Subsemigroups of the bicyclic monoid

L. Descalço and N. Ruškuc
Mathematical Institute, University of St Andrews
St Andrews KY16 9SS, Scotland
Telefax: +44 1334 463748
E-mails: luisd@mcs.st-and.ac.uk, nik@mcs.st-and.ac.uk

September 24, 2002

Abstract

In this paper we give a description of all subsemigroups of the bicyclic monoid **B**. We show that there are essentially five different types of subsemigroups. One of them is the degenerate case, and the remaining four split in two groups of two, linked by the obvious anti-isomorphism of **B**. Each subsemigroup is characterized by a certain collection of parameters. Using our description, we determine the regular, simple and bisimple subsemigroups of **B**. Finally we describe algorithms for obtaining the parameters from the generating set.

2000 Mathematics Subject Classification: 20M10

1 Introduction

The bicyclic monoid **B** is defined by the presentation $\langle b, c \mid bc = 1 \rangle$. A natural set of normal forms is $\{c^i b^j : i, j \geq 0\}$ and we shall identify **B** with this set. The normal forms multiply according to the following rule:

$$c^i b^j c^k b^l = \begin{cases} c^{i-j+k} b^l & \text{if } j \leq k \\ c^i b^{j-k+l} & \text{if } j > k. \end{cases}$$

The bicyclic monoid is one of the most fundamental semigroups. It is one of the main ingredients in the Bruck–Reilly extensions (see [7]), and also the basis of several generalizations; see [1],[2],[5],[6]. In [8, Sec 3.4] references are given to a number of applications of the bicyclic monoid to topics outside semigroup theory. The bicyclic monoid is known to have several remarkable properties, one of which is that it is completely determined by its lattice of subsemigroups; see [10] and [11]. Also, in [9] the authors study properties of a specific subsemigroup of **B**. Slightly surprisingly, there seems to be little other work in literature regarding the subsemigroups of **B**.

In this paper we give a description of all subsemigroups of $\mathbf{B}$. We show that there are essentially five different types of subsemigroups. One of them is the degenerate case of subsets of $\{c^i b^i : i \geq 0\}$, and the remaining four split in two groups of two, linked by the obvious anti-isomorphism $\widehat{\ }: c^i b^j \mapsto c^j b^i$ of $\mathbf{B}$. Each subsemigroup is characterized by a certain collection of parameters. We describe algorithms for obtaining these parameters from the generating set.

The paper is organized as follows. In Section 2 we define a series of distinguished subsets of $\mathbf{B}$, which are then used as a kind of building blocks, and then we state our main theorem in Section 3. Section 4 contains the auxiliary results needed to prove the main theorem. In Sections 5 and 6 we respectively consider the two non-degenerate types of subsemigroups. In Section 7 we determine, using our description, the regular, simple and bisimple subsemigroups of $\mathbf{B}$. Finally, Section 8 contains the algorithms for the computation of parameters. The classification of subsemigroups is used in the forthcoming paper [3] to investigate some properties of subsemigroups of $\mathbf{B}$, such as finite presentability and automaticity.

2 Distinguished subsets

In this section we introduce the notation we will need throughout the paper. In order to define subsets of the bicyclic monoid we find it convenient to represent $\mathbf{B}$ as an infinite square grid, as shown in Figure 1. We start by defining the functions $\Phi, \Psi, \lambda : \mathbf{B} \to \mathbb{N}_0$ by $\Phi(c^i b^j) = i$, $\Psi(c^i b^j) = j$ and $\lambda(c^i b^j) = |j - i|$ and by introducing some basic subsets of $\mathbf{B}$:

$D = \{c^i b^i : i \geq 0\}-$ the *diagonal* ,
$U = \{c^i b^j : j > i \geq 0\}-$ the *upper half*,
$R_p = \{c^i b^j : j \geq p, i \geq 0\}-$ the *right half plane (determined by p)*,
$L_p = \{c^i b^j : 0 \leq j < p, i \geq 0\}-$ the *left strip (determined by p)*,
$M_d = \{c^i b^j : d \mid j - i; i, j \geq 0\} -$ the *λ-multiples of d*,

for $p \geq 0$ and $d > 0$.

We now define the function $\widehat{\ }: \mathbf{B} \to \mathbf{B}$ by $c^i b^j \mapsto \widehat{c^i b^j} = c^j b^i$. Geometrically $\widehat{\ }$ is the reflection with respect to the main diagonal. So, for example, $\widehat{U}$ is the lower half. Algebraically this function is an anti-isomorphism ($\widehat{xy} = \widehat{y}\widehat{x}$), as is easy to check.

By using the above basic sets and functions we now define some further subsets of $\mathbf{B}$ that will be used in our description. For $0 \leq q \leq p \leq m$ we define the *triangle*

$$T_{q,p} = L_p \cap \widehat{R_q} \cap (U \cup D) = \{c^i b^j : q \leq i \leq j < p\},$$

and the *strips*

$$S_{q,p} = R_p \cap \widehat{R_q} \cap \widehat{L_p} = \{c^i b^j : q \leq i < p, j \geq p\},$$
$$S'_{q,p} = S_{q,p} \cup T_{q,p} = \{c^i b^j : q \leq i < p, j \geq i\},$$
$$S_{q,p,m} = S_{q,p} \cap R_m = \{c^i b^j : q \leq i < p, j \geq m\}.$$

	0	1	2	3	
0	1	b	b^2	b^3	$\cdots$
1	c	cb	cb^2	cb^3	$\cdots$
2	c^2	c^2b	c^2b^2	c^2b^3	$\cdots$
3	c^3	c^3b	c^3b^2	c^3b^3	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 1: The bicyclic monoid

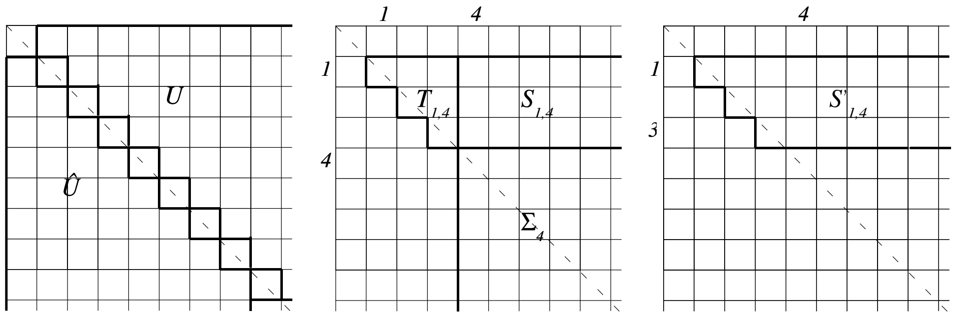

Figure 2: The upper and lower halves $U, \widehat{U}$, the triangle $T_{1,4}$ and the strips $S_{1,4}, S'_{1,4}$

Note that for $q = p$ the above sets are empty. For $i, m \geq 0$ and $d > 0$ we define the *lines*

$$\Lambda_i = \widehat{R_i} \cap \widehat{L_{i+1}} = \{c^i b^j : j \geq 0\},$$
$$\Lambda_{i,m,d} = \Lambda_i \cap R_m \cap M_d = \{c^i b^j : d \mid j - i, j \geq m\}$$

and in general for $I \subseteq \{0, \ldots, m-1\}$,

$$\Lambda_{I,m,d} = \bigcup_{i \in I} \Lambda_{i,m,d} = \{c^i b^j : i \in I, d \mid j - i, j \geq m\}.$$

For $p \geq 0$, $d > 0$, $r \in [d] = \{0, \ldots, d-1\}$ and $P \subseteq [d]$ we define the *squares*

$$\Sigma_p = R_p \cup \widehat{R_p} = \{c^i b^j : i, j \geq p\},$$
$$\Sigma_{p,d,r} = \Sigma_p \cap (\bigcup_{u=0}^{\infty} \Lambda_{p+r+ud}) \cap (\bigcup_{u=0}^{\infty} \widehat{\Lambda_{p+r+ud}}) = \{c^{p+r+ud} b^{p+r+vd} : u, v \geq 0\},$$
$$\Sigma_{p,d,P} = \bigcup_{r \in P} \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \geq 0\}.$$

Some of our subsetes are illustrated in Figures 2 and 3.

Finally, for $X \subseteq S$, we define $\iota(X) = \min(\Phi(X \cap U))$ (if $X \cap U \neq \emptyset$) and $\kappa(X) = \min(\Psi(X \cap \widehat{U}))$ (if $X \cap \widehat{U} \neq \emptyset$).

3 The main theorem

We now state our main theorem, that will be proved in the following sections.

Theorem 3.1 *Let S be a subsemigroup of the bicyclic monoid. Then one of the following conditions holds:*

1. *The subsemigroup is a subset of the diagonal; $S \subseteq D$.*

2. *The subsemigroup is a union of a subset of a triangle, a subset of the diagonal above the triangle, a square below the triangle and some lines belonging to a strip determined by square and the triangle, or the reflection of this union with respect to the diagonal. Formally there exist $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \ldots, p-1\}$ with $q \in I$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$ such that S is of one of the following forms:*

   (i) $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$;

   (ii) $S = F_D \cup \widehat{F} \cup \widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P}$.

3. *There exist $d \in \mathbb{N}$, $I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$ and sets $S_i \subseteq \Lambda_{i,i,d}$ ($i \in I$) such that S is of one of the following forms:*

   (i) $S = F_D \cup \bigcup_{i \in I} S_i$;

   (ii) $S = F_D \cup \bigcup_{i \in I} \widehat{S}_i$;

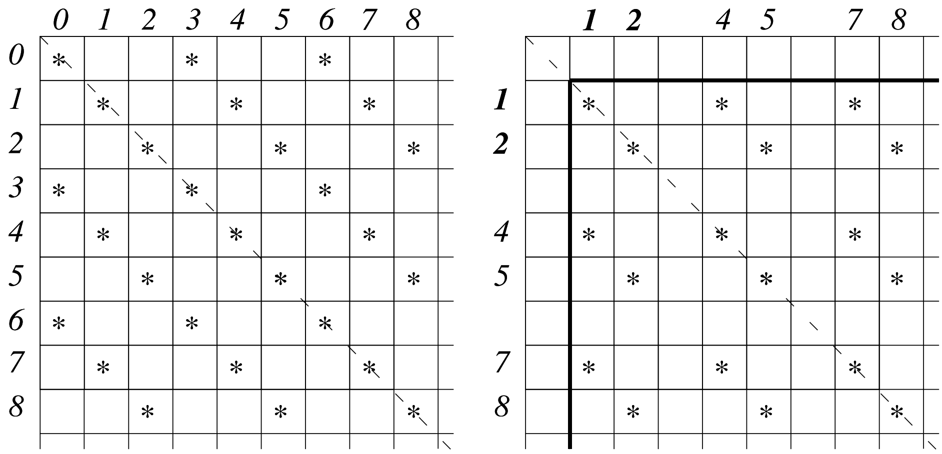

Figure 3: The λ-multiples of 3, M_3, and the square $\Sigma_{1,3,\{0,1\}}$

where each S_i has the form

$$S_i = F_i \cup \Lambda_{i,m_i,d}$$

for some $m_i \in \mathbb{N}_0$ and some finite set F_i, and

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

for some (possibly empty) $R \subseteq \{0, \ldots, d-1\}$, some $N \in \mathbb{N}_0$ and some finite set $I_0 \subseteq \{0, \ldots, N-1\}$.

We start by observing that if $S \subseteq D$ then there is nothing to describe because any idempotent $c^i b^i$ is an identity for the square Σ_i below it.

Condition *2.* corresponds to subsemigroups having elements both above and below the diagonal; we call them *two-sided subsemigroups*. We observe that a subsemigroup defined by condition *2.(ii)* is symmetric to the corresponding subsemigroup given by condition *2.(i)* with respect to the diagonal, and so we can use the anti-isomorphism $\widehat{\ }$ to obtain one from the other. Therefore we only need to consider subsemigroups that fall in one of these two categories. The description of two-sided subsemigroups is obtained in Section 5.

We call *upper subsemigroups* those having all elements above the diagonal and *lower subsemigroups* those having all elements below the diagonal. Condition *3.* corresponds to upper and lower subsemigroups. Again conditions *3.(i)* and *3.(ii)* give subsemigroups symmetric with respect to the diagonal and so only one of them will have to be considered. Upper subsemigroups are dealt with in Section 6.

4 Auxiliary results

In this section we will prove some useful properties of the subsets defined in Section 2.

Lemma 4.1 *For any $d \in \mathbb{N}$ the λ-multiples of d, M_d, is a subsemigroup.*

Proof. Let $c^i b^j, c^k b^l \in M_d$. Then $d \mid i-j$ and $d \mid k-l$. If $j > k$ then $c^i b^j c^k b^l = c^i b^{j-k+l}$ otherwise $c^i b^j c^k b^l = c^{i-j+k} b^l$. In any case $c^i b^j c^k b^l \in M_d$ because $d \mid i-j+k-l$. $\square$

Lemma 4.2 *For any $p \in \mathbb{N}$ the right half plane R_p and the strip $S'_{0,p}$ are subsemigroups.*

Proof. Let $x = c^i b^j, y = c^k b^l \in R_p$ $(j, l \geq p)$. If $j \geq k$ then $xy = c^i b^{j-k+l} \in R_p$ since $j-k+l \geq l \geq p$. If $j < k$ then $xy = c^{i-j+k} b^l \in R_p$ since $l \geq p$. Therefore R_p is a subsemigroup. Let $x = c^i b^j, y = c^k b^l \in S'_{0,p}$ $(i, k < p, j \geq i, l \geq k)$. If $j \geq k$ then $xy = c^i b^{j-k+l} \in S'_{0,p}$ since $i < p$ and $j-k+l \geq j \geq i$. If $j < k$ then $xy = c^{i-j+k} b^l \in S'_{0,p}$ since $i-j+k \leq k < p$ and $l \geq k \geq i-j+k$. Therefore $S'_{0,p}$ is also a subsemigroup. $\square$

In the following result we use the fact that the image of a subsemigroup under an anti-isomorphism is also a subsemigroup.

Lemma 4.3 *For any $q, p, m \in \mathbb{N}_0$ with $q < p \leq m$ the following sets are subsemigroups:*

(i) $S_{q,p}$; *(ii)* $S'_{q,p}$; *(iii)* Σ_p;
(iv) $S_{q,p} \cup \Sigma_p$; *(v)* $S_{q,p,m}$; *(vi)* $S'_{q,p} \cup \Sigma_p$.

Proof. To prove *(i)* – *(v)* we will just write the sets as intersections of subsemigroups given by the previous lemma and their images by the anti-isomorphism $\widehat{\ }$. We have $S_{q,p} = S'_{0,p} \cap \widehat{R_q} \cap R_p$, $S'_{q,p} = S'_{0,p} \cap \widehat{R_q}$, $\Sigma_p = R_p \cap \widehat{R_p}$, $S_{q,p} \cup \Sigma_p = R_p \cap \widehat{R_q}$ and $S_{q,p,m} = S'_{0,p} \cap R_m \cap \widehat{R_q}$. To prove that $S = S'_{q,p} \cup \Sigma_p$ is a subsemigroup, it is sufficient to show that, for $x = c^i b^j \in S'_{q,p}$ ($q \leq i < p, j \geq i$) and $y = c^k b^l \in \Sigma_p$ ($k, l \geq p$), we have $xy, yx \in S$. If $j \geq k$ then $xy = c^i b^{j-k+l} \in S$, because $i \geq q$ and $j - k + l \geq l \geq p$. If $j < k$ then $xy = c^{i-j+k} b^l \in S$, because $i - j + k > i \geq q$ and $l \geq p$. Since $l \geq p > i$ we have $yx = c^k b^{l-i+j} \in \Sigma_p$, because $k \geq p$ and $l - i + j \geq l \geq p$. $\square$

The following lemma establishes some inclusions that will also be useful.

Lemma 4.4 *For any $p, q \in \mathbb{N}_0$ with $q < p$ the following inclusions hold:*

(i) $T_{q,p} S_{q,p} \subseteq S_{q,p}$; *(ii)* $S_{q,p} T_{q,p} \subseteq S_{q,p}$;
(iii) $T_{q,p} \Sigma_p \subseteq S_{q,p} \cup \Sigma_p$; *(iv)* $\Sigma_p T_{q,p} \subseteq \Sigma_p$.

Proof. Let $\alpha = c^i b^j \in T_{q,p}$ ($q \leq i \leq j < p$), $\beta = c^k b^l \in S_{q,p}$ ($q \leq k < p, l \geq p$) and $\gamma = c^u b^v \in \Sigma_p$ ($u, v \geq p$). If $j \geq k$ then $\alpha\beta = c^i b^{j-k+l}$ and, since $j - k + l \geq l \geq p$, $\alpha\beta \in S_{q,p}$. If $j < k$ then $\alpha\beta = c^{i-j+k} b^l$ and, since $l \geq p$ and $S'_{q,p}$ is a subsemigroup, $\alpha\beta \in S_{q,p}$. So *(i)* is proved. We have $\beta\alpha = c^k b^{l-i+j}$ because $i < p \leq l$. Since $l - i + j \geq l \geq p$ we have $\beta\alpha \in S_{q,p}$ and so *(ii)* is proved as well. We have $\alpha\gamma = c^{i-j+u} b^v$ because $j < p \leq u$ and, since $v \geq p$ and $S'_{q,p} \cup \Sigma_p$ is a subsemigroup, $\alpha\gamma \in S_{q,p} \cup \Sigma_p$ and *(iii)* is proved. Finally, $\gamma\alpha = c^u b^{v+j-i}$ because $i < p \leq v$. We have $\gamma\alpha \in \Sigma_p$, because $u \geq p$, and *(iv)* is proved as well. $\square$

Lemma 4.5 *For any $p \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $P \subseteq \{0, \ldots, d-1\}$, the square $\Sigma_{p,d,P}$ is a subsemigroup.*

Proof. Let $\alpha = c^{p+r_1+u_1 d} b^{p+r_1+v_1 d}, \beta = c^{p+r_2+u_2 d} b^{p+r_2+v_2 d} \in \Sigma_{p,d,P}$ where $r_1, r_2 \in P$; $u_1, v_1, u_2, v_2 \in \mathbb{N}_0$. If $p + r_1 + v_1 d \geq p + r_2 + u_2 d$ then $\alpha\beta = c^{p+r_1+u_1 d} b^{p+r_1+(v_1-u_2+v_2)d}$. Since we have $p + r_1 + v_1 d \geq p + r_2 + u_2 d$, it follows that $r_1 + v_1 d - u_2 d \geq r_2 \geq 0$, which implies $r_1 + (v_1 - u_2 + v_2)d \geq 0$. So we have $(v_1 - u_2 + v_2)d \geq -r_1 > -d$ and hence $v_1 + v_2 - u_2 \geq 0$. Therefore $\alpha\beta \in \Sigma_{p,d,P}$. If $p + r_1 + v_1 d < p + r_2 + u_2 d$ then $\alpha\beta = c^{p+r_2+(u_1-v_1+u_2)d} b^{p+r_2+v_2 d}$. Analogously $p + r_2 + u_2 d > p + r_1 + v_1 d$ implies $u_1 - v_1 + u_2 \geq 0$ and so $\alpha\beta \in \Sigma_{p,d,P}$. $\square$

Lemma 4.6 *For any $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$ and $P \subseteq \{0, \ldots, d-1\}$, the set*

$$\Sigma_{p,d,P} \cup (M_d \cap S'_{q,p})$$

is a subsemigroup.

Proof. Let $H = \Sigma_{p,d,P} \cup (M_d \cap S'_{q,p})$. We know from the previous lemma that $\Sigma_{p,d,P}$ is a subsemigroup. From Lemmas 4.1 and 4.3 we know that $M_d \cap S'_{q,p}$ is a subsemigroup as well. Let $\alpha = c^{p+r+ud}b^{p+r+vd} \in \Sigma_{p,d,P}$ and let $\beta = c^i b^{i+sd} \in M_d \cap S'_{q,p}$. We just have to prove that $\alpha\beta, \beta\alpha \in H$. Since $p + r + vd \geq p > i$, $\alpha\beta = c^{p+r+ud}b^{p+r+(v+s)d} \in \Sigma_{p,d,P}$. We have $\beta\alpha = c^i b^{i+sd} c^{p+r+ud} b^{p+r+vd}$. We note that $H \subseteq U = (\Sigma_p \cup S'_{q,p}) \cap M_d$ and, using the same two lemmas, U is a subsemigroup. Therefore, if $i + sd \geq p + r + ud$ then $\beta\alpha \notin \Sigma_p$ and, since U is a subsemigroup, $\beta\alpha \in S'_{q,p} \cap M_d \subseteq H$. If $i + sd < p + r + ud$ and $u - s < 0$ we have again $\beta\alpha \in S'_{q,p} \cap M_d \subseteq H$. Finally, if $i + sd < p + r + ud$ and $u - s \geq 0$ then $\beta\alpha = c^{p+r+(u-s)d}b^{p+r+vd} \in \Sigma_{p,d,P}$. $\square$

Lemma 4.7 *For any $p \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $I \subseteq \{0, \ldots, p-1\}$, the set $\Lambda_{I,p,d}$ is a subsemigroup.*

Proof. Let $\alpha = c^i b^{i+ud}$, $\beta = c^j b^{j+vd} \in \Lambda_{I,p,d}$ $(i, j < p; i + ud, j + vd \geq p)$. Then $\alpha\beta = c^i b^{i+(u+v)d}$ because $i + ud \geq p > j$. Since $i + (u + v)d \geq i + ud \geq p$ we have $\alpha\beta \in \Lambda_{I,p,d}$. $\square$

Lemma 4.8 *Let $p \in \mathbb{N}_0$, $d \in \mathbb{N}$, $\emptyset \neq I \subseteq \{0, \ldots, p-1\}$, $\emptyset \neq P \subseteq \{0, \ldots, d-1\}$, and $q = \min(I)$. The set $H = \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ is a subsemigroup if and only if*

$$I' = \{p + r - ud : r \in P, u \in \mathbb{N}_0, p + r - ud \geq q\} \subseteq I.$$

Proof. We will first assume that H is a subsemigroup and prove that $I' \subseteq I$. Let $c^q b^{q+d_1}, c^{p+r+d}b^{p+r} \in H$ where $r \in P$ and $d_1 > 0$ is a multiple of d. For any $n, m \in \mathbb{N}$ such that $p+r+md-nd_1 \geq q$ we have $(c^q b^{q+d_1})^n (c^{p+r+d}b^{p+r})^m = c^{p+r+md-nd_1}b^{p+r} \in H$ and so $p + r - ud \in I$ for any $r \in P$ and $u \in \mathbb{N}$ such that $p + r - ud \geq q$. Therefore $I' \subseteq I$. Let us assume now that $I' \subseteq I$ and prove that H is a subsemigroup. We know that $\Sigma_{p,d,P}$ is a subsemigroup. Let $\alpha = c^{p+r+ud}b^{p+r+vd} \in \Sigma_{p,d,P}$ $(r \in P; u, v \in \mathbb{N}_0)$ and $\beta = c^i b^{i+d_1} \in \Lambda_{I,p,d}$ $(i \in I, d_1 \in \mathbb{N}, d \mid d_1)$. We have $\alpha\beta = c^{p+r+ud}b^{p+r+vd+d_1} \in \Sigma_{p,d,P}$. If $i + d_1 \geq p + r + ud$ then $\beta\alpha = c^i b^{i+d_1+(v-u)d} \in \Lambda_{I,p,d}$, because $i + d_1 + (v - u)d \geq p + r + vd \geq p$. If $i + d_1 < p + r + ud$ then $\beta\alpha = c^{p+r+ud-d_1}b^{p+r+vd}$. In this case, if $ud - d_1 \geq 0$ then $\beta\alpha \in \Sigma_{p,d,P}$ and if $ud - d_1 < 0$ then $p + r + ud - d_1 \geq q$ because $H \subseteq S_{q,p} \cup \Sigma_p$ and $S_{q,p} \cup \Sigma_p$ is a subsemigroup. Therefore $p + r + ud - d_1 \in I' \subseteq I$, implying $\beta\alpha \in \Lambda_{I,p,d}$. $\square$

5 Two-sided subsemigroups

In this section we describe subsemigroups that have elements both above and below the diagonal. Let S be a subsemigroup of $\mathbf{B}$ with $S \cap U \neq \emptyset$ and $S \cap \widehat{U} \neq \emptyset$. Without loss of generality we can assume that $q = \iota(S) \leq \kappa(S) = p$ observing that the other case is dual to this by using the anti-isomorphism $\widehat{\ }$.

We now state our main result of this section:

Theorem 5.1 *Let S be a subsemigroup of $\mathbf{B}$ such that $S \cap U \neq \emptyset$, $S \cap \widehat{U} \neq \emptyset$ and $q = \iota(S) \leq \kappa(S) = p$. There exist $d \in \mathbb{N}$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$, $I \subseteq \{q, \ldots, p-1\}$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$ such that*

$$S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}.$$

To prove this theorem we need the following elementary result from number theory, the proof of which we include for completeness:

Lemma 5.2 *Let $a_1, \ldots, a_k, b_1, \ldots, b_l, r_0 \in \mathbb{N}_0$ be arbitrary with $a_1 > 0$, $b_1 > 0$ and let*

$$d = \gcd(a_1, \ldots, a_k, b_1, \ldots, b_l).$$

Then there exist numbers $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l \in \mathbb{N}_0$ such that:

1. $\alpha_1 a_1 + \ldots + \alpha_k a_k + \beta_1 b_1 + \ldots + \beta_l b_l = d$;
2. $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l \geq r_0$.

Proof. We start by assuming, without loss of generality, that $a_1, \ldots, a_k, b_1, \ldots, b_l > 0$. Since $d = \gcd(a_1, \ldots, a_k, b_1, \ldots, b_l)$, we can write $d = \sum_{i=1}^k \alpha_i' a_i + \sum_{j=1}^l \beta_j' b_j$ for some integers $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$. Let H be any positive integer and let

$$P = Hkla_1 \ldots a_k b_1 \ldots b_l, \quad Q = P/k, \quad R = P/l.$$

We can then write

$$\begin{aligned} d &= \sum_{i=1}^k \alpha_i' a_i + \sum_{j=1}^l \beta_j' b_j = \sum_{i=1}^k \alpha_i' a_i + P - P + \sum_{j=1}^l \beta_j' b_j \\ &= \sum_{i=1}^k (\alpha_i' a_i + Q) + \sum_{j=1}^l (\beta_j' b_j - R) = \sum_{i=1}^k (\alpha_i' + Q/a_i) a_i + \sum_{j=1}^l (\beta_j' - R/b_j) b_j \\ &= \sum_{i=1}^k \alpha_i a_i + \sum_{j=1}^l \beta_j b_j \end{aligned}$$

It is clear that when H increases all numbers $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_l$ increase as well and so the result holds. $\square$

Proof of Theorem 5.1. Let $F_D = S \cap D \cap L_q$ and $S' = S \backslash F_D$. We have $S' = S \cap (M_d \cap (S'_{q,p} \cup \Sigma_p))$ where $d = \gcd(\lambda(S'))$ and so S' is a subsemigroup. We observe that the elements $c^i b^i \in F_D$ act as identities in S'. Let $x \in S' \cap U$ and $y \in S' \cap \widehat{U}$ such that $\Phi(x) = \iota(S) = q$ and $\Psi(y) = \kappa(S) = p$. Let $Y \subseteq S'$ be a finite set such that:

(i) $x, y \in Y$;

(ii) $\Lambda_i \cap S' \cap S'_{q,p} \neq \emptyset \implies \Lambda_i \cap Y \neq \emptyset$ for $i \in \{q \ldots, p-1\}$ (Y contains at least one representative for each line in the strip with elements in S');

(iii) $\{(i-p) \mod d : \Lambda_i \cap Y \cap \Sigma_p \neq \emptyset\} = \{(i-p) \mod d : \Lambda_i \cap S' \cap \Sigma_p \neq \emptyset\}$ (Y contains at least one representative for each class of lines in the square having a representative in S');

(iv) $\gcd(\lambda(Y)) = d$.

Such Y can be obtained by choosing a finite set Y_1 (with at most $p-q+d$ elements) satisfying *(i)* $-$ *(iii)*, and a finite set Y_2 such that $\gcd(\lambda(Y_2)) = \gcd(\lambda(S'))$, and letting $Y = Y_1 \cup Y_2$. Let $Y \cap (D \cup U) = \{c^{i_1}b^{j_1}, \dots, c^{i_r}b^{j_r}\}$ where $x = c^{i_1}b^{j_1}$, $q = i_1 \leq i_2 \leq \dots \leq i_r, j_1 > i_1, j_2 \geq i_2, \dots, j_r \geq i_r$ and let $Y \cap \widehat{U} = \{c^{k_1}b^{l_1}, \dots, c^{k_s}b^{l_s}\}$ where $y = c^{k_1}b^{l_1}$, $p = l_1 \leq l_2 \leq \dots \leq l_s$ and $k_1 > l_1, \dots, k_s > l_s$.

We are going to show that

$$c^{p+d}b^p, c^pb^{p+d} \in S'.$$

Before proving this we will make an observation showing the importance of these two elements. This observation is illustrated in Figure 4.

Let c^ib^j be an element in $M_d \cap (S_{q,p} \cup \Sigma_p)$. We have $c^ib^jc^pb^{p+d} = c^ib^{j+d}$ which means intuitively that we can move d positions to the right in the grid using the element c^pb^{p+d}. If $i \geq p$ then we also have $c^{p+d}b^pc^ib^j = c^{i+d}c^j$ which means that we can move d positions down. If $j \geq p+d$ then we have $c^ib^jc^{p+d}c^p = c^ib^{j-d}$ which means that we can move left. Finally, if $i \geq p+d$ then we have $c^pb^{p+d}c^ib^j = c^{i-d}b^j$ and so we can move up.

In order to prove that $c^{p+d}b^p, c^pb^{p+d} \in S'$ we first note that $d = \gcd\{j_1 - i_1, \dots, j_r - i_r, k_1 - l_1, \dots, k_s - l_s\}$, by *(iv)*. Since $i_1 - j_1 < 0$ and $k_1 - l_1 > 0$, Lemma 5.2 can be applied and we can write

$$d = \alpha_1(i_1 - j_1) + \dots + \alpha_r(i_r - j_r) + \beta_1(k_1 - l_1) + \dots + \beta_s(k_s - l_s) \tag{1}$$

with $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \geq \max\{i_1, \dots, i_r, l_1, \dots, l_s\}$. We can now consider the product $(c^{i_1}b^{j_1})^{\alpha_1} \dots (c^{i_r}b^{j_r})^{\alpha_r}$ which is equal to

$$(c^{i_1}b^{i_1+\alpha_1(j_1-i_1)})(c^{i_2}b^{i_2+\alpha_2(j_2-i_2)}) \dots (c^{i_r}b^{i_r+\alpha_r(j_r-i_r)}).$$

Since $\alpha_1 \geq \max\{i_1, \dots, i_r\}$ and $j_1 - i_1 \geq 1$ we have $i_1 + \alpha_1(j_1 - i_1) > i_1, \dots, i_r$ and therefore, we can compute the above product working from the left hand side to obtain

$$c^{i_1}b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\dots+\alpha_r(j_r-i_r)}. \tag{2}$$

We now consider the product $(c^{k_s}b^{l_s})^{\beta_s} \dots (c^{k_2}b^{l_2})^{\beta_2}(c^{k_1}b^{l_1})^{\beta_1}$ which is equal to

$$(c^{l_s+\beta_s(k_s-l_s)}b^{l_s}) \dots (c^{l_2+\beta_2(k_2-l_2)}b^{l_2})(c^{l_1+\beta_1(k_1-l_1)}b^{l_1}).$$

Since $\beta_1 \geq \max\{l_1, \dots, l_s\}$ and $k_1 - l_1 \geq 1$ we have $l_1 + \beta_1(k_1 - l_1) > l_1, \dots, l_s$ and we can compute this product from the right to obtain

$$c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\dots+\beta_s(k_s-l_s)}b^{l_1}. \tag{3}$$

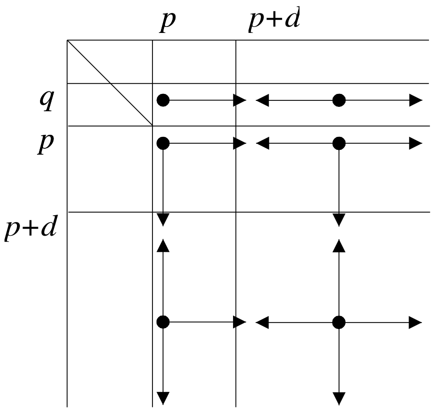

Figure 4: Moving using $c^p b^{p+d}$ and $c^{p+d} b^p$

Multiplying the elements (2) and (3) of S' we obtain

$$c^{i_1}b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)}c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)}b^{l_1}$$
$$= c^{l_1+d}b^{l_1} = c^{p+d}b^p$$

since $q = i_1 \le l_1 = p$ and using equation (1). So $c^{p+d}b^p \in S'$. Since $d \mid (j_1 - i_1)$ we can write $j_1 - i_1 = td$ for some $t \in \mathbb{N}$. Since $p \ge i_1$ we have $p + td \ge j_1$ and therefore $c^{i_1}b^{j_1}(c^{p+d}b^p)^t = c^{i_1-j_1+p+td}b^p = c^p b^p$. We conclude that $c^p b^p \in S'$ as well. We now take the constants $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \ge \max\{i_1, \ldots, i_r, l_1, \ldots, l_s\}$ to be such that

$$d = \alpha_1(j_1 - i_1) + \ldots + \alpha_r(j_r - i_r) + \beta_1(l_1 - k_1) + \ldots + \beta_s(l_s - k_s) \tag{4}$$

and we consider the following element of S':

$$c^p b^p c^{i_1}b^{i_1+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)}c^{l_1+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)}b^{l_1}.$$

Since $i_1 = q \le p = l_1$ this element can be written as

$$c^p b^{p+\alpha_1(j_1-i_1)+\alpha_2(j_2-i_2)+\ldots+\alpha_r(j_r-i_r)}c^{p+\beta_1(k_1-l_1)+\beta_2(k_2-l_2)+\ldots+\beta_s(k_s-l_s)}b^p$$

and it is equal to $c^p b^{p+d}$ by equation (4). Therefore we have $c^p b^{p+d}, c^{p+d}b^p \in S'$ as we wanted to show.

We are now going to prove that $S' \cap \Sigma_p = \Sigma_{p,d,P}$ where $P = \{(i-p) \mod d : L_i \cap Y \cap \Sigma_p \neq \emptyset\}$. We will first show that $\Sigma_{p,d,P} \subseteq S'$. Let $c^{p+r+ud}b^{p+r+vd} \in \Sigma_{p,d,P}$. By definition of Y there is $c^i b^j \in Y \cap \Sigma_p$ such that $(i-p) \mod d = r$. Therefore, since $Y \subseteq S' \subseteq M_d$, we have $c^i b^j = c^{p+r+u'd}b^{p+r+v'd}$. As we have seen we can move from $c^i b^j$ to $c^{p+r+ud}b^{p+r+vd}$ using the elements $c^p b^{p+d}$ and $c^{p+d}b^p$ which means that $c^{p+r+ud}b^{p+r+vd}$ belongs to S', because it can be written as a product of the elements $c^p b^{p+d}$, $c^{p+d}b^p, c^i b^j \in S'$. We will now show that $S' \cap \Sigma_p \subseteq \Sigma_{p,d,P}$. Let $c^i b^j \in S' \cap \Sigma_p$. By definition of P and by *(iii)* in the definition of Y we have $(i-p) \mod d = r \in P$. Since $S' \subseteq M_d$ we have $c^i b^j = c^{p+r+ud}b^{p+r+vd}$ for some $u, v \ge 0$ and so $c^i b^j \in \Sigma_{p,d,P}$. We conclude that $S' \cap \Sigma_p = \Sigma_{p,d,P}$.

We now prove that $S' \cap S_{q,p} = \Lambda_{I,p,d}$ where $I = \{i : q \le i \le p-1; c^i b^j \in S' \text{ for some } j\}$. In fact, from any element $c^i b^j \in S' \cap S_{q,p}$ we can move left and right using the elements $c^p b^{p+d}$ and $c^{p+d}b^p$ in order to obtain the whole line $\Lambda_{i,p,d}$. Since $S' \subseteq M_d$ it follows that $S' \cap S_{q,p} = \Lambda_{I,p,d}$. We conclude that $S' = F \cup \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ where $F = S \cap T_{q,p}$ is a finite set, and this implies $S = F_D \cup F \cup \Sigma_{p,d,P} \cup \Lambda_{I,p,d}$ as required. $\square$

6 Upper subsemigroups

In this section we consider subsemigroups whose elements are above (and on) the diagonal. The case where all elements are below the diagonal is again obtained by using the anti-isomorphism $\widehat{\ }$.

Lemma 6.1 *Let $q, p, d \in \mathbb{N}_0$ with $q < p$ and $d > 0$, and let $X \subseteq S'_{q,p}$ be a finite set with $\iota(X) = q$ and $\gcd(\lambda(X)) = d$. For any $x \in X$ there exists $m \in \mathbb{N}_0$ such that*

$$\Lambda_{\Phi(x),m,d} \subseteq \langle X \rangle.$$

Proof. Let $S = \langle X \rangle$ and let $Y = X \cap U = \{c^{i_1}b^{i_1+d_1}, \ldots, c^{i_n}b^{i_n+d_n}\}$ with $q = i_1 \leq i_2 \leq \ldots \leq i_n; d_1, \ldots, d_n \in \mathbb{N}$. For each $j \in \{1, \ldots, n\}$ choose $\alpha_j \in \mathbb{N}$ such that $i_j + \alpha_j d_j \geq p$ and $d = \gcd(d_1, \ldots, d_n) = \gcd(\alpha_1 d_1, \ldots, \alpha_n d_n)$. We can take $\alpha_1, \ldots, \alpha_n$ to be large enough distinct primes not appearing in the decomposition of d in prime factors. It is well known that given numbers $x_1, \ldots, x_n \in \mathbb{N}$, such that $\gcd\{x_1, \ldots, x_n\} = d$, there is a constant k such that all multiples of d greater then k can be obtained as combinations of $x_1, \ldots, x_n$ with coefficients in $\mathbb{N}$. Let $k \in \mathbb{N}$ be such that

$$\{td : td \geq k, t \in \mathbb{N}\} \subseteq \{\gamma_1(\alpha_1 d_1) + \ldots + \gamma_n(\alpha_n d_n) : \gamma_1, \ldots, \gamma_n \in \mathbb{N}\}.$$

Let $m = p + k$. We are going to prove that $\Lambda_{\Phi(x),m,d} \subseteq S$ for any $x \in X$. Let $x \in X$, $i = \Phi(x) \in \{q, \ldots, p-1\}$ and $t \in \mathbb{N}$ with $i + td \geq m$. Then $td \geq m - i = p + k - i \geq k$. Therefore we can write

$$td = \gamma_1(\alpha_1 d_1) + \ldots + \gamma_n(\alpha_n d_n)$$

with $\gamma_1, \ldots, \gamma_n \in \mathbb{N}$. If $x = c^{i_j}c^{i_j+d_j} \in Y$ then we have

$$c^i b^{i+td} = c^{i_j} b^{i_j+td} = (c^{i_j} b^{i_j+\alpha_j d_j})^{\gamma_j} \cdot \prod_{\substack{1 \leq l \leq n \\ l \neq j}} (c^{i_l} b^{i_l+\alpha_l d_l})^{\gamma_l}.$$

If $x \notin Y$ then $x = c^i b^i$ and so we have $c^i b^{i+td} = c^i b^i (c^{i_1} b^{i_1+\alpha_1 d_1})^{\gamma_1} \ldots (c^{i_n} b^{i_n+\alpha_n d_n})^{\gamma_n} \in S$. □

Theorem 6.2 *Let S be a subsemigroup of $\mathbf{B}$ such that $S \cap \widehat{U} = \emptyset$ and $S \cap U \neq \emptyset$. There exist $d \in \mathbb{N}$, $I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$, and sets $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ such that*

$$S = F_D \cup \bigcup_{i \in I} S_i$$

where each S_i has the form

$$S_i = F_i \cup \Lambda_{i,m_i,d}$$

for some $m_i \in \mathbb{N}_0$ and some finite set F_i, and

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

for some (possibly empty) $R \subseteq \{0, \ldots, d-1\}$, some $N \in \mathbb{N}_0$ and some finite set $I_0 \subseteq \{0, \ldots, N-1\}$.

Proof. Let $q = \iota(S)$, $F_D = S \cap D \cap L_q$, $S' = S \backslash F_D$, so that we have $S = F_D \cup S'$, and let $d = \gcd(\lambda(S'))$. Since $S' \subseteq (U \cup D) \cap M_d$, letting $I = \Phi(S')$, we have $S = F_D \cup \bigcup_{i \in I} S_i$

where $S_i = S' \cap \Lambda_{i,i,d}$ for $i \in I$. For any $i \in I$ we can consider a finite set $X_i \subseteq S'$ with $i \in \Phi(X_i)$ and $\gcd(X_i) = d$ and conclude, by using Lemma 6.1, that $\Lambda_{i,m_i,d} \subseteq S$ for some $m_i \in \mathbb{N}_0$. If I is finite then we can take $R = \emptyset$, $I_0 = I$ and $N = \max(I) + 1$. We will now consider the case where I is infinite. Let $X = \{c^{i_1}b^{i_1+d_1}, \dots, c^{i_k}b^{i_k+d_k}\} \subseteq S'$ such that $d = \gcd(\lambda(X))$, $i_1 \geq i_2 \geq \dots \geq i_k$. By Lemma 6.1, there is a constant M such that $td \geq M$ implies $c^{i_1}b^{i_1+td} \in S'$. Define a set $R \subseteq \{0, \dots, d-1\}$ by

$$r \in R \iff |\{i \in \mathbb{N} : \Lambda_i \cap S' \neq \emptyset \ \& \ i \bmod d = r\}| = \infty.$$

Then there exists a constant K such that

$$c^i b^j \in S' \ \& \ i \geq K \implies (i \bmod d) \in R.$$

Let $N = \max\{i_1, K\}$ and

$$I_0 = \{i : q \leq i \leq N-1, \Lambda_i \cap S' \neq \emptyset\}.$$

We claim that

$$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}.$$

The direct inclusion is obvious, as is $I_0 \subseteq I$. Now consider an arbitrary $r+ud \geq N$, $r \in R$. Choose an arbitrary $c^{r+vd}b^{r+vd+wd} \in S'$ such that $t = v - u \geq M/d$. From $td \geq M$ it follows that $c^{i_1}b^{i_1+td} \in S'$ and so $c^{i_1}b^{i_1+td}c^{r+vd}b^{r+vd+wd} = c^{r+ud}b^{r+vd+wd} \in S'$ because $r + vd = r + ud + td \geq N + td \geq i_1 + td$. We conclude that $r + ud \in I$. □

Remark. In the case where I is finite ($R = \emptyset$), the subsemigroup can be written as a union of two finite sets and finitely many lines all starting from the same column. Formally there exist $q, p, m \in \mathbb{N}_0$ with $q < p \leq m$, finite sets $F_D \subseteq D \cap L_q$, $F \subseteq S'_{q,p} \backslash S_{q,p,m}$ and a set $I \subseteq \{q, \dots, p-1\}$ such that

$$S = F_D \cup F \cup \Lambda_{I,m,d}.$$

7 Corollaries

In this section we use our classification of subsemigroups of $\mathbf{B}$ to describe which of them are regular (and hence inverse), simple or bisimple. We use the standard semigroup theory terminology and notation as found in [7]. In particular, it is well known that in $\mathbf{B}$ we have

$$c^i b^j \ \mathcal{L} \ c^k b^l \Leftrightarrow j = l, \quad c^i b^j \ \mathcal{R} \ c^k b^l \Leftrightarrow i = k, \quad c^i b^j \ \mathcal{H} \ c^k b^l \Leftrightarrow i = k \ \& \ j = l,$$

$$\mathcal{D} = \mathcal{J} = \mathbf{B} \times \mathbf{B}.$$

We see that $\mathbf{B}$ is a bisimple semigroup (i.e. it has a unique $\mathcal{D}$-class, the egg-box of which is the familiar square grid used in the previous sections). Since the idempotents are the elements in the diagonal, an element $c^i b^j$ has the unique inverse $c^j b^i$ and $\mathbf{B}$ is an inverse semigroup. Hence, a subsemigroup S of $\mathbf{B}$ is regular if and only if it is inverse if and only if it satisfies $c^i b^j \in S \implies c^j b^i \in S$. Therefore we have:

Theorem 7.1 *A subsemigroup S of $\mathbf{B}$ is regular (and hence inverse) if and only if it has the form $F_D \cup \Sigma_{p,d,P}$ where F_D is a finite subset of the diagonal and either of F_D or P may be empty.*

Given a regular subsemigroup S of $\mathbf{B}$ we have

$$\mathcal{L}^S = \mathcal{L}^{\mathbf{B}} \cap (S \times S), \quad \mathcal{R}^S = \mathcal{R}^{\mathbf{B}} \cap (S \times S), \quad \mathcal{H}^S = \mathcal{H}^{\mathbf{B}} \cap (S \times S) \tag{5}$$

(see [7, Propositon 2.4.2]) and so we have the following:

Corollary 7.2 *A $\mathcal{D}$-class of a regular subsemigroup of $\mathbf{B}$ is either isomorphic to $\mathbf{B}$ or it is a trivial group.*

Proof. Using (5) we see that a $\mathcal{D}$-class of a regular subsemigroup $S = F_D \cup \Sigma_{p,d,P}$ is either a single idempotent from F_D or a set of the form $\Sigma_{p,d,r}$ with $r \in P$. The latter is isomorphic to $\mathbf{B}$ via $c^{p+r+ud}b^{p+r+vd} \mapsto c^u b^v$. $\square$

To determine the simple subsemigroups of $\mathbf{B}$ we need the following result, proved in [9]:

Lemma 7.3 *A subset of the form $I_p = \{c^i b^j : 0 \leq i \leq j, j \geq p\}$ ($p \in \mathbb{N}_0$) is an ideal of U.*

Theorem 7.4 *The simple subsemigroups of $\mathbf{B}$ are precisely those of the forms $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ and $\widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P}$ (with non-empty P).*

Proof. An upper (or diagonal) subsemigroup S is not simple since, for p large enough the set $S \cap I_p$ is a proper ideal of S; similarly, a lower subsemigroup is not simple. A two-sided subsemigroup S of the form $F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is not simple if $F_D \cup F \neq \emptyset$ because in this case $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is a proper ideal of S. This proves that a simple subsemigroup of $\mathbf{B}$ must be of the form $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ or $\widehat{\Lambda_{I,p,d}} \cup \Sigma_{p,d,P}$. For the converse we will now show that a semigroup S of the form $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is always simple by showing that, for an arbitrary $s = c^k b^l \in S$, we have $S \subseteq S^1 s S^1$. Let $t = c^i b^j \in S$ arbitrary. Taking $\alpha = c^i b^u \in S$ with $u \geq \max(k, j + k - l)$ we have $\alpha s = c^i b^{u-k+l}$. Hence, with $\beta = c^{p+d} b^p \in S$ and $v = (u - k + l - j)/d$, we have $\alpha s \beta^v = c^i b^{u-k+l} c^{p+vd} b^p = t$. $\square$

Theorem 7.5 *A subsemigroup of $\mathbf{B}$ is bisimple if and only if it has the form $\Sigma_{p,d,0}$.*

Proof. Let S be an arbitrary simple subsemigroup of $\mathbf{B}$. Without loss of generality assume that S is of the form $S = \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ with $0 \in P$. If S contains two elements $\alpha = c^i b^j, \beta = c^k b^l$ such that $i - p \mod d \neq k - p \mod d$ then S is not bisimple. Indeed, suppose that that α and β are $\mathcal{D}$-related in S. Then there is $s \in S$ such that $\alpha \ \mathcal{R}^S \ s$ and $s \ \mathcal{R}^S \ \beta$. This implies that $\alpha \ \mathcal{L}^{\mathbf{B}} \ s$ and $s \ \mathcal{R}^{\mathbf{B}} \ \beta$ and so $s = c^k b^j$. But d does not divide $k - j$ and so $s \notin S$, a contradiction. So for S to be bisimple it is necessary that

$P = \{0\}$ and, by using Lemma 4.8, that $I = \{p - ud : p - ud \geq k\}$ for some $k \geq 0$. Next notice that the elements $c^p b^p$ and $c^{p-d} b^p$ are not $\mathcal{L}$-related in S because for any $c^k b^l \in S$ we have $l \geq p$ and so $c^k b^l c^{p-d} b^p = c^k b^{l+d} \neq c^p b^p$. Suppose that $c^p b^p \ \mathcal{D}^S \ c^{p-d} b^p$. Then we would have $c^p b^p \ \mathcal{L}^S s$ and $s \ \mathcal{R}^S c^{p-d} b^p$ for some $s \in S$ and therefore $c^p b^p \ \mathcal{L}^{\mathbf{B}} s$ and $s \ \mathcal{R}^{\mathbf{B}} c^{p-d} b^p$ which implies $s = c^{p-d} b^p$ and so $c^p b^p \ \mathcal{L}^S \ c^{p-d} b^p$, a contradiction. So for S to be bisimple it is in fact necessary that $I = \emptyset$ and $P = \{0\}$. Since $\Sigma_{p,d,0}$ is isomorphic to $\mathbf{B}$ it is bisimple, completing the proof. □

We now describe two-sided subsemigroups as finite unions of semigroups.

Theorem 7.6 *A two-sided subsemigroup is a finite union of copies of* $\mathbf{B}$ *and subsemigroups of* $\mathbb{N}_0$*.*

Proof. Assume without loss of generality that S is of the form $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. We have $S = \bigcup_{i=0}^{p-1} (S \cap \Lambda_{i,i,d}) \cup \bigcup_{r \in P} \Sigma_{p,d,r}$ and, since $\Lambda_{i,i,d}$ is isomorphic to $\mathbb{N}_0$ via $c^i b^{i+ud} \mapsto u$ and $\Sigma_{p,d,r}$ is isomorphic to $\mathbf{B}$, the result follows. □

We say that M is a *special* subsemigroup of $\mathbb{N}_0$ if $M = \{n : n \geq k\}$ for some k.

Corollary 7.7 *A subsemigroup of* $\mathbf{B}$ *is:*

1. *Regular if and only if it is obtained by adjoining successively finitely many identities to a finite union of copies of* $\mathbf{B}$*.*

2. *Simple if and only if it is a finite union of copies of* $\mathbf{B}$ *and special subsemigroups of* $\mathbb{N}_0$*.*

3. *Bisimple if and only if it is isomorphic to* $\mathbf{B}$*.*

8 Computation of parameters and examples

In this section we will show how to compute the parameters that appear in our main theorem, given a finite generating set for the subsemigroup. We will first consider two-sided subsemigroups defined by condition *2.(i)* in the main theorem and then we will consider finitely generated upper subsemigroups defined by condition *3.(i)*, observing again that subsemigroups defined by *2.(ii)* and *3.(ii)* can be obtained from these two by using the anti-isomorphism $\widehat{\ }$. We observe that, given a finite set X, we can determine which kind of subsemigroup it generates:

1) $\langle X \rangle \subseteq D$ if and only if $X \subseteq D$;

2) $\langle X \rangle$ is a two-sided subsemigroup if and only if $X \cap U \neq \emptyset$ and $X \cap \widehat{U} \neq \emptyset$;

3) $\langle X \rangle$ is an upper (respectively lower) subsemigroup if and only if $X \cap U \neq \emptyset$ and $X \cap \widehat{U} = \emptyset$ (respectively $X \cap U = \emptyset$ and $X \cap \widehat{U} \neq \emptyset$).

Theorem 8.1 *Let $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ be a two-sided subsemigroup of $\mathbf{B}$ defined by condition 2.(i) in the main theorem. Let X be a finite generating set for S. Then we have:*

(i) $q = \iota(X)$, $p = \kappa(X)$, $d = \gcd(\lambda(X))$;

(ii) $F_D = X \cap D \cap L_q$;

(iii) $P = \{(i-p) \mod d : \Lambda_i \cap X \cap \Sigma_p \neq \emptyset\}$;

(iv) $F = \bigcup_{i=1}^{M} (X \cap T_{q,p})^i \cap T_{q,p}$ *where* $M = (p-q+1)(p-q)/2$;

(v) Defining

$$I_0 = \{p + r - ud : r \in P, u \in \mathbb{N}_0, p + r - ud \geq q\} \cup \{i : \Lambda_i \cap (F \cup (X \cap S_{q,p})) \neq \emptyset\}$$

and the left action $. : \mathbf{B} \times \mathbb{N}_0 \to \mathbb{N}_0$ *by*

$$c^i b^j.k = \begin{cases} i \text{ if } j \geq k \\ i - j + k \text{ otherwise} \end{cases}$$

we have

$$I = \bigcup_{n=0}^{p-q} F^n.I_0.$$

Proof. Let $q' = \iota(X)$, $p' = \kappa(X)$, $d' = \gcd(\lambda(X))$, $F'_D = X \cap D \cap L_{q'}$ and $X' = X \backslash F'_D$. Then we have $S = F'_D \cup \langle X' \rangle$ and the elements of F'_D act as identities in $\langle X' \rangle$. If $q' \leq p'$ then $X' \subseteq M_d \cap (S'_{q',p'} \cup \Sigma_{p'})$ and, by Lemmas 4.1 and 4.3, this last set is a subsemigroup and so $\langle X' \rangle \subseteq M_d \cap (S'_{q',p'} \cup \Sigma_{p'})$, implying $q = q'$, $p = p'$. If $q' > p'$ then analogous reasoning gives $\langle X' \rangle \subseteq M_d \cap (\widehat{S'_{q',p'}} \cup \Sigma_{p'})$ from which follows that $p = q' < p' = q$ which contradicts our assumption on the shape of S. So we have $q = q'$, $p = p'$ and then it immediately follows that $F_D = F'_D = X \cap D \cap L_{q'} = X \cap D \cap L_q$. Finally, from $S = \langle X \rangle \subseteq M_{d'}$ (since $M_{d'}$ is a subsemigroup) it follows that $d = d'$. This proves *(i)* and *(ii)*.

We know that $P = \{(i-p) \mod d : \Lambda_i \cap S \cap \Sigma_p \neq \emptyset\}$. Let $P' = \{(i-p) \mod d : \Lambda_i \cap X' \cap \Sigma_p \neq \emptyset\}$. It is clear that $P' \subseteq P$. We also have $X' \subseteq \Sigma_{p,d,P} \cup (M_d \cap S'_{q,p}) = T$. But T is a subsemigroup by Lemma 4.6, and so $\langle X' \rangle = S \backslash F_D \subseteq T$. Therefore $S \cap \Sigma_p \subseteq T \cap \Sigma_p$ which is equivalent to $\Sigma_{p,d,P} \subseteq \Sigma_{p,d,P'}$ and so in fact $P = P'$, proving *(iii)*.

To prove *(iv)* we observe that the inclusions in Lemma 4.4 imply that $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is an ideal of S. It then follows that the elements of F can be obtained by forming the appropriate products of the generators of X that belong to $T_{q,p}$. Since $T_{q,p}$ has $(p-q+1)(p-q)/2$ elements the desired formula follows. In practice we do not need to form all these products. Again using the fact that $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is an ideal we see that F can be be computed by the following simple orbit algorithm:

$X_0 := X \cap T_{q,p}$
$F := X_0$
while not $(FX_0 \cap T_{q,p} \subseteq F)$ do
  $F := F \cup (FX_0 \cap T_{q,p})$
od.

To prove *(v)* we will first show that $I_0 \subseteq I$. Since $S \cap (S_{q,p} \cup \Sigma_p) = \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ is a subsemigroup, it follows from Lemma 4.8 that $\{p+r-ud : r \in P, u \in \mathbb{N}_0, p+r-ud \geq q\} \subseteq I$. Given $c^i b^j \in F \cup (X \cap S_{q,p})$ we can multiply this element on the right by a power of an element of the form $c^q b^{q+d_1}$ with $d_1 > 0$ (such an element must exist by definition of q) in order to obtain an element in $S \cap (\Lambda_i \cup S_{q,p})$. From this element we can obtain the whole line $\Lambda_{i,p,d}$ by using the elements $c^p b^{p+d}, c^{p+d} b^p \in T$ and so $I_0 \subseteq I$.

We will now show that $T = \Lambda_{I_0,p,d} \cup \Sigma_{p,d,P}$ is a right ideal ($TS^1 \subseteq T$). We know that T is a subsemigroup, by Lemma 4.8. By the way we have defined I_0 we have $X \cap S_{q,p} \subseteq T$. We also have $X \cap \Sigma_p \subseteq T$ because $S \cap \Sigma_p = \Sigma_{p,d,P} = T \cap \Sigma_p$. It remains to show that $T((X \cap T_{q,p}) \cup F_D) \subseteq T$. Let $c^k b^l \in T, c^i b^{i+d_1} \in (X \cap T_{q,p}) \cup F_D$. Since $l \geq i$, we have $c^k b^l c^i b^{i+d_1} = c^k b^{l+d_1} \in T$. Therefore T is a right ideal. Clearly if $I_0 \subseteq I' \subseteq I$ then $T' = \Lambda_{I',p,d} \cup \Sigma_{p,d,P}$ is a right ideal as well.

Finally we observe that, although multiplying two elements in F we can obtain an element in a line belonging to $I \backslash I_0$, we do not have to consider these products in order to obtain I. If $c^i b^j, c^k b^l \in F$ and $c^i b^j c^k b^l = c^{i-j+k} b^l$ where $i-j+k \in I \backslash I_0$, then I_0 contains line k and so line $i-j+k$ can also be obtained from $F.I_0$. We conclude that I can be obtained by running the orbit algorithm, starting from I_0:

$I := I_0$
while not $(F.I \subseteq I)$ do
  $I := I \cup F.I$
od.

This algorithm must stop in no more then $p-q$ iterations because it generates a strictly ascending chain of sets contained in $\{q, \ldots, p-1\}$ (normally far fewer iterations are necessary), concluding the proof of *(v)*. □

We will now consider finitely generated upper subsemigroups. Let $X \subseteq U \cup D$ be a finite set such that $X \cap U \neq \emptyset$ and let $S = \langle X \rangle$. As already remarked after Theorem 6.2, we are in the case where I is finite ($R = \emptyset$) in condition *3.(i)* of the main theorem, and our subsemigroup has the form

$$S = F_D \cup F \cup \Lambda_{I,m,d}.$$

Similarly as in the proof of Theorem 8.1 we can see that

$$q = \iota(X),\ p = \max(\Phi(X)) + 1,\ I \subseteq \{q, \ldots, p-1\},$$

$$F_D = X \cap D \cap L_q,\ d = \gcd(\lambda(X)).$$

We need to obtain the parameters F, I, and m from the generating set. Since the elements in F_D act as identities in $\langle X' \rangle$, where $X' = X \backslash F_D$, we will assume, without loss of generality, that $F_D = \emptyset$ and so $X = X' \subseteq S'_{q,p}$. We will define an algorithm to obtain these parameters that consists in forming a sequence of unions of powers of the generating set, $X, X \cup X^2, X \cup X^2 \cup X^3, \ldots$, until we have a subsemigroup of the form $F \cup \Lambda_{I,m,d}$. For that we need a sufficient condition, that can be checked by an algorithm, for a finite subset of a strip $S'_{q,p}$ to give us a subsemigroup of this form.

Lemma 8.2 *Let $Y \subseteq S'_{q,p}$ be a finite set with $\gcd(Y) = d$ and $c^q b^{q+d_1} \in Y$ for some $d_1 \in \mathbb{N}$. Suppose that for any $i \in I = \Phi(Y)$ there is $m_i \in \mathbb{N}_0$ such that*

$$c^i b^{m_i}, b^{m_i+d}, \ldots, c^i b^{2m_i-i-d} \in Y, \;\; c^i b^{m_i-d} \notin Y.$$

Let $m = \max\{m_i : i \in I\}$ and $F = Y \cap (S'_{q,p} \backslash S_{q,p,m})$. If $FF \cap (S'_{q,p} \backslash S_{q,p,m}) \subseteq F$ and $F.I \subseteq I$ then $\langle Y \rangle = F \cup \Lambda_{I,m,d}$. Moreover m is minimum such that $\Lambda_{I,m,d} \subseteq \langle Y \rangle$.

Proof. We start by showing that $F \cup \Lambda_{I,m,d} \subseteq \langle Y \rangle = S$. For any $i \in I$, we have $\Lambda_{i,m_i,d} \subseteq \langle c^i b^{m_i}, \ldots, c^i b^{2m_i-i-d} \rangle$, because any element in $\Lambda_{i,m_i,d}$ can be written in the form $c^i b^u (c^i b^{m_i})^k$ for some $k \in \mathbb{N}_0$, and $u \in \mathbb{N}_0$ such that $i + (m_i - i) = m_i \le u \le 2m_i - i - d = i + 2(m_i - i) - d$. We conclude that $\Lambda_{i,m_i,d} \subseteq S$ for any $i \in I$ and therefore $F \cup \Lambda_{I,m,d} \subseteq S$ with $m = \max\{m_i : i \in I\}$. It is clear that $Y \subseteq F \cup \Lambda_{I,m,d}$, because $Y \subseteq M_d$ and $I = \Phi(Y)$, and so to prove the other inclusion we just have to show that $F \cup \Lambda_{I,m,d}$ is a subsemigroup. We have $FF \cap (S'_{q,p} \backslash S_{q,p,m}) \subseteq F$, $F.I \subseteq I$ by hypothesis and, since $\Phi(F) \subseteq I$, we also have $\Phi(FF) \subseteq F.I \subseteq I$ and we conclude that $FF \subseteq F \cup \Lambda_{I,m,d}$. It is also clear that $\Lambda_{I,m,d}(\Lambda_{I,m,d} \cup F) \subseteq \Lambda_{I,m,d}$. Finally, $F.I \subseteq I$ implies $F\Lambda_{I,m,d} \subseteq \Lambda_{I,m,d}$. □

Clearly, it can be checked by an algorithm whether a finite set $Y \subseteq S'_{q,p}$ satisfies the conditions of Lemma 8.2; let us call such a procedure *iscomplete(Y)*. Also, provided that Y does satisfy these conditions, there is a straightforward procedure *parameters(Y)* returning the triple (F, I, m). Given these two procedures, an algorithm to compute the parameters F, I, m given any finite generating set X is:

```
Y := X
while not iscomplete(Y) do
  Y := Y ∪ YX
od
(F, I, m) := parameters(Y).
```

Note that if we are simply interested in the index set I of lines occurring in S, we can use a much more efficient orbit algorithm:

```
I := Φ(X)
while not X.I ⊆ I do
```

$I := I \cup X.I$

od.

We conclude the paper by presenting two examples: one of a two-sided subsemigroup and one of an upper subsemigroup.

Example 8.3 Let S be the subsemigroup of $\mathbf{B}$ generated by the set

$$X = \{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}.$$

Then S is clearly a two-sided subsemigroup of the form $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. From the generating set we see that $F_D = \{cb\}$, $q = 4$, $p = 17$, $d = 3$ and $P = \{0, 1\}$. The remaining parameters have been obtained using our implementation of the above algorithms in the system GAP (see [4]), and they are $I = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15\}$ and

$$F = \{c^4b^7, c^4b^{10}, c^4b^{13}, c^4b^{16}, c^7b^{13}, c^7b^{16}, c^{10}b^{13}, c^{10}b^{16}\}.$$

This subsemigroup is shown in Figure 5.

Example 8.4 Let S to be the subsemigroup of $\mathbf{B}$ generated by the set

$$X = \{cb, c^3b^{13}, c^5b^9, c^{10}b^{16}\}.$$

Then S is clearly an upper subsemigroup of the form $S = F_D \cup F \cup \Lambda_{I,m,d}$ and from the generating set we see that $F_D = \{cb\}$ and $d = 2$. Using again our implementation in GAP we have obtained $m = 20$, $I = \{3, 5, 6, 10\}$ and

$$F = \{c^3b^{13}, c^3b^{17}, c^3b^{19}, c^5b^9, c^5b^{13}, c^5b^{17}, c^5b^{19}, c^6b^{16}, c^{10}b^{16}\}.$$

This subsemigroup is shown in Figure 6.

Acknowledgements. The first author acknowledges the support of the Fundação para a Ciência e a Tecnologia, Portugal (grant number PRAXIS XXI/BD/19730/99).
The second author acknowledges partial support of INTAS 99/1224.

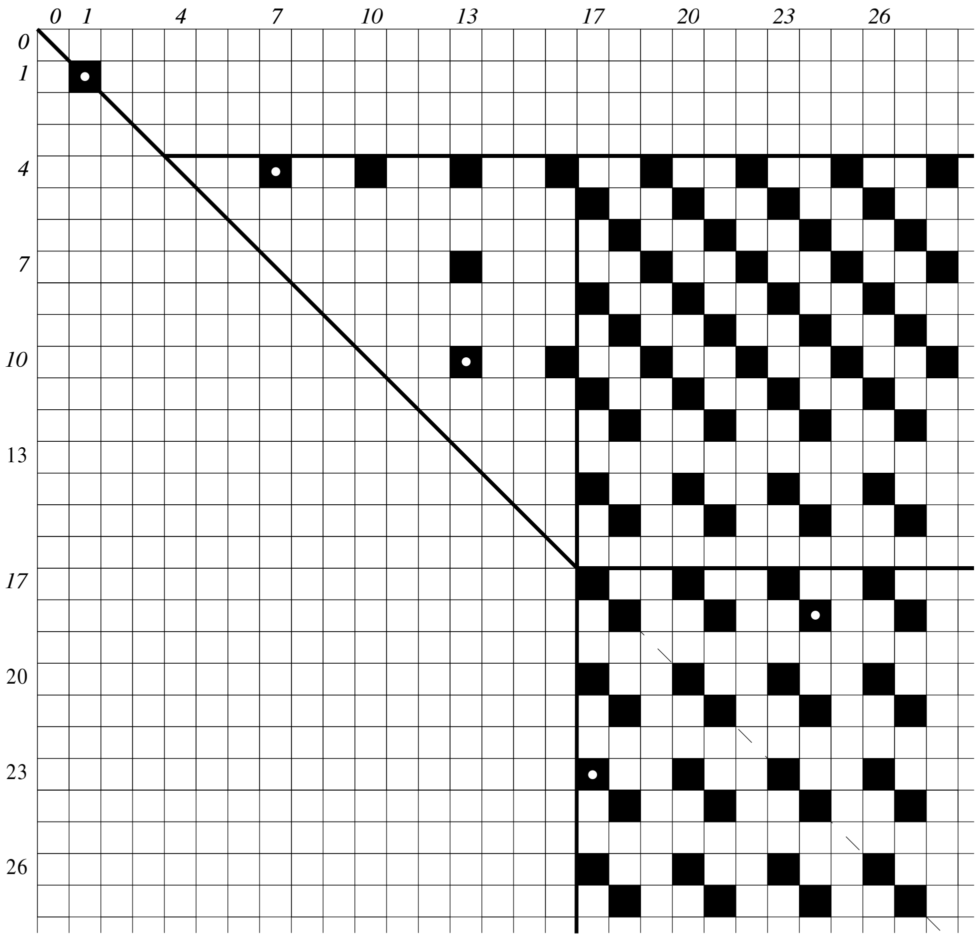

Figure 5: Two-sided subsemigroup generated by $\{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}$.

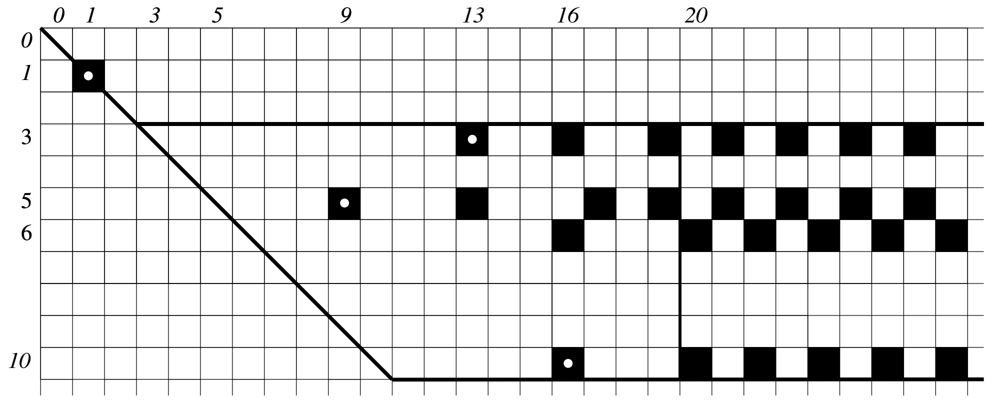

Figure 6: Upper subsemigroup generated by $\{cb, c^3b^{13}, c^5b^9, c^{10}b^{16}\}$.

List of Figures

References

[1] C. L. Adair, *A generalization of the bicyclic semigroup*, Semigroup Forum **21** (1980), 13–25.

[2] K. Byleen, J. Meakin and F. Pastijn, *The fundamental four-spiral semigroup*, J. Algebra **54** (1978), 6–26.

[3] L. Descalço and N. Ruškuc, *Properties of the subsemigroups of the bicyclic monoid*, In preparation.

[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.2*; 2000, (`http://www.gap-system.org`).

[5] P. A. Grillet, *On the fundamental double four-spiral semigroup*, Bull. Belg. Math. Soc. Simon Stevin **3** (1996), 201–208.

[6] J. W. Hogan, *The α-bicyclic semigroup as a topological semigroup*, Semigroup Forum **28** (1984), 265–271.

[7] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, 1991.

[8] M. V. Lawson, *Inverse Semigroups*, World Scientific, 1998.

[9] S. O. Makanjuola and A. Umar, *On a certain subsemigroup of the bicyclic semigroup*, Comm. Algebra **25** (1997), 509–519.

[10] L. N. Shevrin, *The bicyclic semigroup is determined by its subsemigroup lattice*, Simon Stevin **67** (1993), 49–53.

[11] L. N. Shevrin and A. J. Ovsyannikov, *Semigroups and Their Subsemigroup Lattices*, Kluwer Academic Publishers, 1996.