# Good Illumination with Limited Visibility

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**Key words** Computational Geometry, Art Gallery Problems, Illumination and Visibility Problems, Limited Range Visibility.

Most illumination problems are solved considering light sources with unlimited illumination range. In this paper, we consider light sources having a limited illumination range $d$. We associate this restriction with good illumination that was introduced by Canales et al. [1, 4]. We consider two related optimization problems. Given $n$ light sources in the plane, the first one computes the light sources' minimum range so that a given point $p$ is 1-well illuminated. The second problem considers a line segment instead of a point and computes the minimum illumination range to 1-well illuminate all the points in the line segment. We give a $\mathcal{O}(n\log n)$ time algorithm for the first problem and an $\mathcal{O}(n^3\log n)$ worst case time algorithm for the second.



## 1 Introduction, Related Works and Problem Definition

There are lots of different works related to visibility or illumination [3, 7, 8, 10] but most of them cannot be applied to real life, since they deal with ideal concepts. We present some of these illumination problems adding some restrictions to make them more realistic. In fact, cameras or robot vision systems, both have severe visibility range restrictions since they cannot observe with sufficient detail far away objects. We will use a visibility definition due to Ntafos [6]. Two points $x$ and $y$ in the plane are called *d-visible* if they are visible to each other within a range $d$. Let $C(x, d)$ denote the circle of radius $d$ centered at $x$. If there are no obstacles in the plane, the $d$-visible area of a point $x$ is $C(x, d)$.

Consider a set of $n$ fixed light sources in the plane. We assume that there are no obstacles. We will also assume that each light source $f_i$ has a limited illumination range $d_i$. Following this, a point $p$ is $d_i$-illuminated by light source $f_i$ if $d(f_i, p) \leq d_i$ where $d(f_i, p)$ is the euclidean distance between $f_i$ and $p$. Let $F = \{(f_1, d_1), (f_2, d_2), \ldots, (f_n, d_n)\}$ denote the set of $n$ light sources and their respective illumination ranges.

Recently, Canales et al. [1, 4] introduced the concept of *t-good illumination* and presented some related algorithms. By definition, a point $p$ is $t$-well illuminated by $F$ if there are, at least, $t$ light sources in every half-plane with $p$ in its boundary that illuminates $p$, $1 \leq t \leq \frac{n}{2}$. We will restrict our study to *1-good illumination*. Let $T(f_a, f_b, f_c)$ denote the triangle formed by the light sources $f_a, f_b$ and $f_c$. Note that this definition understands $T(f_a, f_b, f_c)$ as a closed region of the plane and its boundary will be denoted as $\partial T(f_a, f_b, f_c)$. A point is 1-well illuminated when it lies in the triangle formed by three light sources [1, 4]. 1-good illumination can also be found under the concept of $\triangle$-guarding [9] or well-covering [5].

Consider three light sources $f_a, f_b$ and $f_c$. If $C(f_i, d_i)$ is the circle of radius $d_i$ centered at $f_i$, then the light source $f_i$ only $d_i$-illuminates points in $C(f_i, d_i), i \in \{a, b, c\}$. Let $A_d(f_a, f_b, f_c)$ denote the $d$-illuminated area by the light sources $f_a, f_b$ and $f_c$ (see Figure 1(a)). It is easy to see that $A_d(f_a, f_b, f_c) = C(f_a, d_a) \cap C(f_b, d_b) \cap C(f_c, d_c)$.

**Definition 1.1** We say that a point $p$ is 1-well $d$-illuminated by the light sources $f_a, f_b$ and $f_c$ if $p \in A^1_d(f_a, f_b, f_c)$. $A^1_d(f_a, f_b, f_c) = A_d(f_a, f_b, f_c) \cap T(f_a, f_b, f_c)$ denotes the 1-well $d$-illuminated area by $f_a, f_b$ and $f_c$.

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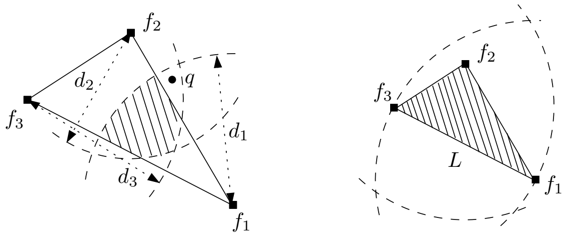
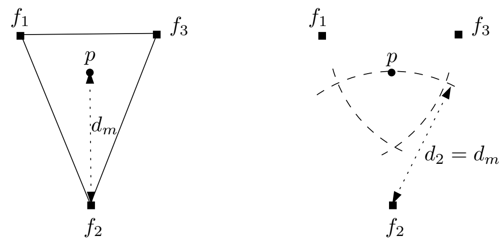
We can see an example of this definition in Figure 1. Notice that point $q$ is not 1-well $d$-illuminated, despite the fact $q \in A_d(f_1, f_2, f_3)$.




(a) (b)

Fig. 1: (a) $A_d^1(f_1, f_2, f_3)$ is the shaded area, so every point that lies in it is 1-well $d$-illuminated by light sources $f_1, f_2$ and $f_3$. (b) All points in $T(f_1, f_2, f_3)$ are 1-well illuminated since $d_i \geq L$, $i = 1, 2, 3$.




(a) (b)

Fig. 2: (a) MIR to 1-well illuminate $p$ is $d_m = d(f_2, p)$. (b) $T(f_1, f_2, f_3) \in M_T(p)$ and $d_i = d_m$, $i = 1, 2, 3$.

This problem is interesting and has real applications such as the wireless sensor networking [5]. As we only consider $d$-illumination, throughout this paper we will refer to it just as illumination. In the next sections we will develop the main problem in this subject: *finding the minimum illumination range (MIR) to 1-well illuminate an object.*

*Our contribution*. This paper is solely focused on limited 1-good illumination and it is organized as follows. In section 2 we propose the algorithm *MIR-Point* to compute the minimum illumination range (MIR) needed to 1-well illuminate a point in the plane. In section 3 we propose the algorithm *MIR-Segment* to calculate the MIR to 1-well illuminate a line segment. All the proofs are omitted due to the lack of space.

## 2 Minimum Illumination Range (MIR) to 1-well Illuminate a Point

Let $F$ be a set of $n$ light sources in the plane and their respective limited illumination ranges. Let $d_m$ denote the MIR to 1-well illuminate a point $p$. Our main goal is to find its value. By definition, if $\exists (f_a, d_a), (f_b, d_b), (f_c, d_c) \in F : d_i \geq d_m, \forall i \in \{a, b, c\}$ and $p \in T(f_a, f_b, f_c)$ then $p$ is 1-well illuminated by $f_a, f_b$ and $f_c$.

Let $p$ be a point in the plane which we want to 1-well illuminate. It is easy to see that we can only 1-well illuminate $p$ if it is in the convex hull of the light sources in $F$, $CH(F)$. Consider any point $p \in T(f_a, f_b, f_c)$. Suppose the light source $f_b$ is such that $d(f_b, p) \geq d(f_i, p), i \in \{a, c\}$. In order to 1-well illuminate $p$, we need to calculate MIR $d_m$ =max$\{d(f_i, p), i \in \{a, b, c\}\}$. In this example, MIR $d_m = d(f_b, p)$ (see Figure 2(a)). This way, we are always sure that $p$ is 1-well illuminated if $d_i \geq d_m, i \in \{a, b, c\}$ (see Figure 2(b)).

Let $M_T(p)$ denote the set of triangles formed by three light sources that 1-well illuminate $p$ and let their MIR be $d_m$. Each triangle in $M_T(p)$ is called a MIR triangle. Note that we can have several solutions to this problem (several MIR triangles), this is, we can find more than one set of three lights whose MIR is $d_m$, but we are only interested in finding one solution. An efficient algorithm to 1-well illuminate a point considers the next proposition.

**Proposition 2.1** *Let $f_p$ be the nearest light source to point $p$. If $M_T(p) \neq \emptyset$ then the light source $f_p$ is a vertex of, at least, one MIR triangle.*

### 2.1 Algorithm to 1-well illuminate a point

We now present an algorithm to find a MIR triangle that 1-well illuminates $p$. Let $F = \{(f_1, d_1), (f_2, d_2), \ldots, (f_n, d_n)\}, n \geq 3$, be the set of $n$ light sources in the plane and their respective illumination ranges. The idea behind this algorithm is to divide the plane in two regions (A and B) and find the two light sources missing to form a MIR triangle, since $f_p$ is already one of its vertices (Proposition 2.1).

**Algorithm** *MIR-Point*

**Input**: *A set $F$ of $n$ light sources in the plane and their respective illumination ranges. Point $p$ which we want to 1-well illuminate.*

**Output**: *$MT$, a MIR triangle formed by three light sources that 1-well illuminates point $p$ and its MIR is $d_m$.*

1. **If** $p \notin CH(F)$ **then** $p$ cannot be 1-well illuminated.
2. Let $f_p \in F$ be the nearest light source to point $p$. Draw the straight line $l$ joining points $p$ and $f_p$. This procedure divides the plane in two regions ($A$ and $B$).
3. Sort all the lights sources in $A$ around $p$ in the CCW direction. Repeat this procedure for the lights sources in $B$.
4. $flag(a_1) \leftarrow a_1$, $a_1$ is the first light source sorted in $A$.
   **For** $i \leftarrow 2$ **to** $|A|$ **do**
      **If** $d(a_i, p) < d(flag(a_{i-1}), p)$ **then** $flag(a_i) \leftarrow a_i$.
      **Else** $flag(a_i) \leftarrow flag(a_{i-1})$.
5. $f_t \leftarrow null$, $d_m \leftarrow -1$.
   **Repeat**
      Rotate $l$ centered at $p$ until it reaches reaches a light source $f$.
      **If** $f \in A$ **then** $f_t \leftarrow flag(f)$.
      **Else If** $f_t \neq null$ **then**
         $d_t \leftarrow max\{d(f,p), d(f_t,p)\}$
         **If** $d_t < d_m$ **or** $d_m = -1$ **then**
         $dm \leftarrow d_t$, $MT \leftarrow T(f_p, f_t, f)$.
   **Until** $f = b_{|B|}$, $b_{|B|}$ is the last light source sorted in $B$.

**Proposition 2.2** *Given a set $F = \{(f_1, d_1), (f_2, d_2), \ldots, (f_n, d_n)\}$, $n \geq 3$, of $n$ light sources in the plane, their respective illumination ranges and a point $p$ in the plane, the algorithm MIR-Point finds one MIR triangle that 1-well illuminates $p$ in $\mathcal{O}(nlogn)$ time.*

## 3 Minimum Illumination Range (MIR) to 1-well Illuminate a Line Segment

Let $d_m$ be the MIR to 1-well illuminate the line segment $\overline{p_l p_r}$ with the light sources $f_1, f_2, \ldots, f_n, n > 3$. Without loss of generality, suppose that $\overline{p_l p_r}$ is an horizontal line segment. Let $p_l$ be the leftmost point and $p_r$ the rightmost point. If we want to 1-well illuminate every point in the line segment with the same three light sources, all we have to do is find three of them that form a MIR triangle containing $\overline{p_l p_r}$. To determine $d_m$ we need to find the greatest distance between one point of the segment and one of the three light sources. This way, we find $d_m$ and the whole line segment $\overline{p_l p_r}$ is in the intersection of the three light sources' illuminated areas, this is, $\overline{p_l p_r}$ is 1-well illuminated (see Figure 3).

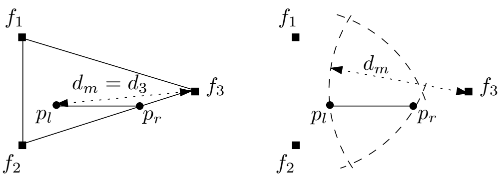



Fig. 3: (a) MIR to 1-well illuminate $p$ is $d_m = d(f_2, p)$. (b) $T(f_1, f_2, f_3) \in M_T(p)$ and $d_i = d_m$, $i = 1, 2, 3$.

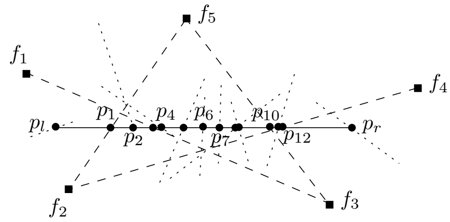



Fig. 4: $\overline{p_l p_r}$ is broken into thirteen segments: $s_0 = \overline{p_l p_1}$, $\ldots$, $s_{12} = \overline{p_{12} p_r}$. Dotted lines represent the light sources' perpendicular bisectors.

We can also break $\overline{p_l p_r}$ into several consecutive line segments $s_0 = \overline{p_l p_1}$, $s_1 = \overline{p_1 p_2}$, ..., $s_{t-1} = \overline{p_{t-1} p_r}$ and compute, for each one of them, the MIR triangle that 1-well illuminates them. This way we will compute MIR $d_j, 0 \leq j \leq t-1$ that each one of the three light sources needs to have to guarantee that $s_i$, $i = 0, \ldots, t-1$ is 1-well illuminated. In the end, it is clear that the MIR to 1-well illuminate the whole line segment is the maximum MIR $d_j$ already found for each part.

**Lemma 3.1** *If the MIR to 1-well illuminate the leftmost point $p_l \in \overline{p_l p_r}$ is $d_0$ then the MIR to 1-well illuminate the whole segment $\overline{p_l p_r}$ is $d_m \geq d_0$.*

If $\overline{p_l p_r} \nsubseteq CH(F)$ then our problem does not have a solution. As it was explained in section 2, we can find $MT \in M_T(p_0)$ with MIR $d_0 = d(f_k, p_l)$, where $f_k$ is the furthest vertex in $MT$ to point $p_l$. If $MT \notin M_T(p_r)$, we need to find intersection points $p_i$ that break $\overline{p_l p_r}$ into $t$ consecutive line segments. Let $F_1$ be the set of all the light sources above $\overline{p_l p_r}$ and $F_2 = F \backslash F_1$. Let $C$ be the set of the intersection points between the line segments connecting $F_1$ and $F_2$ and $\overline{p_l p_r}$. Now let $M$ be the set of all the intersections between $\overline{p_l p_r}$ and the light sources' perpendicular bisectors (see Figure 4). Let $I = C \cup M$ be the sorted union of all the intersection points according to their $x$-coordinate. The set $I$ breaks $\overline{p_l p_r}$ into $t$ consecutive segments: $s_i = \overline{p_i p_{i+1}}, i = 0, \ldots, t-1$ (we assume that $p_0 = p_l$ and $p_t = p_r$). As $p_0$ is already 1-well illuminated, we need to check the next intersection point $p_1$.

Suppose $p_1 \in C$ and $p_1 \in \partial MT$. First we check wether $d_0$ is sufficient to 1-well illuminate $p_1$ with $MT$ and update it if necessary. Then we need to find $MT^* \in M_T(p_1)$ and calculate the MIR $d_1$ as the greatest distance from $p_1$ to one of its three light sources. The next step is to check if the next intersection point is 1-well illuminated. Now suppose that $p_1 \in C$ but $p_1 \notin \partial MT$. If $MT \in M_T(p_1)$, all we have to do is actualize $d_0$ if needed and check the next intersection point. Else if $MT \notin M_T(p_1)$, we need to swap to another MIR triangle. This procedure is the same as the explained above for $p_1 \in \partial MT$. On the other hand, if $p_1 \in M$, this means that exist $f_u$ and $f_v \in F$ so that $p_1 = \overline{p_l p_r} \cap PerpendicularBisector(f_u, f_v)$. Let $f_k \in MT$ be the furthest light source to $p_1$. If $f_k \in \{f_u, f_v\}$ and $\{f_u, f_v\} \notin MT$, we might need to swap to another MIR Triangle. To check this out, we need to find $MT^* \in M_T(p_1)$ using the lights sources in $F \backslash \{f_k\}$. If $MT^*$ 1-well illuminates $p_1$ with the same MIR value then $MT^*$ replaces $MT$. Update the MIR $d_0$ if necessary and calculate the MIR $d_1$ as the greatest distance from $p_1$ to one of the light sources in $MT^*$.

If $p_1 = p_t$ then we are done with the algorithm because $\overline{p_l p_r}$ is 1-well illuminated. Otherwise, we continue applying this process to the next intersection points until we find a MIR triangle that 1-well illuminates $p_r$. When we finally have point $p_r$ 1-well illuminated, $\overline{p_l p_r}$ is partitioned in several line segments. We know which MIR triangles 1-well illuminate each part as well as the MIR to do so. This procedure is interesting because we can 1-well illuminate each part of the line segment without 1-well illuminating the whole line segment $\overline{p_l p_r}$. For example, when an artist is walking on a stage, we know which floodlights we need to turn on and what is the MIR needed depending on the exact position of the artist. This algorithm can also be extended to polygonal lines instead of line segments to simulate roads or itineraries.

**Proposition 3.2** *Given a set $F = \{(f_1, d_1), (f_2, d_2), \ldots, (f_n, d_n)\}$, $n \geq 3$, of $n$ light sources in the plane and their respective illumination ranges and a line segment $\overline{p_l p_r}$, the algorithm MIR-Segment finds a set of MIR triangles that 1-well illuminate the line segment $\overline{p_l p_r}$ in $\mathcal{O}(n^3 log n)$ time.*

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