cap (\{c_{li} : i \in N\} \cdot A^* \cdot \{e_{jr} : j \in N\}).$$

Then we can write

$$L = (\bigcup_{l \in I, r \in J} L^{(l,r)}) \cup D$$

and it is sufficient to prove that for each $l \in I, r \in J$ the set $L^{(l,r)}$ is regular. To do that we define a gsm $\mathcal{A}$ such that $K\eta_{\mathcal{A}} = L^{(l,r)}$. Let $\mathcal{A} = (Q, B, A, \mu, q_0, \{\xi\})$ with $Q = N \cup \{q_0, \xi\}$, where q_0 is the initial state, ξ is the only accept state and μ is a partial function from $Q \times B$ to finite subsets of $Q \times A^+$ defined by:

$$(q_0, b_i)\mu = \{(i, c_{li}), (\xi, c_{li} e_{ir})\} \ (i \in N),$$
$$(i, b_j)\mu = \{(j, d_{ij}), (\xi, d_{ij} e_{jr})\} \ (i, j \in N).$$

Given $u = b_{\alpha_1} \dots b_{\alpha_h} \in K$ it is clear that the only word v in A^+ such that (u, v) corresponds to the label of a successful path in $\mathcal{A}$ is the word $c_{l\alpha_1} d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h} e_{\alpha_h r} \in L^{(l,r)}$. So $K\eta_{\mathcal{A}} = L^{(l,r)}$ and $L^{(l,r)}$ is a regular language.

Given two different words $u_1, u_2 \in K$, they represent two different elements $s_1, s_2 \in V$ because we have assumed that (B, K) is an automatic structure with uniqueness for V. It is clear that the words $u_1\eta_{\mathcal{A}}$ and $u_2\eta_{\mathcal{A}}$ in $L^{(l,r)}$ represent the elements (l, s_1, r) and (l, s_2, r) respectively. It follows that two different words in $L^{(l,r)}$ represent two different elements in $\{l\} \times V \times \{r\}$, and hence that two different words in L represent two different elements in S_1. Therefore $L_= = \Delta_L = \{(w, w)\delta_A : w \in L\}$ is regular.

To conclude that (A, L) is an automatic structure for S_1 (with uniqueness) it remains to show that, given $a \in A$, the language L_a is regular. Let $a\psi = (l_0, s_0, r_0) \in S_1$. Let us fix $l \in I$ and $r \in J$ and prove that the language

$$L_a^{(l,r)} = L_a \cap (L^{(l,r)} \times A^*)\delta_A$$

is regular. Let $\overline{w}$ be the only word in K that represents the element $p_{rl_0}s_0 \in V$. We know from [?, Proposition 3.2] that $K_{\overline{w}}$ is a regular set and we will now show that

$$K_{\overline{w}}\pi_B\zeta_{\mathcal{A}}\delta_A = L_a^{(l,r)},$$

where $\pi_B : (B^* \times B^*)\delta_B \to B^* \times B^*$ is the inverse of δ_B. For $b_{\alpha_1} \dots b_{\alpha_h}, b_{\beta_1} \dots b_{\beta_k} \in K$ we have

$$\begin{aligned}
& (b_{\alpha_1} \dots b_{\alpha_h}, b_{\beta_1} \dots b_{\beta_k})\delta_B \in K_{\overline{w}} \\
\Longleftrightarrow\ & b_{\alpha_1} \dots b_{\alpha_h}\overline{w} = b_{\beta_1} \dots b_{\beta_k} \\
\Longleftrightarrow\ & (l, b_{\alpha_1} \dots b_{\alpha_h}\overline{w}, r_0) = (l, b_{\beta_1} \dots b_{\beta_k}, r_0) \\
\Longleftrightarrow\ & (l, s_{\alpha_1}, \rho_{\alpha_1})(\lambda_{\alpha_1}, s'_{\alpha_1}s_{\alpha_2}, \rho_{\alpha_2}) \dots (\lambda_{\alpha_{h-1}}, s'_{\alpha_{h-1}}s_{\alpha_h}, \rho_{\alpha_h})(\lambda_{\alpha_h}, s'_{\alpha_h}, r)(l_0, s_0, r_0) \\
& = (l, s_{\beta_1}, \rho_{\beta_1})(\lambda_{\beta_1}, s'_{\beta_1}s_{\beta_2}, \rho_{\beta_2}) \dots (\lambda_{\beta_{k-1}}, s'_{\beta_{k-1}}s_{\beta_k}, \rho_{\beta_k})(\lambda_{\beta_k}, s'_{\beta_k}, r_0) \\
\Longleftrightarrow\ & c_{l\alpha_1}d_{\alpha_1\alpha_2} \dots d_{\alpha_{h-1}\alpha_h}e_{\alpha_h r}a = c_{l\beta_1}d_{\beta_1\beta_2} \dots d_{\beta_{k-1}\beta_k}e_{\beta_k r_0} \\
\Longleftrightarrow\ & (b_{\alpha_1} \dots b_{\alpha_h}, b_{\beta_1} \dots b_{\beta_k})\zeta_{\mathcal{A}}\delta_A \in L_a^{(l,r)}.
\end{aligned}$$

We note that in a path in $\mathcal{A}$ each transition outputs a word of length 1 except possibly the last that can output a word of length 1 or 2 and so condition (**??**) in Lemma **??** holds with $C = 1$. Applying Lemma **??** we conclude that $L_a^{(l,r)}$ is a regular set.

The set $\overline{L}_a^{(l,r)} = L_a \cap (\{f_{rhs} : h \in \{1, \dots, m\}\} \times A^*)\delta_A$ is regular because it is finite. But then

$$L_a = (\bigcup_{l\in I, r\in J} L_a^{(l,r)}) \cup (\bigcup_{l\in I, r\in J} \overline{L}_a^{(l,r)}),$$

and so L_a is regular.

We conclude that S_1 is an automatic semigroup. Since S is a subsemigroup of S_1 such that $S_1 \backslash S$ is finite we can use [?, Theorem 1.1] to conclude that S is an automatic semigroup. ∎

Note that Theorem **??** generalises [?, Theorem 7.2], where it is assumed that U is a monoid and that P contains the identity of U. Another interesting application of our theorem arises when U is a simple semigroup:

Corollary 3.3 *If U is an automatic simple semigroup then every Rees matrix semigroup $\mathcal{M}[U; I, J; P]$ (I and J finite) is automatic.* ∎

Example 3.4 The fundamental four spiral semigroup Sp_4 (see Section 8.6 in [**?**]) can be represented as a 2×2 Rees matrix semigroup over the bicyclic monoid B (see [**?**, Exercise 3.8.19] and [**?**]). Since B is simple and automatic ([**?**, Example 4.2]) it follows that Sp_4 is also automatic.

4 AUTOMATICITY OF THE BASE SEMIGROUP IN AN AUTOMATIC REES MATRIX SEMIGROUP

If $S = \mathcal{M}[U; I, J; P]$ is an automatic semigroup we know by the Main Theorem of [**?**] that U must be finitely generated. It is an open question if U is automatic in general. We prove that U is automatic if we assume that there is an element p in the matrix such that the principal right ideal pU^1 generated by p is equal to U. This assumption includes the case where there is an identity in the matrix, considered in [**?**] where the authors are interested in p-automatic monoids. We also prove that if S is p-automatic then U is p-automatic. It is an open question if the definitions of p-automatic and automatic coincide for semigroups as they do for groups and, more generally, for right cancellative monoids; see [**?**, Theorem 8.1].

Theorem 4.1 *Let $S = \mathcal{M}[U; I, J; P]$ be a semigroup, and suppose that there is an entry p in the matrix P such that $pU^1 = U$. If S is automatic then U is automatic.*

Proof. Let $S_1 = \mathcal{M}[U^1; I, J; P]$. Then S is a subsemigroup of S_1 such that $S_1 \backslash S$ is finite. Since S is automatic, by [**?**, Theorem 1.1], S_1 is also automatic. Let (A, L) be an automatic structure for S_1 with uniqueness, where $A = \{a_1, \ldots, a_n\}$ is a generating set for S_1 with respect to

$$\psi : A^+ \to S_1,\ a_h \mapsto (i_h, s_h, j_h)\ (h = 1, \ldots, n).$$

Then

$$B = \{b_1, \ldots, b_n\} \cup \{c_{ji} : j \in J, i \in I\}$$

is a generating set for U^1 with respect to

$$\phi : B^+ \to U^1; b_h \mapsto s_h, c_{ji} \mapsto p_{ji}\ (h = 1, \ldots, n,\ j \in J, i \in I);$$

see [?, Proposition 2.2]. Without loss of generality we can assume that $p_{11} = p$. Let

$$L_{11} = L \cap (\{1\} \times U^1 \times \{1\})\psi^{-1}.$$

This set is regular because

$$\begin{aligned}(\{1\} \times U^1 \times \{1\})\psi^{-1} = &\{a_h \in A : i_h = 1\} \cdot A^* \cdot \{a_h \in A : j_h = 1\} \cup \\ &\{a_h \in A : i_h = j_h = 1\}.\end{aligned}$$

Let

$$f : A^+ \to B^+; a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_h} \mapsto b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \ldots c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h}.$$

We define $K = L_{11} f$ and prove that (B, K) is an automatic structure with uniqueness for U^1 with respect to ϕ. We observe that $f : L_{11} \to K$ is a bijection and K represents the elements of U^1 with uniqueness. In fact, if a word $w \in L_{11}$ represents the element $(1, s, 1) \in \{1\} \times U^1 \times \{1\}$ then the corresponding word wf in K represents the element $s \in U^1$.

Next we show that K is a regular language by defining a gsm $\mathcal{A}$ such that $L_{11}\eta_{\mathcal{A}} = K$. Let $\mathcal{A} = (Q, A, B, \mu, q_0, \{\chi\})$ with $Q = \{q_0, \chi\} \cup J$, where q_0 is the initial state, χ is the only accept state and the transition μ is a partial function from $Q \times A$ to finite subsets of $Q \times B^+$ defined by:

$$\begin{aligned}(q_0, a_h)\mu &= \{(j_h, b_h), (\chi, b_h)\} \ (h \in \{1, \ldots, n\}), \\ (j, a_h)\mu &= \{(j_h, c_{ji_h} b_h), (\chi, c_{ji_h} b_h)\} \ (j \in J, h \in \{1, \ldots, n\}).\end{aligned}$$

Given a word

$$u \equiv a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_h} \in L_{11}$$

there is a unique successful path α in $\mathcal{A}$ such that $\Phi(\alpha) = u$.

This path is

$$q_0 \xrightarrow{(a_{\alpha_1}, b_{\alpha_1})} \chi$$

for $h = 1$ and

$$\begin{aligned}&q_0 \xrightarrow{(a_{\alpha_1}, b_{\alpha_1})} j_{\alpha_1} \xrightarrow{(a_{\alpha_2}, c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2})} j_{\alpha_2} \to \ldots \\ &\ldots \xrightarrow{(a_{\alpha_{h-1}}, c_{j_{\alpha_{h-2}} i_{\alpha_{h-1}}} b_{\alpha_{h-1}})} j_{\alpha_{h-1}} \xrightarrow{(a_{\alpha_h}, c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h})} \chi\end{aligned}$$

for $h > 1$, and its output is

$$\Sigma(\alpha) = b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} \dots c_{j_{\alpha_{h-1}} i_{\alpha_h}} b_{\alpha_h} \equiv uf \in K.$$

We conclude that $K = L_{11}\eta_{\mathcal{A}}$ is regular, as claimed.

We now start proving that K_b is regular for $b \in B$. If $b\phi = 1$ then $K_b = K_=$ and it follows from the uniqueness of K that K_b is regular. If $b\phi \neq 1$ then $b\phi \in U = p_{11}U^1$ and we can write $b\phi = p_{11}s$ for some $s \in U^1$. Since $(1, s, 1)$ is an element of S_1 there is a word $\overline{w} \in L$ that represents the element $(1, s, 1)$. We know by [**?**, Proposition 3.2] that $L_{\overline{w}}$ is a regular language. Let us consider the regular language $H = L_{\overline{w}} \cap (L_{11} \times L_{11})\delta_A$ and prove that $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b$. For $a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h}, a_{\beta_1} a_{\beta_2} \dots a_{\beta_k} \in L_{11}$ we have

$$\begin{aligned}
& (a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h}, a_{\beta_1} a_{\beta_2} \dots a_{\beta_k})\delta_A \in H \\
\Longleftrightarrow\; & a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h} \overline{w} = a_{\beta_1} a_{\beta_2} \dots a_{\beta_k} \\
\Longleftrightarrow\; & (1, s_{\alpha_1}, j_{\alpha_1})(i_{\alpha_2}, s_{\alpha_2}, j_{\alpha_2}) \dots (i_{\alpha_h}, s_{\alpha_h}, 1)(1, s, 1) \\
& = (1, s_{\beta_1}, j_{\beta_1})(i_{\beta_2}, s_{\beta_2}, j_{\beta_2}) \dots (i_{\beta_k}, s_{\beta_k}, 1) \\
\Longleftrightarrow\; & (b_{\alpha_1} c_{j_{\alpha_1} i_{\alpha_2}} b_{\alpha_2} c_{j_{\alpha_2} i_{\alpha_3}} \dots b_{\alpha_h}, b_{\beta_1} c_{j_{\beta_1} i_{\beta_2}} b_{\beta_2} c_{j_{\beta_2} i_{\beta_3}} \dots b_{\beta_k})\delta_B \in K_b \\
\Longleftrightarrow\; & (a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_h}, a_{\beta_1} a_{\beta_2} \dots a_{\beta_k})\zeta_{\mathcal{A}}\delta_B \in K_b.
\end{aligned}$$

Since in any path in $\mathcal{A}$ only the first transition can output a word of length 1 and all the others output words of length 2 we can apply Lemma **??** with $C = 1$ and conclude that K_b is a regular language. So U^1 is an automatic semigroup and, by [**?**, Theorem 7.2], U is automatic. ■

Note that Theorem **??** generalizes [**?**, Theorem 7.4], where it is assumed that U is a monoid and that P has a row and a column consisting entirely of ones.

Theorem 4.2 *Let $S = \mathcal{M}[U; I, J; P]$ be a Rees matrix semigroup. If S is prefix-automatic then U is prefix-automatic.*

Proof. By [**?**, Corollary 5.4] we can fix a prefix-automatic structure with uniqueness (A, L) for S. We define $A, \psi, B, \phi, L_{11}, f, \mathcal{A}$ and K as in the proof of the previous theorem just replacing U^1 by U and S_1 by S in the definitions, and assume that $\psi\restriction_A$ is injective. We will prove that (B, K) is a prefix-automatic structure with uniqueness for U with respect to ϕ. We have proved that K is regular, that $\phi \restriction_L$ is injective and that f is a bijection, without using the fact that U^1 is a monoid. So we just have to prove that

$$K_b = \{(v_1, v_2)\delta_B : v_1, v_2 \in K, v_1 b = v_2\}$$

is a regular language for $b \in B$ to conclude that U is automatic. We start by writing K_b as a finite union of sets which we then prove are regular. We can write

$$K_b = \{(w_1 f, w_2 f)\delta_B : w_1, w_2 \in L_{11}, (w_1 f)b = w_2 f\}.$$

Let $A_1 = \{a_h \in A : j_h = 1\}$. We define

$$K_b^a = \{(w_1 f, w_2 f)\delta_B \in K_b : w_1 \in A^+ a\}$$

for $a \in A_1$. We also define $K_b^1 = K_b \cap (B \times B^*)\delta_B$. It is clear that

$$K_b = (\bigcup_{a \in A_1} K_b^a) \cup K_b^1.$$

The language K_b^1 is regular because it is finite. Let us fix an element $a \in A_1$, with $a\psi = (l, s, 1)$, and prove that K_b^a is a regular set. Let $\overline{w}$ be the only word in L representing $(l, sb, 1)$. The set

$$L'_= = \{(w_1, w_2)\delta_A : w_1 \in L, w_2 \in \mathrm{Pref}(L), w_1 = w_2\}$$

is regular by hypothesis and the set $L_{\overline{w}}$ is regular by [**?**, Proposition 3.2]. So the set

$$D = \{(w_1', w_2)\delta_A : (\exists w_1'' \in A^*)((w_1', w_1'')\delta_A \in L'_=\tau \ \&\ (w_1'', w_2)\delta_A \in L_{\overline{w}})\}$$

is regular by [**?**, Proposition 2.3], where

$$\tau : A(2, \$)^* \to A(2, \$)^*;\ (a, b) \mapsto (b, a)$$

is the homomorphism that swaps coordinates. The set

$$E = \{(w_1' a, w_2)\delta_A : (w_1', w_2)\delta_A \in D\}$$

is also regular, since we can write

$$E = \{(w_1, w_2)\delta_A : (\exists w_1' \in A^*)((w_1, w_1')\delta_A \in F \ \&\ (w_1', w_2)\delta_A \in D)\},$$

where $F = \{(wa, w)\delta_A : w \in A^*\}$ is a regular set.

We now use the regular set

$$H = E \cap (L_{11} \times L_{11})\delta_A$$

to prove that L_b^a is regular by showing that $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b^a$. We note that we can write $H = \{(w_1, w_2)\delta_A : w_1, w_2 \in L_{11} \ \& \ w_1 \equiv w_1'a \ \& \ w_1'\overline{w} = w_2\}$ and so for $a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k} \in L_{11}$ we have

$$\begin{aligned}
&(a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k})\delta_A \in H \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \& \ a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_{h-1}}\overline{w} = a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k} \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \& \ (1, s_{\alpha_1}, j_{\alpha_1})(i_{\alpha_2}, s_{\alpha_2}, j_{\alpha_2})\ldots(i_{\alpha_{h-1}}, s_{\alpha_{h-1}}, j_{\alpha_{h-1}})(i_{\alpha_h}, s_{\alpha_h}b, 1) \\
& \qquad\qquad = (1, s_{\beta_1}, j_{\beta_1})(i_{\beta_2}, s_{\beta_2}, j_{\beta_2})\ldots(i_{\beta_k}, s_{\beta_k}, 1) \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \& \ s_{\alpha_1}p_{j_{\alpha_1}i_{\alpha_2}}s_{\alpha_2}p_{j_{\alpha_2}i_{\alpha_3}}\ldots s_{\alpha_h}b = s_{\beta_1}p_{j_{\beta_1}i_{\beta_2}}s_{\beta_2}p_{j_{\beta_2}i_{\beta_3}}\ldots s_{\beta_k} \\
\Longleftrightarrow\ & a_{\alpha_h} = a \ \& \ ((a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h})f)b = (a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k})f \\
\Longleftrightarrow\ & (a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_h}, a_{\beta_1}a_{\beta_2}\ldots a_{\beta_k})\zeta_{\mathcal{A}}\delta_B \in K_b^a.
\end{aligned}$$

So we have $H\pi_A\zeta_{\mathcal{A}}\delta_B = K_b^a$ and we can use Lemma **??** to conclude that K_b^a is regular. Therefore K_b is regular and U is an automatic semigroup.

To prove that U is prefix-automatic we prove that $K'_=\tau$ is a regular set. We begin by writing it as a union of sets which we then prove are regular:

$$K'_=\tau = (\bigcup_{i,b,c} X_{(i,b,c)}) \cup Y \cup Z,$$

where

$$\begin{aligned}
&X_{(i,b,c)} = \{(w_1, w_2)\delta_B \in K'_=\tau : (\exists \alpha \in \{1, \ldots, n\})(w_1 \in B^+ b_\alpha c_{j_\alpha i} bc)\} \\
&Y = \{(w_1, w_2)\delta_B \in K'_=\tau : (\exists b \in \{b_1, \ldots, b_n\})(w_1 \in B^+ b)\} \\
&Z = \{(w_1, w_2)\delta_B \in K'_=\tau : |w_1| < 5\}
\end{aligned}$$

for $i \in I, b \in \{b_1, \ldots, b_n\}, c \in \{c_{ji} : j \in J, i \in I\}$. Let us fix i, b and c and let $\overline{w}$ be the (unique) word in L representing $(i, bc, 1)$. Defining $L'_{\overline{w}} = \{(u_1, u_2)\delta_A : u_1 \in \mathrm{Pref}(L), u_2 \in L, u_1\overline{w} = u_2\}$ and observing that for $w_1 \in \mathrm{Pref}(K), w_2 \in K$ with $w_1 \equiv b_{\alpha_1}c_{j_{\alpha_1}i_{\alpha_2}}b_{\alpha_2}\ldots b_{\alpha_h}c_{j_{\alpha_h}i}bc$ and $w_2 \equiv b_{\beta_1}c_{j_{\beta_1}i_{\beta_2}}b_{\beta_2}\ldots b_{\beta_k}$ we have

$$\begin{aligned}
&w_1 = w_2 \\
\Longleftrightarrow\ & (1, s_{\alpha_1}, j_{\alpha_1})\ldots(i_{\alpha_h}, s_{\alpha_h}, j_{\alpha_h})(i, bc, 1) = (1, s_{\beta_1}, j_{\beta_1})\ldots(j_{\beta_k}, s_{\beta_k}, 1) \\
\Longleftrightarrow\ & a_{\alpha_1}\ldots a_{\alpha_h}\overline{w} = a_{\beta_1}\ldots a_{\beta_k} \\
\Longleftrightarrow\ & (a_{\alpha_1}\ldots a_{\alpha_h}, a_{\beta_1}\ldots a_{\beta_k})\delta_A \in L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A,
\end{aligned}$$

we can write

$$\begin{aligned}X_{(i,b,c)} = &(\bigcup_{\alpha\in\{1,\ldots,n\}}(B^+\{b_\alpha c_{j_\alpha i}bc\} \times B^+)\delta_B) \cap (\mathrm{Pref}(K) \times K)\delta_B \cap \\ &(\bigcup_{\alpha\in\{1,\ldots,n\}}\{((u_1 f)c_{j_\alpha i}bc, u_2 f)\delta_B : (u_1, u_2)\delta_A \in L'_{\overline{w}} \cap \\ &(\mathrm{Pref}(L_{11}) \times L_{11})\delta_A\}).\end{aligned}$$

The sets $(\mathrm{Pref}(L_{11}) \times L_{11})\delta_A$, $(B^+\{b_\alpha c_{j_\alpha i}bc\} \times B^+)\delta_B$ and $(\mathrm{Pref}(K) \times K)\delta_B$ are regular by [**?**, Proposition 2.2]. The set $L'_{\overline{w}}$ is regular because we can write

$$L'_{\overline{w}} = \{(u_1, u_2)\delta_A : (\exists u_3 \in A^+)((u_1, u_3)\delta_A \in L_= \tau \ \& \ (u_3, u_2)\delta_A \in L_{\overline{w}})\}$$

and use [**?**, Proposition 3.2]. The set $N = (L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A)\pi_A\zeta_{\mathcal{A}}\delta_B$ is regular by Lemma **??** and so for a fixed $\alpha \in \{1, \ldots, n\}$ the set

$$\begin{aligned}&\{(w_1 c_{j_\alpha i}bc, w_2)\delta_B : (w_1, w_2)\delta_B \in N\} \\ &= \{((u_1 f)c_{j_\alpha i}bc, u_2 f)\delta_B : (u_1, u_2)\delta_A \in L'_{\overline{w}} \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A\}\end{aligned}$$

is also regular. Hence $X_{(i,b,c)}$ is regular. We note that $Y = (L'_= \tau \cap (\mathrm{Pref}(L_{11}) \times L_{11})\delta_A)\pi_A\zeta_{\mathcal{A}}\delta_B$ and by Lemma **??** it is regular. Since Z is finite it is proved that $K'_=$ is regular and so U is prefix-automatic. ■

5 REES MATRIX SEMIGROUPS WITH ZERO

In this section we show that the previous results are still valid if we consider Rees matrix semigroups with zero. The Rees matrix semigroup with zero $S = \mathcal{M}^0[U; I, J; P]$ over the semigroup U, where $P = (p_{ji})_{j\in J, i\in I}$ is a matrix with entries in U^0 (U with a zero adjoined to it), is the semigroup with the support set $(I \times U \times J) \cup \{0\}$ and multiplication defined by

$$\begin{aligned}&(l_1, s_1, r_1)(l_2, s_2, r_2) = \begin{cases}(l_1, s_1 p_{r_1 l_2} s_2, r_2) & \text{if } p_{r_1 l_2} \neq 0, \\ 0 & \text{otherwise,}\end{cases} \\ &(l_1, s_1, r_1)0 = 0(l_2, s_2, r_2) = 0 \cdot 0 = 0.\end{aligned}$$

Alternatively, S can be viewed as the Rees quotient S'/M, where $S' = \mathcal{M}[U^0; I, J; P]$ (a Rees matrix semigroup without zero), and $M = I \times \{0\} \times J$ (an ideal). With this in mind, the following result from [**?**] will prove useful:

Proposition 5.1 *If S is an automatic semigroup and if I is a finite ideal of S then S/I is automatic as well.*

In general the converse does not hold; see [**?**]. However, in our context it does:

Proposition 5.2 *If $S = \mathcal{M}^0[U; I, J; P]$ is automatic (resp. prefix-automatic) then so is $T = \mathcal{M}[U^0; I, J; P]$.*

Proof. First we note that S has an automatic (resp. prefix-automatic) structure with uniqueness of the form $(A \cup \{\iota\}, L \cup \{\iota\})$, where ι represents 0, and no element of A or L represents 0. Indeed, let (B, K) be any automatic structure with uniqueness for S, and let w be the only element of K representing 0. Define $A = \{b \in B : b \neq 0\}$ and $L = K \backslash \{w\}$. That $(A \cup \{\iota\}, L \cup \{\iota\})$ is an automatic structure with uniqueness for S follows from Propositions 5.8 and 5.7 and Corollary 5.2 in [**?**]. Moreover, if (B, K) is a prefix-automatic structure for S, then so is $(A \cup \{\iota\}, L \cup \{\iota\})$, because

$$(L \cup \{\iota\})'_= = (K'_= \backslash \{(u, v)\delta_A \in K'_= : v \in \mathrm{Pref}(w)\}) \cup \{(\iota, \iota)\},$$

and the set $\{(u, v)\delta_A \in K'_= : v \in \mathrm{Pref}(w)\}$ is finite.

If $A = \{a_1, \ldots, a_n\}$, and if a_h is mapped onto (i_h, s_h, j_h), then obviously T is generated by the set $C = A \cup \{\iota_{ij} : i \in I, j \in J\}$ under the mapping

$$a_h \mapsto (i_h, s_h, j_h), \ \ \iota_{ij} \mapsto (i, 0, j).$$

Let also

$$M = L \cup \{\iota_{ij} : i \in I, j \in J\}.$$

Clearly, M represents T with uniqueness.

Denoting for a moment the multiplication in T by $*$ we see that

$$(l_1, s_1, r_1) * (l_2, s_2, r_2) = \begin{cases} (l_1, s_1, r_1)(l_2, s_2, r_2) & \text{if } (l_1, s_1, r_1)(l_2, s_2, r_2) \neq 0, \\ (l_1, 0, r_2) & \text{otherwise.} \end{cases}$$

Therefore, using the regular sets

$$L^l = L \cap \{a_h \in A : i_h = l\}A^* \ (l \in I),$$
$$L^{(a_h, 0)} = \{w \in A^+ : (w, \iota)\delta_A \in (L \cup \{\iota\})_{a_h}\} \ (a_h \in A),$$

we see that

$$\begin{aligned} M_{a_h} &= ((L \cup \{\iota\})_{a_h} \cap (A^+ \times A^+)\delta_A) \cup (\textstyle\bigcup_{l\in I}((L^l \cap L^{(a_h,0)}) \times \{\iota_{lj_h}\})\delta_C) \cup \\ &\quad \{(\iota_{ij}, \iota_{ij_h}) : i \in I, j \in J\}, \\ M_{\iota_{ij}} &= (\textstyle\bigcup_{l\in I}(L^l \times \{\iota_{lj}\})\delta_C) \cup \{(\iota_{lr}, \iota_{ij}) : l \in I, r \in J\} \end{aligned}$$

are all regular, and so (C, M) is an automatic structure for T. Moreover, if $(A \cup \{\iota\}, L \cup \{\iota\})$ is a prefix-automatic structure for S then

$$M'_{=} = ((L \cup \{\iota\})'_{=} \cap (A^+ \times A^+)\delta_A) \cup \{(\iota_{ij}, \iota_{ij}) : i \in I, j \in J\}$$

is also regular, so that (C, M) is a prefix-automatic structure for T. ■

Combining the above two propositions with Theorems **??**, **??** and **??**, we obtain the following result:

Theorem 5.3 *Let $S = \mathcal{M}^0[U; I, J; P]$ be a Rees matrix semigroup with zero.*

(i) *If U is automatic and if S is finitely generated then S is automatic as well.*

(ii) *If S is automatic and there is an entry p in the matrix P such that $pU^1 = U$ then U is automatic.*

(iii) *If S is prefix-automatic then U is prefix-automatic.* ■

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