Partitioning Orthogonal Polygons by Extension of All Edges Incident to Reflex Vertices: lower and upper bounds on the number of pieces*

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Abstract. Given an orthogonal polygon P, let $|\Pi(P)|$ be the number of rectangles that result when we partition P by extending the edges incident to reflex vertices towards $\mathrm{INT}(P)$. In [4] we have shown that $|\Pi(P)| \leq 1 + r + r^2$, where r is the number of reflex vertices of P. We shall now give sharper bounds both for $\max_P |\Pi(P)|$ and $\min_P |\Pi(P)|$. Moreover, we characterize the structure of orthogonal polygons in general position for which these new bounds are exact. We also present bounds on the area of grid n-ogons and characterize those having the largest and the smallest area.

1 Introduction

We shall call a *simple polygon* P a region of a plane enclosed by a finite collection of straight line segments forming a simple cycle. Non-adjacent segments do not intersect and two adjacent segments intersect only in their common endpoint. These intersection points are the *vertices* of P and the line segments are the *edges* of P. This paper deals only with simple polygons, so that we simply call just polygons, in the sequel. We will denote the interior of the polygon P by $\mathrm{INT}(P)$ and the boundary by $\mathrm{BND}(P)$. The boundary is thus considered part of the polygon; that is, $P = \mathrm{INT}(P) \cup \mathrm{BND}(P)$. A vertex is called *convex* if the interior angle between its two incident edges is at most π; otherwise it is called *reflex* (or *concave*). We use r to represent the number of reflex vertices of P. A polygon is called *orthogonal* (or *rectilinear*) iff its edges meet at right angles. O'Rourke [3] has shown that $n = 2r + 4$ for every n-vertex orthogonal polygon (n-*ogon*, for short). So, orthogonal polygons have an even number of vertices.

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Definition 1. *A rectilinear cut (r-cut) of an n-ogon P is obtained by extending each edge incident to a reflex vertex of P towards* $\mathrm{INT}(P)$ *until it hits* $\mathrm{BND}(P)$. *We denote this partition by $\Pi(P)$ and the number of its elements (pieces) by $|\Pi(P)|$. Each piece is a rectangle and so we call it a* r-piece.

In [4] we proposed an algorithm to solve the Minimum Vertex Guard problem for polygons, whose main idea is to enclose the optimal solution within intervals that are successively shortened. To find these intervals, it goes on refining a decomposition of the polygon and solving optimization problems that are smaller than the original one. To improve efficiency, it tries to take advantage of the polygon's topology and, in particular, of the fact that some pieces in the decomposition may be dominant over others (i.e., if they are visible so are the dominated ones). The finer the decomposition is, the better the approximation becomes, but the problem that the algorithm has to solve at each step might become larger. For the case of orthogonal polygons, we could start from different partitions, one of which is $\Pi(P)$ and so, we were interested in establishing more accurate bounds for the number of pieces that $\Pi(P)$ might have in general.

The paper is structured as follows. We then introduce some preliminary definitions and useful results. In Sect. 2, we present the major result of this paper, that establishes lower and upper bounds on $|\Pi(P)|$, and improves an upper bound we gave in [4]. Finally, Sect. 3 contains some interesting results about lower and upper bounds on the area of grid n-ogons, although they do not extend to generic orthogonal polygons.

1.1 Preliminaries

Generic orthogonal polygons may be obtained from a particular kind of orthogonal polygons, that we called grid orthogonal polygons, as depicted in Fig. 1. (The reader may skip Definition 2 and Lemmas 1 and 2 if he/she has read [5].)

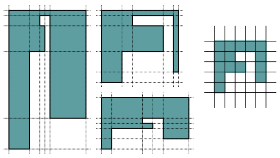


Fig. 1. Three 12-ogons mapped to the same grid 12-ogon.

Definition 2. *An n-ogon P is in general position iff every horizontal and vertical line contains at most one edge of P, i.e., iff P has no collinear edges. We call "grid n-ogon" to each n-ogon in general position defined in a $\frac{n}{2} \times \frac{n}{2}$ square grid.*

Lemma 1 follows immediately from this definition.

Lemma 1. *Each grid n-ogon has exactly one edge in every line of the grid.*

Each n-ogon which is not in general position may be mapped to an n-ogon in general position by ϵ-perturbations, for a sufficiently small constant $\epsilon > 0$. Consequently, we shall first address n-ogons in general position.

Lemma 2. *Each n-ogon in general position is mapped to a unique grid n-ogon through top-to-bottom and left-to-right sweep. And, reciprocally, given a grid n-ogon we may create an n-ogon that is an instance of its class by randomly spacing the grid lines in such a way that their relative order is kept.*

The number of classes may be further reduced if we group grid n-ogons that are symmetrically equivalent. In this way, the grid n-ogons in Fig. 2 represent the same class.

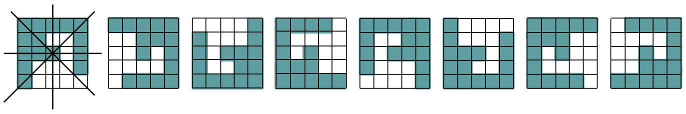


Fig. 2. Eight grid n-ogons that are symmetrically equivalent. From left to right, we see images by clockwise rotations of $90°$, $180°$ and $270°$, by flips wrt horizontal and vertical axes and flips wrt positive and negative diagonals.

Given an n-ogon P in general position, $\textsc{Free}(P)$ represents any grid n-ogon in the class that contains the grid n-ogon to which P is mapped by the sweep procedure described in Lemma 2.

The following result is a trivial consequence of the definition of $\textsc{Free}(P)$.

Lemma 3. *For all n-ogons P in general position, $|\Pi(P)| = |\Pi(\textsc{Free}(P))|$.*

2 Lower and Upper Bounds on $|\Pi(P)|$

In [4] we have shown that $\Pi(P)$ has at most $1 + r + r^2$ pieces. Later we noted that this upper bound is not sufficiently tightened. Actually, for small values of r, namely $r = 3, 4, 5, 6, 7$, we experimentally found that the difference between $1 + r + r^2$ and $\max |\Pi(P)|$ was 1, 2, 4, 6 and 9, respectively.

Definition 3. *A grid n-ogon Q is called* $\textsc{Fat}$ *iff $|\Pi(Q)| \geq |\Pi(P)|$, for all grid n-ogons P. Similarly, a grid n-ogon Q is called* $\textsc{Thin}$ *iff $|\Pi(Q)| \leq |\Pi(P)|$, for all grid n-ogons P.*

The experimental results supported our conjecture that there was a single $\textsc{Fat}$ n-ogon (except for symmetries of the grid) and that it had the form illustrated in Fig. 3.

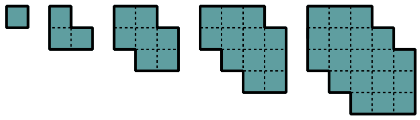


Fig. 3. The unique FAT n-ogons (symmetries excluded), for $n = 4, 6, 8, 10, 12$.

Clearly, each r-piece is defined by four vertices. Each vertex is either in $\mathrm{INT}(P)$ (*internal vertex*) or is in $\mathrm{BND}(P)$ (*boundary vertex*). Similar definitions hold for the edges. An edge e of r-piece R is called an *internal edge* if $e \cap \mathrm{INT}(P) \neq \emptyset$, and it is called a *boundary edge* otherwise.

Lemma 4. *The total number $|V_i|$ of internal vertices in $\Pi(P)$, when the grid n-ogon P is as illustrated in Fig. 3 is given by (1) where r is the number of reflex vertices of P.*

$$|V_i| = \begin{cases} \frac{3r^2-2r}{4} & \text{for } r \text{ even.} \\ \frac{(3r+1)(r-1)}{4} & \text{for } r \text{ odd.} \end{cases} \tag{1}$$

Proof. By construction, $|V_i|$ is defined by (2).

$$|V_i| = \begin{cases} 2\displaystyle\sum_{k=1}^{\frac{r}{2}}(r-k) & \text{iff } r \text{ is even.} \\ (r - \dfrac{r+1}{2}) + 2\displaystyle\sum_{k=1}^{\frac{r-1}{2}}(r-k) & \text{iff } r \text{ is odd.} \end{cases} \tag{2}$$

□

Proposition 1. *If P is any n-vertex orthogonal polygon such that the number of internal vertices of $\Pi(P)$ is given by (1), then P has at most a single reflex vertex in each horizontal and vertical line.*

Proof. We shall suppose first that P is a grid n-ogon. Then, let $v_{L_1} = (x_{L_1}, y_{L_1})$ and $v_{R_1} = (x_{R_1}, y_{R_1})$ be one of the leftmost and one of the rightmost reflex vertices of P, respectively. The horizontal chord with origin at v_{L_1} can intersect at most $x_{R_1} - x_{L_1}$ vertical chords, since we shall not count the intersection with the vertical chord defined by v_{L_1}. The same may be said about the horizontal chord with origin at v_{R_1}. There are exactly r vertical and r horizontal chords, and thus $x_{R_1} - x_{L_1} \leq r - 1$. If there were c vertical edges such that both extreme points are reflex vertices then $x_{R_1} - x_{L_1} \leq r - 1 - c$. This would imply that the number of internal vertices of $\Pi(P)$ would be strictly smaller than the value defined by (1). In fact, we could proceed to consider the second leftmost vertex (for $x > x_{L_1}$), say v_{L_2}, then second rightmost vertex (for $x < x_{R_1}$), and so forth. The horizontal chord that v_{L_2} defines either intersects only the vertical chord defined by v_{L_1} or it does not intersect it at all. So, it intersects at most $r-2-c$ vertical chords. In sum, c should be null, and by symmetry, we would conclude that there is exactly one reflex vertex in each vertical grid line (for $x > 1$ and $x < \frac{n}{2} = r+2$).

Now, if P is not a grid n-ogon but is in general position, then $\Pi(P)$ has the same combinatorial structure as $\Pi(\textsc{Free}(P))$, so that we do not have to prove anything else. If P is not in general position, then let us render it in general position by a sufficiently small ϵ-perturbation, so that the partition of this latter polygon does not have less internal vertices than $\Pi(P)$. □

Corollary 1. *For all grid n-ogons P, the number of internal vertices of $\Pi(P)$ is less than or equal to the value established by (1).*

Proof. It results from the proof of Proposition 1. □

Theorem 1. *Let P be a grid n-ogon and $r = \frac{n-4}{2}$ the number of its reflex vertices. If P is* Fat *then*

$$|\Pi(P)| = \begin{cases} \frac{3r^2+6r+4}{4} & \textit{for } r \textit{ even} \\ \\ \frac{3(r+1)^2}{4} & \textit{for } r \textit{ odd} \end{cases}$$

and if P is Thin *then $|\Pi(P)| = 2r+1$.*

Proof. Suppose that P is a grid n-ogon. Let V, E and F be the sets of all vertices, edges and faces of $\Pi(P)$, respectively. Let us denote by V_i and V_b the sets of all internal and boundary vertices of the pieces of $\Pi(P)$. Similarly, E_i and E_b represent the sets of all internal and boundary edges of such pieces. Then, $V = V_i \cup V_b$ and $E = E_i \cup E_b$. Since P is in general position, each chord we draw to form $\Pi(P)$ hits $\mathrm{BND}(P)$ in the interior of an edge and no two chords hit $\mathrm{BND}(P)$ in the same point. Hence, using O'Rourke's formula [3] we obtain $|E_b| = |V_b| = (2r+4) + 2r = 4r+4$. We may easily see that to obtain Fat n-ogons we must maximize the number of internal vertices.

By Corollary 1,

$$\max_P |V_i| = \begin{cases} \frac{3r^2-2r}{4} & \text{for } r \text{ even} \\ \\ \frac{(3r+1)(r-1)}{4} & \text{for } r \text{ odd} \end{cases}$$

and, therefore, $\max_P |V| = \max_P(|V_i| + |V_b|)$ is given by

$$\max_P |V| = \begin{cases} \frac{3r^2+14r+16}{4} & \text{for } r \text{ even} \\ \\ \frac{3r^2+14r+15}{4} & \text{for } r \text{ odd} \end{cases}$$

From Graph Theory [1] we know that the sum of the degrees of the vertices in a graph is twice the number of its edges, that is, $\sum_{v \in V} \delta(v) = 2|E|$. Using the definitions of grid n-ogon and of $\Pi(P)$, we may partition V as

$$V = V_c \cup V_r \cup (V_b \setminus (V_c \cup V_r)) \cup V_i$$

where V_r and V_c represent the sets of reflex and of convex vertices of P, respectively. Moreover, we may conclude that $\delta(v) = 4$ for all $v \in V_r \cup V_i$, $\delta(v) = 3$ for all $v \in V_b \setminus (V_c \cup V_r)$ and $\delta(v) = 2$ for all $v \in V_c$. Hence,

$$\begin{aligned} 2|E| &= \textstyle\sum_{v\in V_r\cup V_i} \delta(v) + \sum_{v\in V_c} \delta(v) + \sum_{v\in V_b\setminus(V_c\cup V_r)} \delta(v) \\ &= 4|V_i| + 4|V_r| + 2|V_c| + 3(|V_b| - |V_r| - |V_c|) = 4|V_i| + 12r + 8 \end{aligned}$$

and, consequently, $|E| = 2|V_i| + 6r + 4$.

Similarly, to obtain THIN n-ogons we must minimize the number of internal vertices of the arrangement. There are grid n-ogons such that $|V_i| = 0$, for all n (such that $n = 2r + 4$ for some $r \geq 0$). Thus, for THIN n-ogons $|V| = 4r + 4$.

Finally, to conclude the proof, we have to deduce the expression of the upper and lower bound of the number of faces of $\Pi(P)$, that is of $|\Pi(P)|$. Using Euler's formula $|F| = 1 + |E| - |V|$, and the expressions deduced above, we have $\max_P |F| = 1 + 2(\max_P |V_i|) + 6r + 4 - \max_P |V|$. That is, $\max_P |F| = \max_P |V_i| + 6r + 5$, so that

$$\max_P |F| = \begin{cases} \frac{3r^2+6r+4}{4} & \text{for } r \text{ even} \\ \\ \frac{3(r+1)^2}{4} & \text{for } r \text{ odd} \end{cases}$$

and $\min_P |F| = 1+2(\min_P |V_i|)+6r+4-\min_P |V| = 1+6r+4-4r-4 = 2r+1$. The existence of FAT and THIN grid n-ogons, for all n, follows from Lemma 4 and from the construction indicated in Fig. 6, respectively. □

Fig. 4 shows some THIN n-ogons.

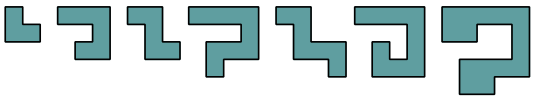


Fig. 4. Some grid n-gons with $|V_i| = 0$.

The area of a grid n-ogon is the number of grid cells in its interior. Corollary 2 gives some more insight into the structure of FATs, although the stated condition is not sufficient for a grid ogon to be FAT.

Corollary 2. *If P is a* FAT *grid n-ogon then each r-piece in $\Pi(P)$ has area 1.*

Proof. By Pick's Theorem (see, e.g. [2]), the area $A(P)$ of grid n-ogon P is given by (3)

$$A(P) = \frac{b(P)}{2} + i(P) - 1 \tag{3}$$

where $b(P)$ and $i(P)$ represent the number of grid points contained in BND(P) and INT(P), respectively. Using (3) and the expressions deduced in Theorem 1, we conclude that if P is FAT then

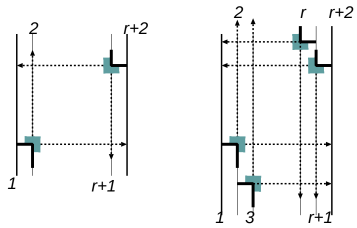
$$A(P) = \begin{cases} \dfrac{4r+4}{2} + \dfrac{3r^2-2r}{4} - 1 = \dfrac{3r^2+6r+4}{4} & \text{for } r \text{ even} \\[2ex] 2r+1+\dfrac{(3r+1)(r-1)}{4} = \dfrac{3(r+1)^2}{4} & \text{for } r \text{ odd} \end{cases}$$

so that $A(P) = |\Pi(P)|$. Hence, each r-piece has area 1. □

Nevertheless, based on the proof of Proposition 1, we may prove the uniqueness of FATs and fully characterize them.

Proposition 2. *There is a single* FAT *n-ogon (except for symmetries of the grid) and its form is illustrated in Fig. 3.*

Proof. We saw that FAT n-ogons must have a single reflex vertex in each vertical grid-line, for $x > 1$ and $x < \frac{n}{2}$. Also, the horizontal chords with origins at the reflex vertices that have $x = 2$ and $x = \frac{n}{2} - 1 = r+1$, determine $2(r-1)$ internal points (by intersections with vertical chords). To achieve this value, they must be positioned as illustrated below on the left.



Moreover, the reflex vertices on the vertical grid-lines $x = 3$ and $x = r$ add $2(r-2)$ internal points. To achieve that, we may conclude by some simple case reasoning, that v_{L_2} must be below v_{L_1} and v_{R_2} must be above v_{R_1}, as shown above on the right. And, so forth… □

FAT n-grid ogons are not the grid n-ogons that have the largest area, except for small values of n, as we may see in Fig 5. Some more details are given in the

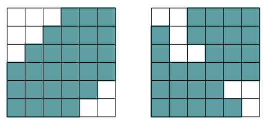


Fig. 5. On the left we see the FAT grid 14-ogon. It has area 27, whereas the grid 14-ogon on the right has area 28, which is the maximum for $n = 14$.

following section, where we shall also prove that the set of grid ogons that have the smallest area is a proper subset of the THIN grid ogons.

3 Lower and Upper Bounds on the Area

In [5] we proposed an iterative method that constructs a grid n-ogon from the unit square by applying a transformation we called INFLATE-PASTE r times. Based on this method we may show the following result.

Lemma 5. *Each $(i+2)$-grid ogon P is obtained by adding at least two grid cells to a i-grid ogon (dependent of P), for all even $i \geq 4$.*

Proof. Each INFLATE-PASTE transformation increases the area of the grid ogon constructed up to a given iteration by at least two (i.e., it glues at least two grid cells to the polygon) and the INFLATE-PASTE method is complete. □

Another concept is needed for our proof of Proposition 3, stated below. A *pocket* of a nonconvex polygon P is a maximal sequence of edges of P disjoint from its convex hull except at the endpoints. The line segment joining the endpoints of a pocket is called its *lid*. Any nonconvex polygon P has at least one pocket. Each pocket of an n-ogon, together with its lid, defines a simple polygon without holes, that is almost orthogonal except for an edge (lid). It is possible to slightly transform it to obtain an orthogonal polygon, say an *orthogonalized pocket*.

We may now prove the following property about the area of grid ogons. We note that, for $r = 1$, there is a single grid ogon (except for symmetries of the grid) which is necessarily the one with the smallest and the largest area.

Proposition 3. *Let P_r be a grid n-ogon and $r = \frac{n-4}{2}$ the number of its reflex vertices. Then $2r + 1 \leq A(P_r) \leq r^2 + 3$, for $r \geq 2$.*

Proof. From Lemma 5, we may conclude that $A(P_r) \geq 2r + 1$, for all P_r and all $r \geq 1$. The INFLATE-PASTE method starts from the unit square (that is P_0) and applies r INFLATE-PASTE transformations to construct P_r. In each transformation it glues two cells (at least) to the polygon being constructed, so that $A(P_r) \geq 2r+1$. Fig. 6 may provide some useful intuition to this proof. To prove

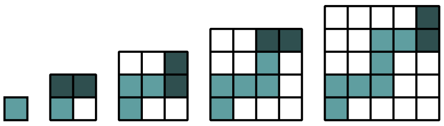


Fig. 6. Constructing the grid ogons of the smallest area, for $r = 0, 1, 2, 3, 4, \ldots$. The area is $2r + 1$.

that $A(P_r) \leq r^2 + 3$, we imagine that we start from a square (not grid ogon) of area $(r+1)^2$. This is equivalent to saying that our $\frac{n}{2} \times \frac{n}{2}$ square grid (that consists of $(r+1)^2$ unit square cells) is initially completely filled. Then, we start removing grid cells, to create reflex vertices, while keeping the orthogonal polygon in the grid in general position. Each time we introduce a new reflex vertex, we are either creating a new pocket or increasing a pocket previously created. To keep the polygon in general position, only two pockets may start at the corners (indeed opposite corners) and to start each one we delete one cell (at least). To create any other pocket we need to delete at least three cells. On the other hand, by Lemma 5, to augment an already created pocket, we have to delete at least two cells. In sum, to obtain a polygon with the maximal area we have to remove the smallest number of cells, so that only two pockets may be created. Each one must be a grid ogon with the smallest possible area. In Fig. 7 we show a family of polygons that have the largest area, $A(P_r) = r^2 + 3$. $\square$

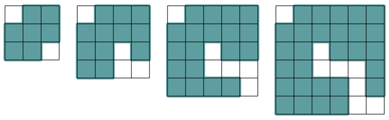


Fig. 7. A family of grid n-ogons with MAX-AREA (the first elements: $r = 2, 3, 4, 5, \ldots$).

Definition 4. *A grid n-ogon P is a* MAX-AREA *grid n-ogon iff $A(P) = r^2 + 3$ and it is a* MIN-AREA *grid n-ogon iff $A(P) = 2r + 1$.*

There exist MAX-AREA grid n-ogons for all n, as indicated in Fig. 7, but they are not unique, as we may see in Fig. 8. Regarding MIN-AREA n-ogons, it

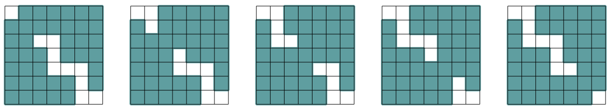


Fig. 8. A sequence of MAX-AREA n-ogons, for $n = 16$.

is obvious that they are THIN grid n-ogons, because $|\Pi(P)| = 2r + 1$ holds only for THIN grid n-ogons. This condition is not sufficient for a grid n-ogon to be a MIN-AREA grid n-ogon (see for example the rightmost grid n-ogon in Fig. 4). Based on Proposition 3 and on the INFLATE-PASTE method, we may prove the uniqueness of MIN-AREA grid n-ogons.

Proposition 4. *There is a single* MIN-AREA *grid n-ogon (except for symmetries of the grid) and it has the form illustrated in Fig. 6.*

Proof (Sketch). It is strongly based on the INFLATE-PASTE construction. The idea is to proceed by induction on r and by case analysis to see which are the convex vertices v_i that allow to increase the area by just two units (see Fig. 9). □

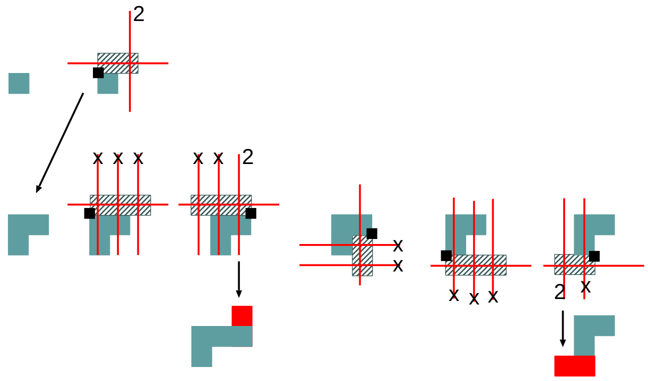


Fig. 9. Uniqueness of MIN-AREA grid n-ogons related to INFLATE-PASTE construction.

4 Further Work

We are now investigating how the ideas of this work may be further exploited to obtain better approximate solutions to the MINIMUM VERTEX GUARD problem, where the goal is to find the minimum number of vertex guards that are necessary to completely guard a given polygon. Our strategy is to establish bounds for families of grid ogons and to see how these bounds apply to the orthogonal polygons in the class of a given n-ogon.

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