Characterizing and Covering Some Subclasses of Orthogonal Polygons ⋆

Ana Mafalda Martins[1] and António Leslie Bajuelos[2]

[1] Escola Superior de Ciências e Tecnologia, Universidade Católica Portuguesa, Campus Viseu, Portugal
ammartins@crb.ucp.pt
[2] Dept. of Mathematics & CEOC - Center for Research in Optimization and Control, University of Aveiro, Portugal
leslie@mat.ua.pt

Abstract. A *grid n-ogon* is a n-vertex orthogonal polygon that may be placed in a $\frac{n}{2} \times \frac{n}{2}$ unit square grid and that does not have collinear edges. Given a grid n-ogon P, let $|\Pi(P)|$ be the number of rectangles that results when we partition P by extending the edges incident to reflex vertices towards its interior. P is called FAT if $|\Pi(P)|$ is maximal for all grid n-ogons; P is called THIN if $|\Pi(P)|$ is minimal for all grid n-ogons. THINs with area $2r+1$ are called MIN-AREA. We will show that $\lceil \frac{n}{6} \rceil$ vertex guards are necessary to guard a MIN-AREA grid n-ogon and present some problems related to THINs.

1 Introduction

Art Gallery problems represent a classic and very interesting field of Computational Geometry. The original art gallery problem was introduced by V. Klee in 1973 in a conference of Mathematics. He posed the following problem to V. Chvtal: How many stationary guards are needed to cover an art gallery room with n walls? Informally the floor plan of the art gallery room is modeled by a *simple polygon* (simple closed polygon with its interior) P and a guard is considered a fixed point in P with 2π range visibility. We say that a point x *sees* point y (or y is visible to x) if the line segment connecting them lies entirely in P. A set of guards covers P, if each point of P is visible by at least one guard. Thus, the Art Galery Problem deals with setting a minimal number of guards in a gallery room whose floor plan has polygonal shape, so that they could see every point in the room. Two years later Chvtal established the well known *Chvátal Art Gallery Theorem*: $\lfloor \frac{n}{3} \rfloor$ *guards are occasionally necessary and always sufficient to cover a simple polygon of n vertices.*

Many variants of the original art gallery problem have been considered and studied over the years, see [4, 5, 9] for comprehensive surveys. An interesting variant is the *Orthogonal Art Gallery Theorem*. This theorem was first formulated

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and proved by Kahn et al, in 1983. It states that $\lfloor \frac{n}{4} \rfloor$ *guards are occasionally necessary and always sufficient to cover an orthogonal simple polygon of* n *vertices*. *Orthogonal simple polygons* (simple polygons whose edges meet at right angles) are an important subclass of polygons. Indeed, they are useful as approximations to polygons; and they arise naturally in domains dominated by Cartesian coordinates, such as raster graphics, VLSI design, or architecture. Efficient algorithms, based on the proofs of the above theorems, were developed to cover both arbitrary and orthogonal simple polygons with $\lfloor \frac{n}{3} \rfloor$ and $\lfloor \frac{n}{4} \rfloor$ guards, respectively. While this number of guards is necessary in some cases, often it is far more than it is needed to cover a particular simple polygon. For instance, it is known that convex polygons only require one guard. Similarly, depending on the structure of a simple polygon the minimum number of guards may be smaller than the estimated. A variant of this problem is the Minimum Vertex Guard (MVG) problem, that is the problem of finding the minimum number of guards placed on vertices (*vertex guards*) needed to cover a given simple polygon. This is a NP-hard problem both for arbitrary and orthogonal simple polygons [2, 6].

Our contribution. This paper has as intention to introduce a subclass of orthogonal polygons that presents sufficiently interesting characteristics that we are studying and formalizing, in particular the way they can be guarded. Of these polygons two classes stand out: the Fats and Thins. We think Fats and Thins are representative of extremal behavior and they are used experimentally to evaluate some approximated methods of resolution of the MVG problem [8]. The paper is structured as follows: in the next section we will present some introductory definitions and useful results. In section 3, we will study the MVG problem for a subclass of Thin grid n-ogons (the Min-Area) and in section 4 we will refer to some problems related to Thins.

2 Conventions, Definitions and Results

For every n-vertex orthogonal simple polygon (n*-ogon* for short), $n = 2r + 4$, where r denotes the number of reflex vertices, e.g. [4]. A *rectilinear cut* (r-cut) of an n-ogon P is obtained by extending each edge incident to a reflex vertex of P towards the interior of P until it hits P's boundary. We denote this partition by $\Pi(P)$ and the number of its pieces by $|\Pi(P)|$. Each piece is a rectangle and so we call it a r*-piece*. A n-ogon that may be placed in a $\frac{n}{2} \times \frac{n}{2}$ square grid and that does not have collinear edges is called *grid* n*-ogon*. We assume that the grid is defined by the horizontal lines $y = 1, \ldots, y = \frac{n}{2}$ and the vertical lines $x = 1, \ldots, x = \frac{n}{2}$ and that its northwest corner is $(1, 1)$. Grid n-ogons that are symmetrically equivalent are grouped in the same class [1]. A grid n-ogon Q is called Fat iff $|\Pi(Q)| \geq |\Pi(P)|$, for all grid n-ogons P. Similarly, a grid n-ogon Q is called Thin iff $|\Pi(Q)| \leq |\Pi(P)|$, for all grid n-ogons P. Let P be a grid n-ogon with r reflex vertices, in [1] is proved that, if P is Fat then $|\Pi(P)| = \frac{3r^2+6r+4}{4}$, for r even and $|\Pi(P)| = \frac{3(r+1)^2}{4}$, for r odd; if P is Thin then $|\Pi(P)| = 2r + 1$. There is a single Fat n-ogon (except for symmetries of the grid) and its form is illustrated in fig. 1(a). However, the Thin n-ogons are not unique (see fig. 1(b)).

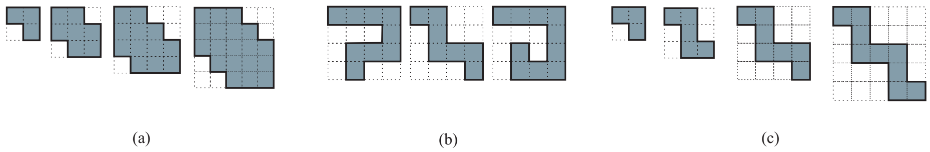


Fig. 1. (a) The unique FAT n-ogon, for $n = 6, 8, 10, 12$; (b) Three THIN 10-ogons; (c) The unique MIN-AREA grid n-ogons, for $n = 6, 8, 10, 12$.

The area of a grid n-ogon is the number of grid cells in its interior. In [1] it is proved that for all grid n-ogon P, with $n \geq 8$, $2r + 1 \leq A(P) \leq r^2 + 3$. A grid n-ogon P is a MAX-AREA grid n-ogon iff $A(P) = r^2 + 3$ and it is a MIN-AREA grid n-ogon iff $A(P) = 2r + 1$. There exist MAX-AREA grid n-ogons for all n, but they are not unique. However, there is a single MIN-AREA grid n-ogon (except for symmetries of the grid) and it has the form illustrated in fig. 1(c). Regarding MIN-AREA grid n-ogons, it is obvious that they are THIN n-ogons, because $|\Pi(P)| = 2r + 1$ holds only for THIN n-ogons. However, this condition is not sufficient for a grid n-ogon to be a MIN-AREA grid n-ogon.

Our aim is to study the MVG problem for grid n-ogons. Since THIN and FAT n-ogons are the classes for which the number of r-pieces is minimum and maximum, we think that they can be representative of extremal behavior, so we started with them. We already proved that to cover any FAT grid n-ogon it is always sufficient two $\frac{\pi}{2}$ vertex guards (*vertex guards with $\frac{\pi}{2}$ range visibility*) and established where they must be placed [3]. THINs are much more difficult to cover, on the contrary of what we might think once they have much fewer pieces than FATs. Since THIN grid n-ogons are not unique, we intend to characterize structural properties of classes of THINs that allow to simplify the problem study. Up to now the only quite characterized subclass is the MIN-AREA grid n-ogons: the subclass for which the number of grid cells is minimum.

3 Guarding Min-Area grid n-ogons

Given P, a MIN-AREA, we will denote by $g(P)$ the minimum number of vertex guards that is needed to cover P. We will show not only that $g(P) = \lceil \frac{r+2}{3} \rceil$ but also in which vertices these guards must be placed.

Lemma 1. *Two vertex guards are necessary to cover the Min-Area* grid 12-ogon*. Moreover, the only way to do so is with the vertex guards $v_{2,2}$ and $v_{5,5}$.*

Proof (Sketch).

This demonstration is based on the fact that the unit squares Q_0 and Q_1 will have to be guarded and that the only vertex guards that can do it and simultaneously guard all the polygon are $v_{2,2}$ and $v_{5,5}$ (see fig. 2(a)).

□

Proposition 1. *If we "merge" $k \geq 2$* Min-Area *grid 12-ogons, we will obtain the* Min-Area *grid n-ogon with $r = 3k + 1$. More, $k + 1$ vertex guards are necessary to cover it, and the only way to do so is with the vertex guards: $v_{2+3i,2+3i}, i = 0, 1, \ldots, k$.*

Proof. Let P be the Min-Area *grid n-ogon* with $r = 7$ reflex vertices. P can be obtained "merging" two Min-Area *grid 12-ogons* (see fig. 2(b)).

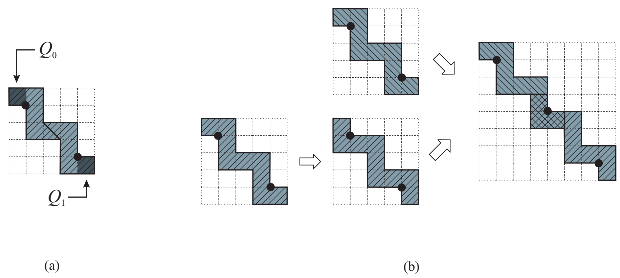


Fig. 2. (a) Min-Area grid 12-ogon; (b) Construction of the Min-Area grid 18-ogon from two Min-Area grid 12-ogons.

By lemma 2 and as we can see, 3 vertex guards are necessary to cover P, and the only way to do that is with $v_{2,2}$, $v_{5,5}$ and $v_{8,8}$. Thus, for $k = 2$, the proposition is true. Let $k \geq 2$, we will show that the proposition is true for $k+1$ (*induction thesis*), assuming that it is true for k (*induction hypotheses*).

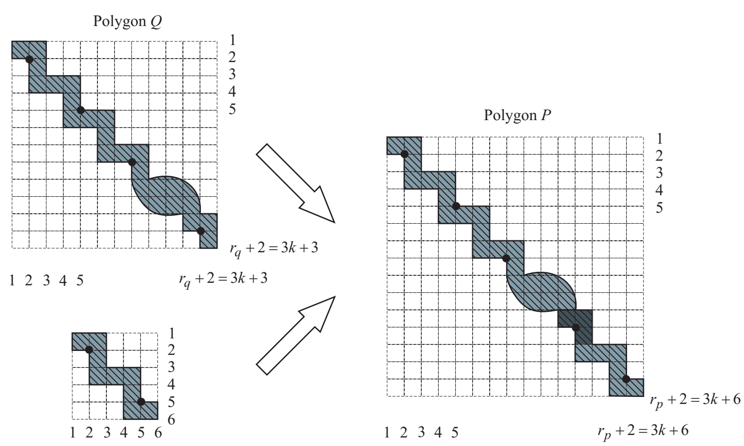


Fig. 3. Polygon P ("merge" of Q with the Min-Area grid 12-ogon).

By induction hypothesis, "merging" k Min-Area grid 12-ogons we obtain Q, the Min-Area grid n-ogon with $r_q = 3k + 1$ reflex vertices. If we "merge" Q

with the MIN-AREA grid 12-ogon, we will obtain a polygon P (see fig. 3). P has $r_p = 3k+4$ reflex vertices and $A(P) = 2r_p + 1$. Therefore, "merging" $k+1$ MIN-AREA grid 12-ogons we obtain P, the MIN-AREA grid n-ogon with $r = 3k+4$. Furthermore, by induction hypotheses and from what we can observe in fig. 3, we can conclude that $k+2$ vertex guards are necessary to cover P. Moreover, the only way to do so is with the vertex guards: $v_{2+3i,2+3i}, i = 0, 1, \ldots, k+1$.

□

Proposition 2. $\lceil \frac{r+2}{3} \rceil$ *vertex guards are always necessary to guard a* MIN-AREA *grid n-ogon with r reflex vertices.*

Proof. Let P_n be a MIN-AREA grid n-ogon with $r_n = \frac{n-4}{2}$ reflex vertices. We may easily check that $1, 2$ and 2 vertex guards are necessary to guard MIN-AREA grid n-ogons with $r_n = 1, 2, 3$, respectively (see fig. 4).

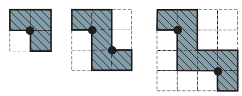


Fig. 4. Min-Area grid n-ogons with $r = 1, 2, 3$.

Let $r_n \geq 4$. If $r_n \equiv 1\ (mod\ 3)$ then, by proposition 1, the $\lceil \frac{r_n+2}{3} \rceil$ vertex guards $v_{2+3i,2+3i}, i = 0, 1, \ldots, \frac{r_n-1}{3}$, are necessary to cover P_n. Thus, we just need to prove the following cases: $r_n \equiv 2\ (mod\ 3)$ and $r_n \equiv 0\ (mod\ 3)$.

In any case, P_n can be obtained, by INFLATE-PASTE (a complete process to generate grid n-ogons, well described in [7]), from a Min-Area Q_m with $r_m = \frac{m-4}{2}$ and such that $r_m = 3k_m + 1$ (see fig. 5). The first case corresponds to polygon Q_{m+2}, in fig. 5, and $r_n = r_m + 1$. The second case corresponds to polygon Q_{m+4}, in fig. 5, and $r_n = r_m + 2$.

As we can see, in any case, is always necessary one more vertex guard, which can be v_{r_n+1,r_n+1}. Thus, $\lceil \frac{r_m+2}{3} \rceil + 1 = \lceil \frac{r_n+2}{3} \rceil$ vertex guards are necessary to guard P_n.

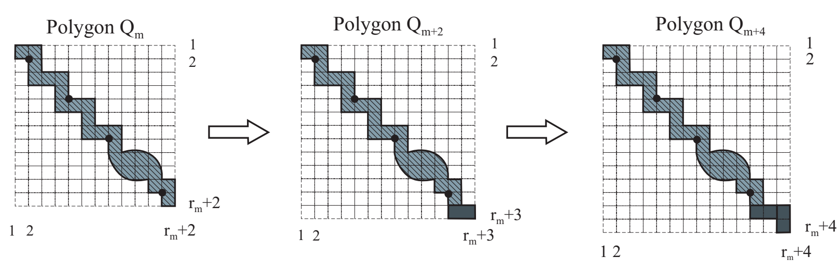


Fig. 5. Min-Area grid n-ogons Q_m, Q_{m+2} and Q_{m+4}.

□

Proposition 2 not only gives the guarantee of that $\lceil \frac{r+2}{3} \rceil$ vertex guards are required to guard the Min-Area grid n-ogon with r reflex vertices, but also establishes a possible positioning.

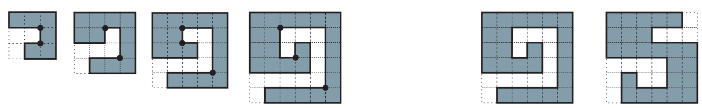
4 Some problems related to Thin n-ogons

As we saw in section 1, on the contrary of the FAT the Thin grid n-ogons are not unique. In fact, 1 Thin 6-ogon exists, 2 Thin 8-ogons exist, 30 Thin 10-ogons exist, 149 Thin 12-ogons exist, etc. Thus, it is interesting to evidence that the number of Thin grid n-ogons ($|\text{Thin}(n)|$) grows exponentially. Will it exist some expression that relates n to $|\text{Thin}(n)|$? Also, we can question on the value of the area of the Thin grid n-ogon with maximum area (Max-Area-Thin n-ogon) and if the Max-Area-Thin n-ogon is unique.

Denote by MA_r the value of the area of "the" Max-Area-Thin n-ogon with r reflex vertices. By observation we concluded that $MA_2 = 6, MA_3 = 11, MA_4 = 17$ and $MA_5 = 24$ (see Fig. 6(a)). Note that, $MA_2 = 6$, $MA_3 = MA_2 + 5$, $MA_4 = MA_3 + 6 = MA_2 + 5 + 6$ and $MA_5 = MA_4 + 7 = MA_2 + 5 + 6 + 7$. From these observations it follows:

Conjecture 1. $MA_r = MA_2 + 5 + 6 + 7 + \ldots + (r + 2) = \frac{r^2+5r-2}{2}$.

If conjecture 1 is true we can say the Thin grid n-ogon with maximum area is not unique (see Fig. 6(b)).



(a) (b)

Fig. 6. (a) From left to right $MA_2 = 6, MA_3 = 11, MA_4 = 17, MA_5 = 24$; (b) Two Thin 14-ogons with area 24, $MA_5 = 24$.

Definition 1. *A* Thin *n-ogon is called* Spiral-Thin *if its boundary consists of two polygonal chains: a chain of reflex vertices and a chain of convex vertices.*

From left to right in fig. 6(a), the second Spiral-Thin can be obtained from the first by Inflate-Paste, the third Spiral-Thin can be obtained from the second... So we believe that a Max-Area-Thin grid $(n + 2)$-ogon can always be obtained from a Max-Area-Thin grid n-ogon. We intend to use the following results and the Spiral-Thin grid n-ogons illustrated in fig. 6(a) to prove conjecture 1.

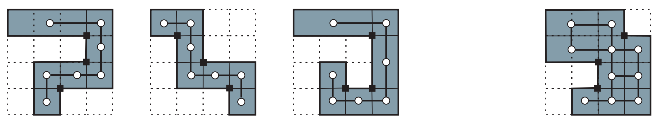
The dual graph of $\Pi(P)$ captures the adjacency relation between pieces of $\Pi(P)$. Its nodes are r-pieces and its non-oriented edges connect adjacent r-pieces, i.e., r-pieces with a common edge. We will denote the dual graph of $\Pi(P)$ by $DG(\Pi(P))$

Lemma 2. *Let P be a* Thin *$(n+2)$-ogon. Then every grid n-ogon that yields P by* Inflate-Paste *is also* Thin.

The proof of this lemma is strongly based on the reverse process of Inflate-Paste.

Proposition 3. *Let P be a* Thin *grid n-ogon with $r = \frac{n-4}{2} \geq 1$ reflex vertices, then $DG(\Pi(P))$ is a path graph (i.e., a tree with two nodes of vertex degree 1, called leaves, and the other nodes of vertex degree 2) (see examples in fig. 7(a)).*

The proof of this proposition is done by induction on r and uses lemma 2.



(a) (b)

Fig. 7. (a) Three Thin grid 10-ogon and respective dual graphs; (b) A grid 10-ogon and respective dual graph.

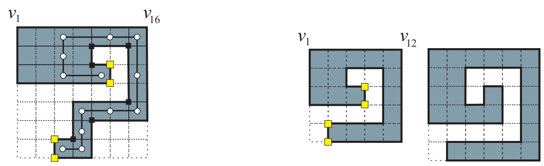
Proposition 4. *Let P be a grid n-ogon. If P is not* Thin *then $DG(\Pi(P))$ is not a tree (see example in fig 7(b)).*

Proposition 5. *The unique convex vertices of a* Thin *grid n-ogon that could be used to yield a* Thin *grid $(n+2)$-ogon, by Inflate-Paste, are those which belong to the r-pieces associated to the leaves of $DG(\Pi(P))$.*

Lemma 2 and proposition 5 can be very useful in the generation, by Inflate-Paste, of Thin grid n-ogons ($n \geq 8$). Lemma 2 says that we must take a Thin grid $(n-2)$-ogon, and proposition 5 establishes that the only convex vertices that can "work" are those which belong to the r-pieces associated to the leaves of $DG(\Pi(P))$ (which are in number of 4). In this way we do not need to apply Inflate-Paste to all the convex vertices of a Thin and then to check which of the produced polygons are Thins. We just need to apply Inflate-Paste to 4 convex vertices and then check which of the produced polygons are Thins. So the number of case analysis is significantly reduced (see fig 8).

Conjecture 2. There exists at least a Thin grid n-ogon for which $\lfloor \frac{r}{2} \rfloor + 1$ vertex guards are necessary to cover it.

It seems to us, with some certainty, that the Spiral-Thin grid n-ogons, illustrated in fig. 6(a), require $\lfloor \frac{r}{2} \rfloor + 1$ vertex guards.



(a) (b)

Fig. 8. (a) The only convex vertices that could yield, by Inflate-Paste, Thin grid 14-ogons are v_3, v_4, v_{11} and v_{12} (in CCW order); (b) The only convex vertices, from the first Spiral-Thin, that could yield the second are v_3, v_4, v_9 and v_{10}.

5 Conclusions and Further Work

We defined a particular type of polygons - grid n-ogons - and presented some results and problems related to them. Of these problems, the guarding problems are the ones that motivate us more. We proved that $\lceil \frac{r+2}{3} \rceil$, i.e., $\lceil \frac{n}{6} \rceil$ vertex guards are necessary to guard any Min-Area grid n-ogon with r reflex vertices. Moreover, we showed where these vertex guards could be placed. We are investigating now how the ideas of this work may be further exploited to obtain better approximate solutions to MVG problem. The next step is to characterize structural properties of classes of Thins with the aim of simplifying our next objective: study MVG problem for Thin grid n-ogons.

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