Automatic subsemigroups of free products

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Abstract

We consider the automaticity of subsemigroups of free products of semigroups, proving that subsemigroups of free products, with all generators having length greater than one in the free product, are automatic. As a corollary, we show that if S is a free product of semigroups that are either finite or free, then any finitely generated subsemigroup of S is automatic. In particular, any finitely generated subsemigroup of a free product of finite or monogenic semigroups is automatic.

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1 introduction

The notion of automatic group has recently been extended to semigroups and the basic properties of this new class of semigroups have been established in [3]. The notion of automatic semigroup does not correspond to a nice geometric property as in the case of groups where being automatic is the same as having the fellow traveller property (see [1, 2]). Nevertheless it is a natural class of semigroups where we have some interesting computational properties, for example the word problem is solvable in quadratic time (see [3]), and several results concerning automaticity of semigroups have been established (see, for example [4, 7, 8, 9, 10]).

We are interested in the following general question:

When is a subsemigroup of an automatic semigroup automatic as well?

A general result concerning this problem was established in [9], where the authors have proved the following:

Proposition 1.1 *Let S be a semigroup and let T be a subsemigroup of S such that $S\backslash T$ is finite. Then S is automatic if and only if T is automatic.*

A description of the subsemigroups of the bicyclic monoid was obtained in [5] and, using this description, the question above was answered (in [6]) for the bicyclic monoid and its subsemigroups:

Proposition 1.2 *All finitely generated subsemigroups of the bicyclic monoid are automatic.*

The question was also solved (in [3]) for free semigroups and their subsemigroups where the following was shown:

Proposition 1.3 *If F is a free semigroup and S is a finitely generated subsemigroup of F, then S is automatic.*

In this paper we extend this last result by considering subsemigroups of free products of semigroups. We show that some subsemigroups of free products of arbitrary semigroups, including in particular finitely generated subsemigroups of free semigroups, are automatic.

We start by introducing the definitions we require. Given a finite set A, which we call an *alphabet*, we denote by A^+ the free semigroup generated by A consisting of finite sequences of elements of A, which we call *words*, under the concatenation, and by A^* the free monoid generated by A consisting of A^+ together with the empty word ϵ. Let S be a semigroup and $\psi : A \to S$ a mapping. We say that A is a *finite generating set for S with respect to ψ* if the unique extension of ψ to a semigroup homomorphism $\psi : A^+ \to S$ is surjective. For $u, v \in A^+$ we write $u \equiv v$ to mean that u and v are equal as words and $u = v$ to mean that u and v represent the same element in the semigroup i.e. that $u\psi = v\psi$. We say that a subset L of A^+, usually called a *language*, is *regular* if there is a finite state automaton accepting L. To be able to deal with automata that accept pairs of words and to define automatic semigroups we need to define the set $A(2,\$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) \backslash \{(\$,\$)\}$ where \$ is a symbol not in A (called the *padding symbol*) and the function $\delta_A : A^* \times A^* \to A(2,\$)^*$ defined by

$$(a_1 \ldots a_m, b_1 \ldots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n \\ (a_1, b_1) \ldots (a_m, b_m) & \text{if } 0 < m = n \\ (a_1, b_1) \ldots (a_m, b_m)(\$, b_{m+1}) \ldots (\$, b_n) & \text{if } 0 \leq m < n \\ (a_1, b_1) \ldots (a_n, b_n)(a_{n+1}, \$) \ldots (a_m, \$) & \text{if } m > n \geq 0. \end{cases}$$

Let S be a semigroup and A a finite generating set for S with respect to $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S (with respect to ψ)* if

- L is a regular subset of A^+ and $L\psi = S$,
- $L_= = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha = \beta\}\delta_A$ is regular in $A(2,\$)^+$, and
- $L_a = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha a = \beta\}\delta_A$ is regular in $A(2,\$)^+$ for each $a \in A$.

We say that a semigroup is *automatic* if it has an automatic structure.

Given an alphabet A and a set $K \subseteq A^+$ we define

$$\begin{aligned} \mathrm{Pref}(K) &= \{w \in A^* : ww' \in K \text{ for some } w' \in A^*\}, \\ \mathrm{Suff}(K) &= \{w \in A^* : w'w \in K \text{ for some } w' \in A^*\}, \\ \mathrm{Subw}(K) &= \{w \in A^* : w_1 w w_2 \in K \text{ for some } w_1, w_2 \in A^*\} \end{aligned}$$

to be the sets of prefixes, suffixes and subwords of words in K, respectively.

If $S_1, \ldots, S_n$ are semigroups with presentations $\langle A_1 \mid R_1 \rangle, \ldots, \langle A_n \mid R_n \rangle$ then their *free product*, $S = S_1 * \ldots * S_n$, is the semigroup defined by the presentation $\langle A_1 \cup \ldots \cup A_n \mid R_1 \cup \ldots \cup R_n \rangle$. Any element $s \in S$ can be identified with a sequence

$$s_1 \ldots s_m \ (m > 1)$$

of elements of $\bigcup_{k=1}^n S_k$ such that,

$$s_i \in S_k \implies s_{i+1} \notin S_k \ (i = 1, \ldots, m-1; k = 1, \ldots, n);$$

such a sequence we call a *reduced sequence (of elements of* $\bigcup_{k=1}^n S_k$*)*. Given two elements $s = s_1 \ldots s_m, s' = s'_1 \ldots s'_p \in S$, their product ss' is the following: if the elements s_m, s'_1 do not belong to a common factor S_k then the product ss' is the concatenation of sequences and in this case we say simply that *the product* ss' *is the concatenation*; otherwise we have $s_m, s'_1 \in S_k$ for some k and the product ss' is the reduced sequence $s_1 \ldots s_{m-1} s'_0 s'_2 \ldots s'_p$ where $s'_0 = s_m s'_1$ in S_k.

2 Main Result

Our main result is the following:

Theorem 2.1 *Let S be a free product of finitely many semigroups. Let H be a subsemigroup of S generated by a finite set X such that no element of X belongs to a non free factor of S. Then H is automatic.*

This result has the following equivalent formulation:

Theorem 2.2 *Let S be a free product of finitely many semigroups*

$$S = S_1 * \ldots * S_n * T_1 * \ldots * T_m$$

where $T_1, \ldots, T_m$ are free semigroups on finite sets $Y_1, \ldots, Y_m$ respectively. Let $H =< t_1, \ldots, t_l >$ be a subsemigroup of S where

$$t_1, \ldots, t_l \in S \backslash (S_1 \cup \ldots \cup S_n).$$

Then H is an automatic semigroup.

Proof. Let us denote T_i by S_{n+i} for $i = 1, \ldots, m$ and let $Y = Y_1 \cup \ldots \cup Y_m$. Each generator t_i such that $t_i \notin T_1 \cup \ldots \cup T_m$ can be written as a reduced sequence of elements of $\bigcup_{k=1}^{m+n} S_k$:

$$t_i = s_{i,1} s_{i,2} \ldots s_{i,p(i)},$$

with $p(i) \geq 2$. For each $k \in \{1, \ldots, n\}$ we define

$$A_k = \{ {}^k a_1, \ldots, {}^k a_{r_k} \}$$

to be an alphabet in bijection with the following finite subset of S_k:

$$F_k = \bigcup_{i=1}^{l} (\{s_{i,j} \in S_k : j = 1, \ldots, p(i)\}) \cup \{s_{i,p(i)} s_{j,1} \in S_k : i, j \in \{1, \ldots, l\}\},$$

and let $f_k : A_k \to F_k$ be that bijection (we assume that the alphabets are disjoint). The elements in F_k are all those from S_k that may appear in a reduced sequence corresponding to an element from H (here it is essential that no generator belongs to S_k). They are finitely many and each one corresponds now to a letter from A_k.

We define the alphabet

$$A = A_1 \cup \ldots \cup A_n \cup Y$$

and the language $L \subseteq A^+$ by

$$\begin{aligned} L = \{y_1 \ldots y_k : \quad & y_i \in A_1 \cup \ldots \cup A_n \cup Y_1^+ \cup \ldots \cup Y_m^+, \\ & y_i \in A_j \implies y_{i+1} \notin A_j \ (i = 1, \ldots, k-1; j = 1, \ldots, n), \\ & y_i \in Y_j^+ \implies y_{i+1} \notin Y_j^+ \ (i = 1, \ldots, k-1; j = 1, \ldots, m)\}. \end{aligned}$$

The bijections f_k induce a homomorphism

$$f : A^+ \to S$$

and we will now show that any element in H has a unique representative in L. Given an element $h \in H$ it can be written as a product of the generators $t_1, \ldots, t_l$. Hence, when we write h as a reduced sequence of elements of $\bigcup_{j=1}^{n+m} S_j$: $h = u_1 \ldots u_r$, each element u_i is either some $s_{k,l}$ or a product $s_{k,p(k)} s_{l,1}$ or belongs to a free semigroup T_j. It follows from the definition of the alphabets $A_1, \ldots, A_n$ and from the definition of L that there is a unique word $w \in L$ such that $wf = h$.

Let $\gamma_1, \ldots, \gamma_l$ be the unique words in L such that $\gamma_i f = t_i$, $i = 1, \ldots, l$. Let $X = \{x_1, \ldots, x_l, 1\}$ be a new alphabet and ρ be the homomorphism defined by

$$\rho : (X \cup \{\$\})^+ \to A^*; x_i \mapsto \gamma_i; 1, \$ \mapsto \epsilon.$$

We define the partial function

$$\begin{aligned} \lambda : A^* \to L \cup \{\epsilon\}; \quad & \epsilon \mapsto \epsilon, \\ & w \mapsto \overline{w} \in L \text{ if there is } \overline{w} \in L \text{ such that } \overline{w} = w \text{ in } S, \end{aligned}$$

$$
\begin{tikzcd}
X^+ \arrow[r, "\rho"] & (X^+\rho)\backslash\{\varepsilon\} \arrow[r, "f"] \arrow[d, "\lambda"] & H \\
& (\overline{X^+\rho})\backslash\{\varepsilon\} \arrow[ur, leftrightarrow, "f"'] &
\end{tikzcd}
$$

Figure 1: Diagram with ρ, f and λ

which maps each word in A^+ to the corresponding unique "reduced word" in L if such word exists. The domain of this partial function is not A^* because there may for example exist $a, b \in A_k$ for some k, such that $(af)(bf) \notin F_k$ and in this case there is no word $w \in L$ such that $w = ab$ in S. Nevertheless, since we have

$$X^+\rho\backslash\{\epsilon\} = \{\gamma_{\alpha_1} \dots \gamma_{\alpha_k} : k \in \mathbb{N}; \alpha_1, \dots, \alpha_k \in \{1, \dots, l\}\},$$

the partial function λ is defined on $X^+\rho$, and more generally, it is easy to see that it is also defined on

$$\mathrm{Subw}((X^+\rho \cup \overline{X^+\rho})^+).$$

We observe that the set $\overline{X^+\rho}\backslash\{\epsilon\} \subseteq L \subseteq A^+$ is in bijection with H since, given an arbitrary $h \in H$ we have $h = t_{\alpha_1} \dots t_{\alpha_k}$ if and only if $h = \overline{(x_{\alpha_1} \dots x_{\alpha_k})\rho f}$, and we have already seen that there is a unique word in L representing h. Therefore, we can identify the subsemigroup H with the set $\overline{X^+\rho}\backslash\{\epsilon\}$ which is a semigroup, defining the product of two words $w_1, w_2 \in \overline{X^+\rho}\backslash\{\epsilon\}$, representing two elements $s_1, s_2 \in H$, to be the word $\overline{w_1 w_2} \in \overline{X^+\rho}\backslash\{\epsilon\}$, which represents the element $s_1 s_2 \in H$. This semigroup is generated by the words $\gamma_1, \dots, \gamma_l$. We observe that this product may be simply the concatenation or not, depending on the words w_1, w_2, but if it is not the concatenation, it means that the last letter in w_1 multiplies by the first letter from w_2 and we have $|\overline{w_1 w_2}| = |w_1 w_2| - 1$. Figure 1 illustrates the use of our functions by showing a diagram with the relevant subsets of their domains and ranges.

Let us consider the language $K \subseteq X^+$ defined by

$$\begin{aligned} K = \{ \quad & x_{\alpha_1} 1^{|\gamma_{\alpha_1}|-1} x_{\alpha_2} 1^{r(\alpha_1,\alpha_2)} x_{\alpha_3} \dots 1^{r(\alpha_{t-2},\alpha_{t-1})} x_{\alpha_t} 1^{r(\alpha_{t-1},\alpha_t)} : \\ & t \geq 1, \alpha_i \in \{1, \dots, l\}, i = 1, \dots, t\} \end{aligned}$$

where

$$r(i,j) = \begin{cases} |\gamma_j| - 1 & \text{if } |\overline{\gamma_i \gamma_j}| = |\gamma_i \gamma_j| \\ |\gamma_j| - 2 & \text{if } |\overline{\gamma_i \gamma_j}| = |\gamma_i \gamma_j| - 1. \end{cases}$$

We observe that $|w| = |\overline{w\rho}|$ for any word $w \in K$. We can easily define a finite deterministic automaton that recognizes the language K and so K is a regular language.

We denote by H^1 the monoid obtained by adjoining an identity 1_H to H and we identify this monoid with the monoid $\overline{X^+\rho}$, obtained from the semigroup $\overline{X^+\rho}\backslash\{\epsilon\}$ defined above, by adding the identity ϵ. Hence, we consider X as a generating set for $\overline{X^+\rho}$ with respect to the unique extension of the function $\varphi : X \to \overline{X^+\rho};\ x \mapsto x\rho$ to an homomorphism $\varphi : X^+ \to \overline{X^+\rho} \cong H^1$. We will show that (X, K^1) is an automatic structure for H^1, where K^1 is the regular language $K \cup \{1\} \subseteq X^+$.

We have

$$K^1_= = K^1_1 = K_= \cup \{(1,1)\},$$
$$K^1_{x_i} = K_{x_i} \cup \{(1,w)\delta_X : w \in K, \overline{w\rho} \equiv \gamma_i\}.$$

The sets $\{(1,1)\}$ and $\{(1,w)\delta_A : w \in K, \overline{w\rho} \equiv \gamma_i\}$ are finite, since $\overline{w\rho} \equiv \gamma_i$ implies $|w| = |\gamma_i|$, and so we just have to prove that $K_=$ and K_{x_i}, for each i, are regular languages.

Denoting by ${}^i\!a$, ${}^i\!b$, …generic elements in A_i, for $w_1, w_2 \in A^*$ we write $w_1 \bowtie w_2$ if one of the following situations occur:

$$\begin{aligned}
&(w_1 \in \mathrm{Pref}(w_2)\ \&\ w_1 \in A^*Y) \text{ or}\\
&(w_2 \in \mathrm{Pref}(w_1)\ \&\ w_2 \in A^*Y) \text{ or}\\
&(w_1 \equiv w\,{}^i\!a \text{ and } w_2 \equiv w\,{}^i\!bw') \text{ for some } i \text{ or}\\
&(w_1 \equiv w\,{}^i\!aw' \text{ and } w_2 \equiv w\,{}^i\!b) \text{ for some } i.
\end{aligned}$$

For $w_1 \bowtie w_2$ we define

$$\mathrm{Rem}(w_1, w_2) = \begin{cases} (\epsilon, w) & (w_2 \equiv w_1 w, w_1 \in A^*Y)\\ (w, \epsilon) & (w_1 \equiv w_2 w, w_2 \in A^*Y)\\ ({}^i\!a, {}^i\!bw') & (w_1 \equiv w\,{}^i\!a, w_2 \equiv w\,{}^i\!bw', i \in \{1, \ldots, k\})\\ ({}^i\!aw', {}^i\!b) & (w_1 \equiv w\,{}^i\!aw', w_2 \equiv w\,{}^i\!b, i \in \{1, \ldots, k\}). \end{cases}$$

Intuitively, for two words $w_1, w_2 \in L$ we have $w_1 \bowtie w_2$ if one of the words is almost a prefix of the other, in the sense that it may be possible to multiply the shorter word by a word from L in order to obtain the longer word. The function Rem (which stands for remainder) gives us the remainders of the two words: the two suffixes not belonging to the common prefix.

The following result tells us that there is a finite set where we can store the remainders, if we are dealing with words from our languages.

Claim 1 *There is a finite set $W \subseteq A^*$ such that $(w_1, w_2)\delta_X \in K_= \cup (\bigcup_{i=1}^{l} K_{x_i})$ implies that, for all $t \in \mathbb{N}$, we have $\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}$ and Rem$(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) \in W \times W$.*

Proof. We take

$$N = \max\{|\gamma_i| : i = 1, \ldots, l\}$$

and we will prove that the result holds with

$$W = \{w \in \mathrm{Suff}(\overline{X^+\rho}) : |w| \leq N + 1\}.$$

Let $w_1, w_2 \in K$ and $t \leq |w_1|, |w_2|$. By the definition of K, we can write $t \leq |\overline{w_j(t)\rho}| \leq t + N$ $(j = 1, 2)$ and so we have

$$||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N.$$

If $(w_1, w_2)\delta_X \in K_=$ then $\overline{w_1\rho} \equiv \overline{w_2\rho}$ and therefore

$$\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}.$$

Let $\mathrm{Rem}(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) = (\eta_1, \eta_2)$ where $\eta_1, \eta_2 \in A^*$. Since $w_1, w_2 \in K \subseteq X^+$ we have $w_1(t)\rho, w_2(t)\rho \in X^+\rho$ and so $\overline{w_1(t)\rho}, \overline{w_2(t)\rho} \in \overline{X^+\rho}$. Therefore, by definition of Rem, $\eta_1, \eta_2 \in \mathrm{Suff}(\overline{X^+\rho})$. Since $||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N$, again by definition of Rem, we have $|\eta_1|, |\eta_2| \leq N + 1$ and we conclude that $(\eta_1, \eta_2) \in W \times W$.

Suppose now that $(w_1, w_2)\delta_A \in K_{x_i}$. Then it is $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$ and so

$$\overline{w_1(t)\rho} \bowtie \overline{w_2(t)\rho}$$

for any $t \in \mathbb{N}$. Since we have $|\overline{w_1\rho}| = |w_1|$ and $|\overline{w_2\rho}| = |w_2|$ it may be $|w_2| = |w_1| + |\gamma_i|$ or $|w_2| = |w_1| + |\gamma_i| - 1$ according to whether $\overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ or not. For $t \leq |w_1|$ we have as above $t \leq |\overline{w_j(t)\rho}| \leq t + N$ $(j = 1, 2)$ and so $||\overline{w_1(t)\rho}| - |\overline{w_2(t)\rho}|| \leq N$. For $|w_1| < t \leq |w_2|$ we have

$$|\overline{w_1(t)\rho}| = |\overline{w_1\rho}| = |w_1|, \quad t \leq |\overline{w_2(t)\rho}| \leq |w_1| + |\gamma_i| \leq |w_1| + N$$

and so $||\overline{w_2(t)\rho}| - |\overline{w_1(t)\rho}|| \leq N$. Again $\overline{w_1(t)\rho}, \overline{w_2(t)\rho} \in \overline{X^+\rho}$, since $w_1, w_2 \in K \subseteq X^+$, and we have $\mathrm{Rem}(\overline{w_1(t)\rho}, \overline{w_2(t)\rho}) \in W \times W$. $\square$

From now on we assume that a set W satisfying the conditions of Claim 1 is fixed and we will use this set to construct automata that allow us to prove the regularity of our languages. We will prove that there is an automaton $\mathcal{M}$ such that $K_= = \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_X$ and automata $\mathcal{M}_i$ such that $K_{x_i} = \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_X$. Let

$$\mathcal{M} = (Q, B, (\epsilon, \epsilon), \mu, T)$$

where $Q = W \times W$ is the set of states, $B = (X \cup \{\$\}) \times (X \cup \{\$\})$ is the alphabet, (ϵ, ϵ) is the initial state, $T = \{(a, a) : a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}\}$ is the set of terminal states and the transition μ is the partial function from $Q \times B$ to Q defined by

$$(\alpha, \beta) \xrightarrow{(x,y)}_{\mu} \mathrm{Rem}(\overline{\alpha(x\rho)}, \overline{\beta(y\rho)}) \text{ if } \begin{array}{l} \overline{\alpha(x\rho)} \bowtie \overline{\beta(y\rho)} \text{ and} \\ \mathrm{Rem}(\overline{\alpha(x\rho)}, \overline{\beta(y\rho)}) \in W \times W \end{array}$$

for $(\alpha, \beta) \in Q$ and $(x, y) \in B$. For $i \in \{1, \ldots, l\}$ we define

$$M_i = (Q, B, (\epsilon, \epsilon), \mu, T_i)$$

where the set of terminal states T_i is defined as follows. If $\gamma_i \equiv {}^j a\gamma_i'$ for some word $\gamma_i' \in A^+$ then we define

$$T_i = \{({}^j b, {}^j c\gamma_i') : {}^j b\, {}^j a = {}^j c \text{ in } S_j\} \cup \{({}^k b, {}^k b\gamma_i) : k \neq j\} \cup \{(\epsilon, \gamma_i)\}.$$

If $\gamma_i \in YA^*$ then we define

$$T_i = \{(\,^k a,\ ^k a\gamma_i)\} \cup \{(\epsilon, \gamma_i)\}.$$

For $w_1 \equiv x_1 \dots x_n$, $w_2 \equiv y_1 \dots y_n$, with $x_1, \dots, x_n, y_1, \dots, y_n \in X \cup \{\$\}$, and $(\alpha, \beta), (\alpha', \beta') \in W \times W$ we write

$$(\alpha, \beta) \xrightarrow{(w_1, w_2)}_{\mu} (\alpha', \beta')$$

and we say that there is a path in the automaton from (α, β) to (α', β') labelled by (w_1, w_2), if there are $(\alpha_0, \beta_0) = (\alpha, \beta), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n) = (\alpha', \beta') \in W \times W$ such that $(\alpha_{i-1}, \beta_{i-1}) \xrightarrow{(x_i, y_i)} (\alpha_i, \beta_i)$, $i = 1, \dots, n$.

The following result, relates a path in the automata with the remainders of the pair of words labelling the path.

Claim 2 *For any $w_1, w_2 \in (X \cup \{\$\})^+$, with $|w_1| = |w_2|$, we have*

$$(\alpha, \beta) \xrightarrow{(w_1, w_2)}_{\mu} \quad (\theta_1, \theta_2) \quad \Longrightarrow \text{Rem}(\overline{\alpha(w_1\rho)}, \overline{\beta(w_2\rho)}) = (\theta_1, \theta_2). \tag{1}$$

Proof. We will prove this claim by induction on $m = |w_1| = |w_2|$. For $m = 1$ the implication follows from the definition of μ. Suppose the claim holds for words of length m and let w_1, w_2 be words of length $m + 1$ with $(\alpha, \beta) \xrightarrow{(w_1, w_2)}_{\mu} (\theta_1, \theta_2)$. Then we can write $w_1 \equiv w_1' x$ and $w_2 \equiv w_2' y$ where w_1' and w_2' are words of length m. We have $(\alpha, \beta) \xrightarrow{(w_1', w_2')}_{\mu} (\eta_1, \eta_2)$ and $(\eta_1, \eta_2) \xrightarrow{(x, y)}_{\mu} (\theta_1, \theta_2)$ for some words $\eta_1, \eta_2 \in W$. By the induction hypothesis and by definition of μ it is $(\eta_1, \eta_2) = \text{Rem}(\overline{\alpha(w_1'\rho)}, \overline{\beta(w_2'\rho)})$ and $(\theta_1, \theta_2) = \text{Rem}(\overline{\eta_1(x\rho)}, \overline{\eta_2(y\rho)})$. We can then write

$$\begin{aligned} \overline{\alpha(w_1'\rho)} &\equiv w''\eta_1, & \qquad \overline{\eta_1(x\rho)} &\equiv w'\theta_1, \\ \overline{\beta(w_2'\rho)} &\equiv w''\eta_2, & \qquad \overline{\eta_2(y\rho)} &\equiv w'\theta_2, \end{aligned}$$

for some words $w', w'' \in A^*$.

We will now show that

$$\overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1.$$

The equation holds trivially for $\theta_1 \equiv \epsilon$. If $w' \neq \epsilon$ the equation holds as well, since $w'\theta_1 \in L$. We will now consider the case where $\theta_1 \neq \epsilon$ and $w' \equiv \epsilon$. If $w'' \in A^*Y \cup \{\epsilon\}$ then the equation clearly holds. Otherwise we have $w'' \equiv w'''\,^i a$ for some i, it must be $\eta_1 \neq \epsilon$ by the definition of Rem and, since $w''\eta_1 \in L$, we have either $\eta_1 \in YA^*$ or $\eta_1 \equiv\, ^j b\eta_1'$ with $i \neq j$. Since $\overline{\eta_1(x\rho)} \equiv \theta_1$ we have $\theta_1 \in YA^*$ or $\theta_1 \in A_jA^*$ with $i \neq j$ as well and, in either case, $\overline{w''\theta_1} \equiv \overline{w''}\theta_1$ yielding again $\overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1$. A similar argument shows that $\overline{w''w'\theta_2} \equiv \overline{w''w'}\theta_2$.

Therefore we have

$$\begin{gathered} \overline{\alpha(w_1\rho)} \equiv \overline{\alpha(w_1'\rho)(x\rho)} \equiv \overline{\overline{\alpha(w_1'\rho)}(x\rho)} \equiv \\ \overline{(w''\eta_1)(x\rho)} \equiv \overline{w''\overline{\eta_1(x\rho)}} \equiv \overline{w''w'\theta_1} \equiv \overline{w''w'}\theta_1 \end{gathered}$$

and

$$\begin{aligned}&\overline{\beta(w_2\rho)} \equiv \overline{\beta(w_2'\rho)(y\rho)} \equiv \overline{\overline{\beta(w_2'\rho)}(y\rho)} \equiv \\ &\overline{(w''\eta_2)(y\rho)} \equiv \overline{w''\overline{\eta_2(y\rho)}} \equiv \overline{w''w'\theta_2} \equiv \overline{w''w'}\theta_2.\end{aligned}$$

Hence $\mathrm{Rem}(\overline{\alpha(w_1\rho)}, \overline{\beta(w_2\rho)}) = (\theta_1, \theta_2)$ which concludes the proof of the claim. $\square$

We will now use the two claims to prove that

$$\begin{aligned}K_= &= \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_X, \\ K_{x_i} &= \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_X \ (i = 1, \ldots, l),\end{aligned}$$

by showing each of the four inclusions separately.

To prove that $K_= \subseteq \mathcal{L}(\mathcal{M})$ let $(w_1, w_2)\delta_X \in K_=$ arbitrary. We have $\overline{w_1\rho} \equiv \overline{w_2\rho}$, $|w_1| = |\overline{w_1\rho}| = |\overline{w_2\rho}| = |w_2|$ and we can write $w_1 \equiv y_1 \ldots y_k$ and $w_2 \equiv z_1 \ldots z_k$ with $y_1, \ldots, y_k, z_1 \ldots, z_k \in X$. Using the two claims and by definition of μ we can construct a unique path labeled by (w_1, w_2),

$$(\epsilon, \epsilon) \xrightarrow{(y_1,z_1)}_\mu (\eta_1, \eta_1') \xrightarrow{(y_2,z_2)}_\mu (\eta_2, \eta_2') \xrightarrow{(y_3,z_3)}_\mu \ldots \xrightarrow{(y_k,z_k)}_\mu (\eta_k, \eta_k'),$$

with all $\eta_i, \eta_i' \in W$. By Claim 2 it must be $(\eta_k, \eta_k') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. Since $\overline{w_1\rho} \equiv \overline{w_2\rho}$, by definition of Rem we have $(\eta_k, \eta_k') = (a, a)$ with $a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}$, which means that $(w_1, w_2)\delta_A \in \mathcal{L}(\mathcal{M})$.

To prove that $\mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_X \subseteq K_=$ let w_1, w_2 be arbitrary words in K such that $(w_1, w_2)\delta_X \in \mathcal{L}(\mathcal{M})$. We can write $w_1 \equiv y_1 \ldots y_q$ and $w_2 \equiv z_1 \ldots z_r$ where $y_1, \ldots, y_q, z_1, \ldots, z_r \in X$. So there is a path

$$(\epsilon, \epsilon) \xrightarrow{(y_1 \ldots y_k, z_1 \ldots z_k)} (a, a)$$

in $\mathcal{M}$ where $k = max\{q, r\}$, $y_{q+1} = \ldots = y_k = z_{r+1} = \ldots = z_k = \$$ and $a \in A_1 \cup \ldots \cup A_n \cup \{\epsilon\}$. By Claim 2 and since $\$\rho = \epsilon$, it is $(a, a) = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$ which implies that $w_1 = w_2$ as elements of H and so $(w_1, w_2)\delta_X \in K_=$.

To prove that $K_{x_i} \subseteq \mathcal{L}(\mathcal{M}_i)$ let $(w_1, w_2)\delta_A \in K_{x_i}$ be arbitrary. We have $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$ and we write $w_1 \equiv y_1 \ldots y_k$, $w_2 \equiv z_1 \ldots z_r$ with $y_1, \ldots, y_k, z_1, \ldots, z_r \in X$. We note that $r = |w_2| = |\overline{w_2\rho}| = |\overline{(w_1\rho)\gamma_i}| > |\overline{w_1\rho}| = |w_1| = t$. Using the previous claims and by definition of μ we can construct a unique path in $\mathcal{M}_i$ labeled by $(w_1\$^{r-k}, w_2)$,

$$\begin{aligned}&(\epsilon, \epsilon) \xrightarrow{(y_1,z_1)}_\mu (\eta_1, \eta_1') \xrightarrow{(y_2,z_2)}_\mu (\eta_2, \eta_2') \to \ldots \\ &\xrightarrow{(y_k,z_k)}_\mu (\eta_k, \eta_k') \xrightarrow{(\$,z_{k+1})}_\mu (\eta_{k+1}, \eta_{k+1}') \to \ldots \xrightarrow{(\$,z_r)}_\mu (\eta_r, \eta_r'),\end{aligned}$$

with all $\eta_j, \eta_j' \in W$. By Claim 2, $(\eta_r, \eta_r') = \mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. If $\gamma_i \in YA^*$ then, it can be $\overline{w_1\rho} \in A^*Y$ and so $(\eta_r, \eta_r') = (\epsilon, \gamma_i) \in T_i$, or $\overline{w_1\rho} \equiv w\ {}^k a$ and then $(\eta_r, \eta_r') = ({}^k a, {}^k a\gamma_i) \in T_i$ as well. Otherwise we have $\gamma_i \equiv {}^j a\gamma_i'$ and, since $\overline{(w_1\rho)\gamma_i} \equiv \overline{w_2\rho}$, there are three possibilities: it may be $\overline{w_1\rho} \equiv w'\ {}^j b$ and $\overline{w_2\rho} \equiv w'\ {}^j c\gamma_i'$ with ${}^j b\, {}^j a = {}^j c$ in S_j, and so $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^j b, {}^j c\gamma_i') \in T_i$; it can also be $\overline{w_1\rho} \equiv w'\ {}^k b$ $(k \neq j)$ and $\overline{w_2\rho} \equiv w'\ {}^k b\gamma_i$ and then $\mathrm{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^k b, {}^k b\gamma_i) \in T_i$; finally it can be $\overline{w_1\rho} \in A^*Y$

and then $\text{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = (\epsilon, \gamma_i) \in T_i$. In any case $(\eta_r, \eta'_r) = \text{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) \in T_i$ and so $(w_1, w_2)\delta_X \in \mathcal{L}(\mathcal{M}_i)$.

To prove that $\mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_X \subseteq K_{x_i}$ let $w_1, w_2 \in K$ arbitrary such that $(w_1, w_2)\delta_X \in \mathcal{L}(\mathcal{M}_i)$. We can write $w_1 \equiv y_1 \dots y_q$ and $w_2 \equiv z_1 \dots z_r$ where $y_1, \dots, y_q$, $z_1, \dots, z_r \in X$. There is a path

$$(\epsilon, \epsilon) \xrightarrow{(y_1 \dots y_k, z_1 \dots z_k)} (\eta, \eta')$$

in $\mathcal{M}_i$ where $k = max\{q, r\}$, $y_{q+1} = \dots = y_k = z_{r+1} = \dots = z_k = \$$ and $(\eta, \eta') \in T_i$. By Claim 2 we have $(\eta, \eta') = \text{Rem}(\overline{w_1\rho}, \overline{w_2\rho})$. If $\gamma_i \equiv {}^j a \gamma'_i$ then, by definition of T_i, we have either $(\eta, \eta') = ({}^j b, {}^j c\gamma'_i)$ with ${}^j b\, {}^j a = {}^j c$ in S_j, or $(\eta, \eta') = ({}^k b, {}^k b\gamma_i)$ with $k \neq j$, or $(\eta, \eta') = (\epsilon, \gamma_i)$. In the first case we have $\overline{w_1\rho} \equiv w\, {}^j b$ and $\overline{w_2\rho} \equiv w\, {}^j c\gamma'_i$ for some word $w \in A^*$ and so we can write $\overline{(w_1\rho)\gamma_i} \equiv \overline{w\, {}^j b\, {}^j a\gamma'_i} \equiv \overline{w\, {}^j c\gamma'_i} \equiv \overline{w_2\rho}$ which means that $w_1 x_i = w_2$ in H. In the second case we have $\overline{w_1\rho} \equiv w\, {}^k b$ and $\overline{w_2\rho} \equiv w\, {}^k b\gamma_i$ for some word $w \in A^*$ and so we can write $\overline{(w_1\rho)\gamma_i} \equiv \overline{w\, {}^k b\gamma_i} \equiv \overline{w_2\rho}$ and again $w_1 x_i = w_2$ in H. In the third case we have $\overline{w_1\rho} \in A^*Y$ and so $\overline{w_2\rho} \equiv \overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ which implies $w_2 = w_1 x_i$ in H. If we have $\gamma_i \in YA^*$ then, by definition of T_i, it may be $\text{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = ({}^k a, {}^k a\gamma_i)$ or $\text{Rem}(\overline{w_1\rho}, \overline{w_2\rho}) = (\epsilon, \gamma_i)$. In the first case we have $\overline{w_1\rho} \equiv w\, {}^k a$ and $\overline{w_2\rho} \equiv w\, {}^k a\gamma_i$ which implies that $\overline{(w_1\rho)\gamma_i} \equiv \overline{w\, {}^k a}\gamma_i \equiv w\, {}^k a\gamma_i \equiv \overline{w_2\rho}$ and therefore $w_1 x_i = w_2$ in H. In the second case we have $\overline{w_1\rho} \in A^*Y$ and $\overline{w_2\rho} \equiv \overline{w_1\rho}\gamma_i \equiv \overline{(w_1\rho)\gamma_i}$ which implies again $w_2 = w_1 x_i$ in H. So in any case $(w_1, w_2)\delta_X \in K_{x_i}$ and the inclusion is proved.

To conclude the proof of the theorem we observe that, since $K_= = \mathcal{L}(\mathcal{M}) \cap (K \times K)\delta_A$ and $K_{x_i} = \mathcal{L}(\mathcal{M}_i) \cap (K \times K)\delta_A$, $K_=$ and K_{x_i} are regular languages and so H^1 is automatic which implies that H is automatic. □

3 Corollaries and Questions

We have the following consequence of our result, which concerns free products of free and finite semigroups:

Corollary 3.1 *If S is a free product of semigroups that are either finite or free then any finitely generated subsemigroup of S is automatic.*

Proof. Let $S = S_1 * \dots * S_n * T_1 * \dots * T_m$ where $S_1, \dots, S_n$ are finite semigroups and $T_1, \dots, T_m$ are free semigroups. Let H be an (infinite) subsemigroup of S. Suppose that H is generated by $A = \{t_1, \dots, t_l\} \subseteq S$ and, without loss of generality, that $A \cap S_1 = \{t_1, \dots, t_k\}$ $(0 < k < l)$. Since the semigroup $U =< t_1, \dots, t_k >$ is a subsemigroup of S_1 it is finite. Let H' be the semigroup generated by the finite set

$$A' = \{U^1 t_{k+1} U^1, U^1 t_{k+2} U^1, \dots, U^1 t_l U^1\}.$$

We observe that

$$A \cap (S_1 \cup \dots \cup S_n) \supsetneq A' \cap (S_1 \cup \dots \cup S_n), \quad A' \cap S_1 = \emptyset$$

and $H \backslash H' = U$ is finite. If A' contains elements from S_2 we can remove them the same way obtaining a semigroup H'' generated by a set A'' that does not contain elements from $S_1 \cup S_2$ and such that $H \backslash H''$ is finite. Repeating this process for every S_i that contains generators we will obtain a semigroup V generated by a set B such that $B \cap (S_1 \cup \ldots \cup S_n) = \emptyset$ and $H \backslash V$ is finite. Since V is in the conditions of the previous theorem it is automatic. Since $H \backslash V$ is finite we can use Proposition 1.1 and conclude the H is automatic. □

Corollary 3.2 *Any finitely generated subsemigroup of a free product of finite semigroups is automatic.*

Proof. This is a particular case of the previous corollary, worth stating separately. □

We say that a semigroup is *monogenic* if it is generated by a single element and we have the following result:

Corollary 3.3 *Any finitely generated subsemigroup of a free product of monogenic semigroups is automatic.*

Proof. A monogenic semigroup is either free or finite and so we can use Corollary 3.1.□

Defining a semigroup to be *strongly automatic* if all its finitely generated subsemigroups are automatic we may ask the following question:

Question 3.4 Is the free product of strongly automatic semigroups always strongly automatic?

The answer to the same question for groups is ”yes” because we can use the Kurosh Subgroup Theorem: If H is a subgroup of $G_1 * G_2$ then H is isomorphic to $F * H_1 * H_2$ where F is a free group, H_1 is isomorphic to a subgroup of G_1 and H_2 is isomorphic to a subgroup of G_2. For semigroups it is still an open question.

By Proposition 1.2 the bicyclic monoid is strongly automatic and so we may also consider the following question:

Question 3.5 Does Theorem 2.1 still hold if we allow generators to belong to factors isomorphic to the bicyclic monoid?

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