Vertex Guards in a Subclass of Orthogonal Polygons

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Summary
We call *grid n-ogon* each *n*-vertex orthogonal simple polygon, with no collinear edges, that may be placed in a $(n/2)\times(n/2)$ unit square grid. In this paper we consider the Minimum Vertex Guard problem for this class of orthogonal polygons. As a step for the resolution of this general problem, we are going to study it for an interesting subclass of grid *n*-ogons: the SPIRAL grid *n*-ogons, which are the grid *n*-ogons whose boundary can be divided into a reflex chain and a convex chain.

Key words:
Computational Geometry, Art Gallery Problem and Theorem, Orthogonal polygons, Spiral polygons.

1. Introduction

Visibility problems have been studied extensively in the Computational Geometry literature, and the so-called *Art Gallery Problems* ([10,14]) form an important subcategory within this field. Legend has it that during a conference in 1973, Victor Klee started the study by posing the following problem, which today is known as the original art gallery problem: How many guards are needed to see every point in the interior of an art gallery? In the abstract version of this problem, the input is a simple polygon in the plane, representing the floor plan of the art gallery, and the visibility is, of course, limited to the interior of the polygon. In 1975, Chvátal [2] proved that $\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient to cover a simple polygon of *n* vertices. Two natural outgrowths of the proof of the *Chvátal Art Gallery Theorem* were to see how the ideas in the proof could be extended to get more general results, and to see how the number of guards needed might change when working with polygons with special characteristics. Orthogonal polygons, that is, simple polygons whose edges are either horizontal or vertical, are an important subclass of polygons. Interesting results on this class of simple polygons include the *Orthogonal Art Gallery Theorem*, proved by Kahn et al [4], in 1983. It states that $\lfloor n/4 \rfloor$ guards are occasionally necessary and always sufficient to cover an orthogonal polygon of *n* vertices (*n-ogon*, for short). Efficient algorithms were developed to cover both arbitrary and orthogonal simple polygons with $\lfloor n/3 \rfloor$ and $\lfloor n/4 \rfloor$ guards, respectively. In contrast, the MINIMUM VERTEX GUARD (MVG) problem, that is the problem of finding the minimum number of guards placed on vertices (*vertex guards*) needed to cover a given simple polygon, is much harder. This is a NP-hard problem both for arbitrary and orthogonal simple polygons [5,12]. Spiral polygons (simple polygons whose boundary can be divided into a reflex chain and a convex chain) are a subclass of polygons that have been usefully distinguished in the literature. These polygons can be recognized in linear time and they have arisen in "practice". For instance, Feng and Pavlidis studied decomposition of polygons into spiral pieces for its application to character recognition [3,11]. Besides, spiral polygons form the first level of a hierarchy that contains all simple polygons, the so called *k*-spiral polygons; that are the polygons having *k* reflex chains.

Our contribution: In this paper we address the MVG problem for an interesting subclass of orthogonal polygons, the SPIRAL *grid n-ogons*. The paper is structured as follows: in the next section we will define the grid *n*-ogons, state useful results related to them and formalize the problem. In section 3 we will define and present some properties of the SPIRAL grid *n*-ogons, a subclass of THIN grid *n*-ogons, and in section 4 we will study the MVG problem for it. Finally, in section 5 we will draw conclusion and further work.

2. Preliminaries

For convenience, we will assume that the vertices of a polygon *P* are given in counterclockwise (CCW) order. A vertex of a polygon *P* is called *convex* if the interior angle between its two incident edges is at most π, otherwise is called *reflex*. We use *r* to represent the number of reflex vertices of *P*. It has been shown by O'Rourke (see [10]) that $n = 2r + 4$, for every *n*-ogon. A *rectilinear cut* (*r-cut*) of an *n*-ogon *P* is obtained by extending each edge incident to a reflex vertex of *P* towards the interior of *P* until it hits *P*'s boundary. By drawing all *r*-cuts, we partition *P* into rectangles (called *r*-

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pieces). This partition is denoted by $\prod(P)$ and the number of its pieces by $|\prod(P)|$. A *n*-ogon that may be placed in a square grid and that does not have collinear edges is called *grid n-ogon*. We assume that the grid is defined by the horizontal lines $y=1,\ldots,y=n/2$ and the vertical lines $x=1,\ldots,x=n/2$ and that its northwest corner is $(1,1)$. A correct and complete method to generate grid *n*-ogons, well described in [13] and briefly explained here, is the INFLATE-PASTE. Let $v_i=(x_i,y_i)$, for $i=1,\ldots,n$ be the vertices of a grid *n*-ogon *P*. INFLATE takes *P* and a pair of integers (p,q) with $p,q\in[0,n/2]$, and yields a new *n*-ogon $\tilde{P}$ with vertices $\tilde{v}_i=(\tilde{x}_i,\tilde{y}_i)$ given by $\tilde{x}_i=x_i$ if $x_i\le p$ and $\tilde{x}_i=x_i+1$ if $x_i>p$, and $\tilde{y}_i=y_i$ if $y_i\le q$ and $\tilde{y}_i=y_i+1$ if $y_i>q$, for $i=0,\ldots,n$. To transform *P* by INFLATE-PASTE, we first imagine it merged in a $(n/2)\times(n/2)$ square grid, with top, bottom, leftmost, and rightmost grid free lines. The northwest corner of this extended grid is the point $(0,0)$. Let $e_H(v_i)$ be the horizontal edge of *P* to which v_i belongs.

Definition 1. *Given a grid n-ogon merged into a* $(n/2)\times(n/2)$ *square grid and a convex vertex* v_i *of P, the free staircase neighborhood of* v_i *, denoted by* FSN(v_i)*, is the largest staircase polygon in this grid that has* v_i *as vertex, that does not intersect the interior of P and whose base contains* $e_H(v_i)$ *(see fig.* 1).

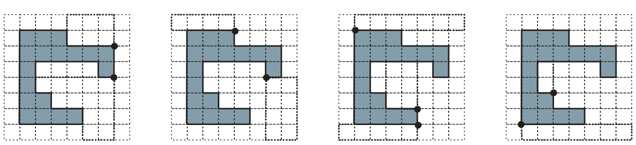

Fig. 1 A grid *n*-ogon merged into a square grid and the free staircase neighborhood for each of its convex vertices.

Now, we first take a convex vertex v_i of *P*, select a cell *C* in FSN(v_i), with center *c* and northwest corner , and apply INFLATE to *P* using (p,q). Its center is mapped to $\tilde{c}=(p+1,q+1)$, which will now be a convex vertex of the new polygon. PASTE glues the rectangle defined by $\tilde{v}_i$ and $\tilde{c}$ to $\tilde{P}$, increasing the number of vertices by two. Fig. 2 illustrates this transformation.

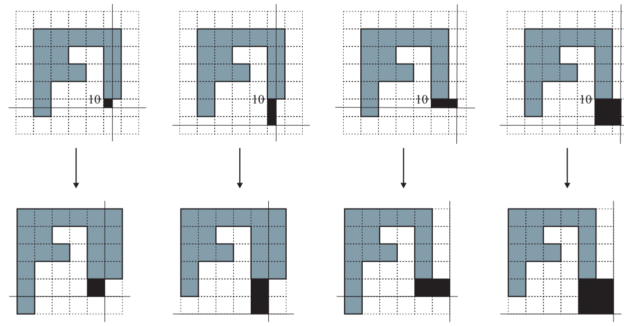

Fig. 2 The four grid 14-ogon that we may construct if we apply INFLATE-PASTE to the given 12-ogon, to extend the vertical edge that ends in vertex 10.

Grid *n*-ogons that are symmetrically equivalent are grouped in the same class [1]. A grid *n*-ogon *Q* is called FAT iff $\prod(Q)\ge\prod(P)$, for all grid *n*-ogons *P*; and is called THIN iff $\prod(Q)\le\prod(P)$, for all grid *n*-ogons *P*. Let *P* be a grid *n*-ogon with *r* reflex vertices. In [1] it is proved that, if *P* is FAT then $\prod(P)=\dfrac{3r^2+6r+4}{4}$, for *r* even and $\prod(P)=\dfrac{3(r+1)^2}{4}$, for *r* odd; if *P* is THIN then $\prod(P)=2r+1$. There is a single FAT grid *n*-ogon; however, THIN grid *n*-ogons are not unique (see fig. 3 (a)). We already proved that, if *P* is a THIN grid *n*-ogon, $n\ge6$, then every grid (*n*-2)-ogon that yields it, by INFLATE-PASTE, is also THIN. Moreover, denoting by DG($\prod(P)$) the dual graph of $\prod(P)$, we showed that DG($\prod(P)$) is a *path graph*, i.e., a tree with two nodes of vertex degree 1, called leaves, and the other nodes of vertex degree 2 (see fig.3(b)). We also showed that, the unique convex vertices of a THIN grid (*n*-2)-ogon that could be used to yield a THIN grid *n*-ogon are those which belong to the *r*-pieces associated to the leaves of DG($\prod(P)$), which are in number of 4 (see fig. 3(c)).

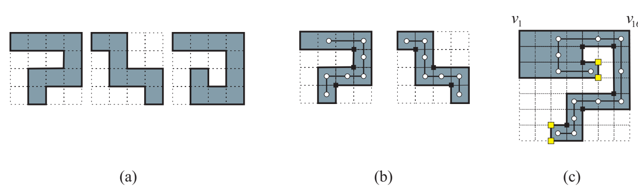

Fig. 3 (a) Three different THIN grid 10-ogons; (b) Two THIN grid 10-ogons and respective dual graphs; (c) The only convex vertices that could yield THIN grid 14-ogons are v_3,v_4,v_{11} and v_{12}.

The area of a grid *n*-ogon *P*, $A(P)$, is the number of grid cells in its interior. In [1] it is proved that for all grid *n*-ogon *P*, with $n\ge8$, $2r+1\le A(P)\le r^2+3$. A grid *n*-

ogon P is a MAX-AREA grid n-ogon iff $A(P) = r^2 + 3$ and it is a MIN-AREA grid n-ogon iff $A(P) = 2r + 1$. There are MAX-AREA grid n-ogons for all n, but they are not unique. However, there is a single MIN-AREA grid n-ogon and its form is illustrated in fig. 4. Regarding Min-Area grid n-ogons, it is obvious that they are THIN grid n-ogons, because $\prod(P) = 2r + 1$ holds only for THIN grid n-ogons. However, this condition is not sufficient for a THIN grid n-ogon to be a MIN-AREA grid n-ogon.

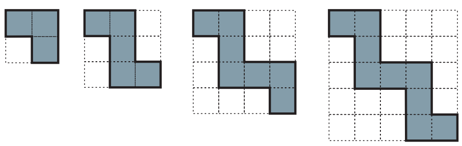

Fig. 4 The Min-Area for $n = 6,8,10,12$

Our main goal is to study the MVG problem for grid n-ogons . Since THIN and FAT grid n-ogons are the classes for which the number of r-pieces is minimum and maximum, we think that they can be representative of extreme behavior, so we started with them. For FAT grid n-ogons we already solved the problem [7]. Unfortunately, THIN grid n-ogons are not so easy to cover, in spite of their having much fewer r-pieces than FAT grid n-ogons. Besides, they are not unique and it seems that the number of THIN grid n-ogons grows exponentially with n. Thus we are trying to identify subclasses of THIN grid n-ogons with the aim of simplifying the problem's study. In this section, we already characterized the MIN-AREA grid n-ogon subclass. In [8], we proved that $\lceil n/6 \rceil$ vertex guards are always necessary to cover a MIN-AREA grid n-ogon, and we also established a possible positioning for those guards. In this paper we will characterize another subclass: the SPIRAL grid n-ogons and study the MVG problem for them. From the k-spiral viewpoint, given a THIN grid n-ogon with r reflex vertices, it can have at least 1 reflex chain and at most r reflex chains. We already know that there is a THIN grid n-ogon with r reflex chains (each consisting of one reflex vertex), the MIN-AREA grid n-ogon, and for this the MVG problem is already solved. Now we will show that there are THIN grid n-ogons with 1 reflex chain, which we will call SPIRAL grid n-ogons, and we will study the problem concerning them.

3. SPIRAL grid *n*-ogons

In this section, we will define the SPIRAL grid n-ogon subclass, we will prove that for all $n \geq 6$ there is, at least, a SPIRAL grid n-ogon, and, finally, we will show that a SPIRAL grid n-ogon is a THIN grid n-ogon.

Definition 2. *A grid n-ogon is called* SPIRAL *grid n-ogon if its boundary can be divided into a reflex chain and a convex chain.*

A polygonal chain is called *reflex* if its vertices are all reflex (all except the vertices at the end of the chain) with respect to the interior of the polygon; and is called *convex* if its vertices are all convex with respect to the interior of the polygon. Note that, a SPIRAL grid n-ogon P can be expressed as an ordered sequence of vertices $u_1, u_2, \ldots, u_r, c_1, c_2, \ldots, c_{n-r}$ where the u_i's are reflex and the c_i's are convex. Thus, the reflex chain is the polygonal chain $c_{n-r}, u_1, \ldots, u_r, c_1$ and the convex chain is the polygonal chain $c_1, c_2, \ldots, c_{n-r}$. We will denote by $e_0, e_1, \ldots, e_{r-1}, e_r$ the edges of the reflex chain, where: $e_0 \equiv \overline{c_{n-r} u_1}$; $e_i \equiv \overline{u_i u_{i+1}}$, $1 \leq i \leq r-1$; and $e_r \equiv \overline{u_r c_1}$.

Proposition 1. *There is, at least, a* SPIRAL *grid n-ogon with* $r = \dfrac{n-4}{3}$ *reflex vertices, for all* $r \geq 1$.

This proposition establishes that there are SPIRAL grid n-ogons, for all $n \geq 6$; however, they are not unique, as we may see in fig. 5. Now we will prove that a SPIRAL grid n-ogon is a THIN grid n-ogon. To show this result, we will first establish lemma 1.

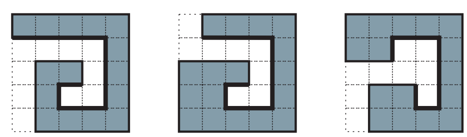

Fig. 5 Three different SPIRAL grid n-ogons with $n = 12$ (reflex chain in bold)

Lemma 1. *Only* SPIRAL *grid n-ogons can yield, by* INFLATE-PASTE, SPIRAL grid (n+2)-*ogons*.

Proof. Let P be a grid n-ogon and $v_1, v_2, \ldots, v_n$ its vertices. Take a convex vertex $v_i = (x_i, y_i)$ of P and apply INFLATE-PASTE, this would yield a grid (n+2)-ogon Q. Suppose that $e_H(v_i) \equiv \overline{v_i v_{i+1}}$, there are two possibilities for v_{i+1}: it can be a reflex or a convex vertex. If v_{i+1} is a reflex vertex, then the form of the rectangle glued by PASTE to yield Q is illustrated in fig. 6 (*Case 1*); otherwise we will have one of the two forms illustrated in fig. 6 (*Case 2.1* and *Case 2.2*).

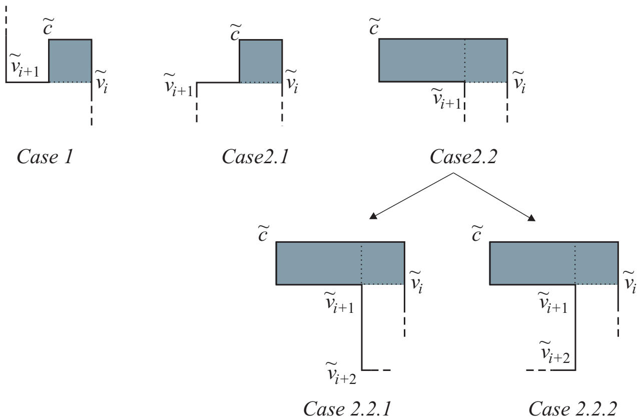

Fig. 6 Rectangles that might be glued by PASTE to yield Q

In Case 1, it's easy to verify that Q is a SPIRAL grid (n+2)-ogon only if P is a SPIRAL grid n-ogon. In Case 2.1, Q is never a SPIRAL grid (n+2)-ogon, independently of P being a SPIRAL grid n-ogon or not, since a reflex vertex is inserted between two convex ones. In Case 2.2, to be able to draw conclusions about Q, we have to split in two cases: Case 2.2.1, when v_{i+2} is convex, and Case 2.2.1, when v_{i+2} is reflex (see fig 6). Such as in Case 2.1, in Case 2.2.1, Q is never a SPIRAL grid (n+2)-ogon. In Case 2.2.2, it's easy to verify that Q is a SPIRAL grid (n+2)-ogon only if P is a SPIRAL grid n-ogon. Thus, if $e_H(v_i) \equiv \overline{v_i v_{i+1}}$ and P is not a SPIRAL grid n-ogon then Q is never a SPIRAL grid (n+2)-ogon in any case. If $e_H(v_i) \equiv \overline{v_i v_{i+1}}$ and P is a SPIRAL grid n-ogon then Q is a SPIRAL grid (n+2)-ogon in cases 1 and 2.2.2 and it is not a SPIRAL (n+2)-ogon in cases 2.1 and 2.2.1.

Suppose, now, that $e_H(v_i) \equiv \overline{v_{i-1} v_i}$, in analogous way, we can prove that Q is a SPIRAL grid (n+2)-ogon only if P is a SPIRAL grid n-ogon and:

i) v_{i-1} is a reflex vertex and we select any cell C in FSN(v_i) (see fig. 7(a)); or

ii) $v_{i-1} = (x_{i-1}, y_{i-1})$ is convex, v_{i-2} is reflex and we select a cell C in FSN(v_i) such that its center $c = (c_x, c_y)$ verifies $|c_x - x_i| > |x_{i-1} - x_i|$ (see fig. 7(b)).

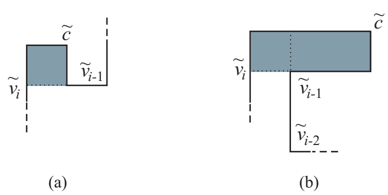

Fig. 7 Rectangles glued to P

Consequently, only SPIRAL grid n-ogons can yield SPIRALS grid (n+2)-ogons, by INFLATE-PASTE.

q.e.d.

Proposition 2. *Every* SPIRAL *grid n-ogon, with* $r \geq 1$ *reflex vertices, is a* THIN *grid n-ogon.*

Proof. The proposition is true for $r = 1$, because there is only one grid n-ogon with $r = 1$ and it is SPIRAL and THIN. Let $r \geq 1$, we will prove that the proposition is true for $r+1$, assuming that it is true for r. Let Q be a SPIRAL grid n-ogon with $r+1$ reflex vertices. We already know, by lemma 1, that Q can have been generated only from one SPIRAL grid n-ogon P with r reflex vertices. Moreover, the convex vertex v_i taken to yield Q has to be such that:

- if $e_H(v_i) \equiv \overline{v_i v_{i+1}}$, then: a) v_{i+1} is reflex or b) v_{i+1} is convex and v_{i+2} is reflex;
- if $e_H(v_i) \equiv \overline{v_{i-1} v_i}$, then c) v_{i-1} is reflex or d) v_{i-1} is convex and v_{i-2} is reflex.

Since P is a SPIRAL it comes: in Case a) $v_i = c_{n-r}$, in Case b) $v_i = c_{n-r-1}$, in Case c) $v_i = c_1$ and in Case d) $v_i = c_2$. Furthermore, by induction hypothesis, P is a THIN grid n-ogon then DG($\prod(P)$) is a path graph, so it has two leaves. Each leaf has three adjacent vertices of P: one reflex vertex preceded or followed by two convex vertices. Thus, we can conclude that u_r, c_1, c_2 and c_{n-r-1}, c_{n-r}, u_1 belong to the leaves, since they are the only vertices of P in the above stated condition. Therefore, the four cases a), b), c), and d) are illustrated in fig. 8.

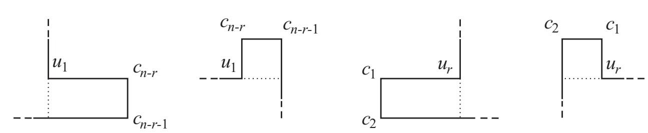

Fig. 8 From left to right: Case a), Case b), Case c) and Case d).

In lemma 1 we also proved that the rectangle glued to P, by PASTE, to yield Q is of the four forms illustrated in fig. 9.

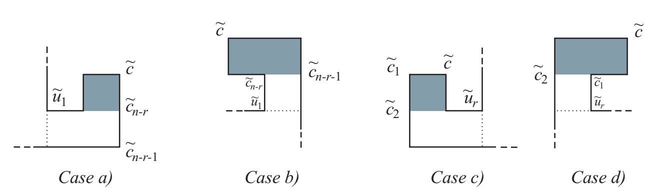

Fig. 9 Rectangles glued, by PASTE, to yield Q.

In any case, P is THIN grid n-ogon then $|\prod(P)| = 2r+1$. And we can easily check that only two r-pieces are added to yield Q, thus $|\prod(Q)| = 2(r+1)+1$. Therefore, Q is a THIN grid n-ogon.

q.e.d.

4. Guarding SPIRAL grid *n*-ogons

Nilsson and Wood proved that *a collection of guards (mobile or stationary) see a spiral polygon iff they see all edges of the reflex chain* [9]. It is easy to prove that this result remains true if guards are replaced by vertex guards in spiral n-ogons. Then we have the following lemma:

Lemma 2. *A collection of vertex guards covers a spiral n-ogon iff they see all the edges of the reflex chain.*

Let P be a spiral polygon having n vertices, k of which are reflex, and having its vertices labeled according to our previously described conventions for SPIRAL grid n-ogons. Nilsson and Wood also established that for a guard to be able to see an edge of the reflex chain e_i, $i \in \{0,\dots,k\}$, it has to be placed in a particular convex region, CR_i, defined in the following way:

1. if $i=0$ ($i=k$), e_i is extended through u_1 (u_k) until it intersects the convex chain. In this case, CR_i is the region bounded by $\overline{c_{n-r}x_1}$ ($\overline{x_k u_k}$), x_1 (x_k) is the intersection point with the convex chain, and the subchain of the boundary of P from x_1 (u_k) to c_{n-r} (x_k) in CCW order (see fig. 10(a)).

2. if $i \neq 0,k$, e_i is extended through u_i and u_{i+1} until it intersects the convex chain. In this case, CR_i is the region bounded by $\overline{x'_i x_i}$, x'_i and x_i are the intersection points with the convex chain, and the subchain of the boundary of P from x_i to x'_i in CCW order (see fig. 10(b)).

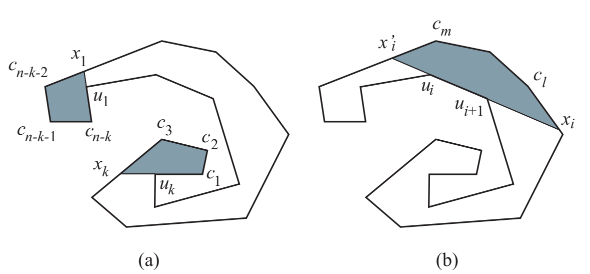

Fig. 10 (a) CR_0, CR_k; (b) CR_i, with $i \neq 0,k$.

They also provided an algorithm to find the minimum number of stationary guards necessary to guard a spiral polygon. Their algorithm computes an optimum guard cover in a spiral polygon, however it does not give an explicit number of guards and it deals with guards and not vertex guards, which is a different problem. Based on their algorithm, particularizing for spiral n-ogon and adapting for vertex guards, we will prove that $\lfloor r/2 \rfloor + 1$ vertex guards are necessary to cover any spiral n-ogon with r reflex vertices.

Let P be a spiral n-ogon with r reflex vertices, we want to determine the minimum number of vertex guards that is needed to guard P. By lemma 2, it is only necessary to consider the visibility of the edges of the reflex chain. Moreover, being e_i an edge of the reflex chain we already know that for a guard to be able to see e_i it has to be placed in CR_i, as we are dealing with vertex guards, we can conclude that for a vertex guard to be able to see e_i it has to be placed in a vertex of P that belongs to CR_i. In the case of spiral n-ogons, these convex regions have a particular shape, they are rectangles [6] (see fig. 11).

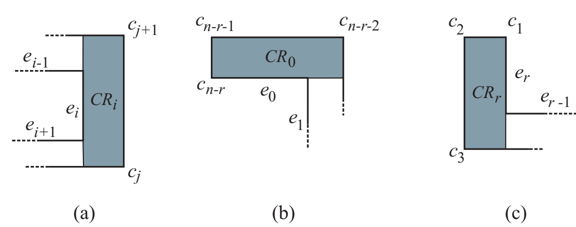

Fig. 11 (a) CR_i, $i \in \{1,\dots,r-1\}$; (b) CR_0;.(c) CR_r

Lemma 3. *Let P be a spiral n-ogon with r reflex vertices. A vertex guard that sees the edge* e_i, *with* $0<i<r$, *can also see* e_{i-1} *or* e_{i+1}, *but not both.*

Proof. For a vertex guard to be able to see $e_i \equiv \overline{u_i u_{i+1}}$ ($0<i<r$) it has to be placed in a vertex of P that belongs to CR_i. As we saw before, being P a spiral n-ogon, the only vertices of P that belong to CR_i are: u_i, u_{i+1}, c_j or c_{j+i} (see fig. 11(a)). Thus the guard has to be placed in one of those vertices. If we choose u_i or c_{j+1}, the vertex guard also sees e_{i-1}, but it does not see e_{i+1}. If we select u_{i+1} or c_j, he also sees e_{i+1}, but it does not see e_{i-1}.

q.e.d.

In the previous lemma we proved that a vertex guard that sees an edge of the reflex chain, different from the first one and from the last one, only manages to see one of its adjacent edges. Let's see what happens with a vertex guard that sees the first or the last edge of the reflex chain:

i) for a vertex guard to be able to see e_0 it has to be placed in c_{n-r-2}, c_{n-r-1}, c_{n-r} or u_1 (see fig. 11(b)). From these positions we can choose one that also sees e_1, which is c_{n-r-2} or u_1;

ii) for a vertex guard to be able to see e_r it has to be placed in u_r, c_1, c_2 or c_3 (see fig. 11(c)). From these positions we can choose one that also sees e_{r-1}, which is c_3 or u_r.

Therefore, from i), ii) and lemma 3 we can conclude that a vertex guard sees at most two edges of the reflex chain.

Proposition 3. $\lfloor r/2 \rfloor + 1$ *vertex guards are necessary to cover any spiral n-ogon with r reflex vertices.*

Proof. Let *P* be a spiral *n*-ogon with *r* reflex vertices, its reflex chain has $r+1$ edges: $e_0, e_1, \ldots, e_r$. Two cases can happen: *r* is odd or *r* is even. If *r* is odd, place guards at the following vertices: u_{1+2k}, with $k = 0,1,\ldots,\frac{r-1}{2}$ (see fig. 12).

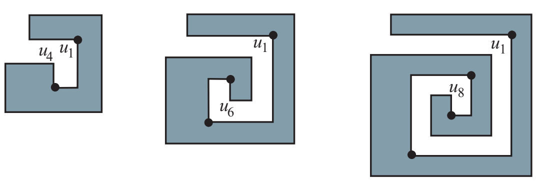

Fig. 12 Spiral *n*-ogons with r odd.

These guards see all the edges of the reflex chain. In fact, u_{1+2k} is the reflex vertex common to edges e_{2k} and e_{1+2k}, thus these edges are seen by the vertex guard placed at u_{1+2k}. Consequently, these guards cover *P* since they see all the edges of the reflex chain, and by lemma 2 this is enough. Thus, $\frac{r+1}{2}$ vertex guards cover *P*. Suppose, now, that there is a set of vertex guards *S*, with $|S| \leq \frac{r+1}{2} - 1$, that cover P. We know that each vertex guard sees at most 2 edges of the reflex chain, thus at most $2 \times |S| \leq r-1$ edges are seen by those vertex guards. As the reflex chain has $r+1$ edges, at least two edges of the reflex chain are not seen, as a consequence *P* is not covered by the vertex guards in *S*.

If r is even, place guards at the following vertices: u_{1+2k}, with $k = 0,1,\ldots,\frac{r}{2}-1$, and c_1 (see fig. 13).

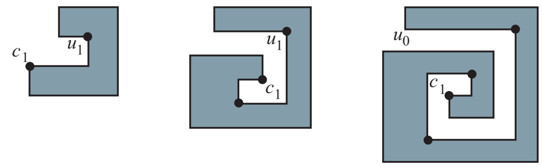

Fig. 13 Spiral *n*-ogons with r even.

These guards see all the edges of the reflex chain. In fact, as in the previous case, the edges e_{2k} and e_{1+2k}, with $k = 0,1,\ldots,\frac{r}{2}-1$ are seen by the guard placed at u_{1+2k}. And the edge e_r is seen by the guard placed at c_1, since c_1 is an endpoint of e_r. Therefore, as in the previous case, these guards cover *P*. Thus, $\frac{r}{2}+1$ vertex guards cover *P*. Suppose, now, that there is a set of vertex guards *S*, with $|S| \leq \frac{r}{2}$, that cover *P*. We know that each vertex guard sees at most 2 edges of the reflex chain, thus at most $2 \times |S| \leq r$ edges are seen by those vertex guards. As the reflex chain has $r+1$ edges, at least one edge of the reflex chain is not seen, as a consequence *P* is not covered by the vertex guards in *S*. In any case, *r* odd or *r* even, $\lfloor r/2 \rfloor + 1$ vertex guards are necessary to cover *P*.

q.e.d.

Corollary 1. $\lfloor n/4 \rfloor$ *vertex guards are necessary to cover any SPIRAL grid n-ogon.*

This corollary is a consequence of the previous proposition, since a SPIRAL grid *n*-ogon is a spiral *n*-ogon and $\lfloor n/4 \rfloor = \lfloor r/2 \rfloor + 1$. As we already know, by the Orthogonal Art Gallery Theorem, that $\lfloor n/4 \rfloor$ vertex guards are sufficient to cover any grid *n*-ogon we can conclude that SPIRAL grid *n*-ogons gives us the worst scenario within the THIN grid *n*-ogons.

5. Conclusions and Further Work

We defined a particular type of polygons - grid *n*-ogons - and presented some results related to them. Of the problems related to the grid *n*-ogons, the MVG is the one that motivates us more. We proved that $\lfloor n/4 \rfloor$ vertex guards are necessary to guard any SPIRAL grid *n*-ogon with *r* reflex vertices. Moreover, we established a possible positioning for those guards. We are investigating now how the ideas of this work may be further exploited to obtain better approximate solutions to MVG problem for orthogonal polygons merged in a grid. The next step will be to identify more subclasses of THIN grid *n*-ogons with the aim of simplifying our next objective: to study the MVG problem for the THIN grid *n*-ogons.

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