Summary

- Random Processes
  Stationary and ergodic
- Correlation (auto and cross) Function
- Covariance Function
- Estimates of the function: using samples
- Correlation and Covariance Matrices
Random vectors

• A random vector is

\[ \mathbf{X} = [X_1 \quad X_2 \quad \cdots \quad X_N]^T \]

where each entry is random variable. When entries \( X_i \) \( i = 1, 2, \cdots N \)

• are independent random variables

The joint pdf of \( \mathbf{X} \) is

\[ f_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) \cdots f_{X_N}(x_N) \]

Product of the probability density functions.

• i.i.d: independent and identically distributed

\[ f_{X_i}(x_i) = f(x), i = 1, 2 \cdots N \]

Example: Joint Gaussian Random Vector

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \text{VAR}[X_1] & \text{COV}(X_1, X_2) & \cdots & \text{COV}(X_1, X_n) \\ \text{COV}(X_2, X_1) & \text{VAR}[X_2] & \cdots & \text{COV}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{COV}(X_n, X_1) & \text{COV}(X_n, X_2) & \cdots & \text{VAR}[X_n] \end{bmatrix} \]

\[ f_{\mathbf{x}}(\mathbf{x}) = \frac{\exp(-0.5(\mathbf{x} - \mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x} - \mathbf{m}))}{(2\pi)^{n/2} |\mathbf{K}|^{1/2}} \]

\( \mathbf{K} \) - covariance matrix

\(|\mathbf{K}|\) - determinant of \( \mathbf{K} \)
Example I: 2D-vector

```matlab
>> x1=1.5*randn(1,10000);
>> x2=0.6*randn(1,10000);
>> plot(x1,x2,'*')
>> axis([-6 6 -6 6])
```

Mean Vector

\[
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix} = \begin{bmatrix}
-0.0223 \\
-0.0016
\end{bmatrix}
\]

Covariance matrix

\[
K = \begin{bmatrix}
2.2607 & 0.0200 \\
0.0200 & 0.3671
\end{bmatrix}
\]

Example II: 2D-vector

```matlab
>> X=randn(2,10000)
>> plot(X(1,:),X(2,:),'*')
```

Mean Vector

\[
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix} = \begin{bmatrix}
-0.0066 \\
-0.0037
\end{bmatrix}
\]

Covariance matrix

\[
K = \begin{bmatrix}
0.9824 & 0.0198 \\
0.0198 & 1.0350
\end{bmatrix}
\]

What is the difference between EXAMPLE I and EXAMPLE II?
Ex2

Ex1
Random Process

Realization

Generalization of the concept of random vectors: each entry is the value of a random variable in time instant

Random Processes: types

Time continuous or discrete, random variable continuous or discrete
Example I: Bernoulli

Each sample \( n \) (along realizations) is a random variable (Bernoulli)

Example II: sinusoid with random phase

\[ X[n] = \cos(2\pi(0.1)n + \theta) \]

Phase: uniformly distributed random variable \([0, 2\pi]\)
Example III: sinusoid with random amplitude

Amplitude: uniformly distributed random variable $[-0.5, 0.5]$

$X[n] = A \cos(2\pi 100t)$

5 Realizations of random process.

Expected Value and Variance

$m_X(t) = E[X(t)] \quad VAR[X(t)] = E[(X(t) - m_X(t))^2]$  
$m_X[n] = E[X[n]] \quad VAR[X[n]] = E[(X[n] - m_X(n))^2]$
Continuos time: Autocorrelation and Autocovariance Functions

Second order Descriptors:

\[ X(t_1), X(t_2) \]

\[ R_X(t_1, t_2) = E[X(t_1), X(t_2)] \]

\[ C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \]

Discrete time: Autocorrelation and Autocovariance

Second order descriptors

\[ X(n_1), X(n_2) \]

\[ R_X(n_1, n_2) = E[X(n_1), X(n_2)] \]

\[ C_X(n_1, n_2) = R_X(n_1, n_2) - m_X(n_1)m_X(n_2) \]
Stationary Processes

Independent  (same as an independent random vector)

\[ X = [X(n_1) \; X(n_2) \; \ldots \; X(n_N)] \]
\[ P_{X[n_1] \; x[n_2] \; \ldots \; x[n_N]}(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} P_{X[n_i]}(x_i) \]

Stationary

\[ X = [X(n_0 + n_1) \; X(n_0 + n_2) \; \ldots \; X(n_0 + n_N)] \]
\[ P_{X[n_0+n_1] \; x[n_0+n_2] \; \ldots \; x[n_0+n_N]} = P_{X[n_0]} \; x(n_2) \; \ldots \; x(n_N) \]

Independent and identically distributed (IID) is stationary.

Examples: first and second order description

Sinusoid 1: amplitude uniformly distributed [0,1].

Sinusoid 2: phase uniformly distributed [0, 2\pi].

\[ X(t) = A \text{sen}(2\pi t) \]
\[ m_x(t) = 0.5 \text{sen}(2\pi) \]
\[ R_x(t_1, t_2) = E[A^2] \text{sen}(2\pi t_1) \text{sen}(2\pi t_2) \]
\[ C_x(t_1, t_2) = \text{VAR}[A] \text{sen}(2\pi t_1) \text{sen}(2\pi t_2) \]

Example 2:
• Expected value is constant (not depending on t)
• Autocovariance/Autocorrelation: functions of time differences.
WSS- Wide Sense Stationary

The expected value (mean) is constant

\[ m_x(t) = \text{cte} \quad \forall t \text{(continuous)} \quad \text{ou} \quad m_x(n) = \text{cte} \quad \forall n \text{(discrete)} \]

The autocorrelation/ auto-covariance:

\[ t_1 = t \land t_2 = t + \tau \quad \text{ou} \quad n_1 = n \land n_2 = n + k \]

\[ R_x(\tau) = g(\tau) \quad \text{(continous)} \quad \text{ou} \quad R_x(k) = g(k) \quad \text{(discrete)} \]

Function of the difference between time instants.

Ergodic Process

- All statistics can be estimated using one realization of the process
- The operator expectation is substituted by averages along the time. For instance,

\[
r_{xx}(m) = \frac{1}{2N + 1} \sum_{n=\infty}^{\infty} x[n]x[n + m]
\]

\[
R_{xx}(m) = \lim_{N \to \infty} r_{xx}(m) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} x[n]x[n + m]
\]
Ergodic process: mean

Mean along time
(average of the samples of one realization)

mean along realizations
(group mean)

\[ \hat{m}_X = \langle X[n] \rangle = \frac{1}{N} \sum_{i=0}^{N-1} x[n] \]

\[ \langle X(t) \rangle = \frac{1}{2T} \int_{-T}^{T} x(t) dt \]

---

Estimates along time (1/3)

Autocorrelation function
• Given N samples the estimate is

\[ \hat{R}_X(m) = S \sum_{n=0}^{N-1-|m|} x[n]x[n+m] \]

\[ S = 1 \quad S = \frac{1}{N} \quad S = \frac{1}{N-|m|} \quad S = \frac{1}{R_X(0)} \]

S- scaling factor: none, biased, unbiased and normalized (maximum of the function equal to 1)
Estimates along time (2/3)

Autocovariance Function

\[ \tilde{C}_X (m) = S \sum_{n=0}^{N-1-[m]} (x[n] - \tilde{m}_X)(x[n+m] - \tilde{m}_X) \]

\[ S = 1 \quad S = \frac{1}{N} \quad S = \frac{1}{N-[m]} \quad S = \frac{1}{C_X (0)} \]

S- scaling factor: none, biased, unbiased and normalized (maximum equal to 1)

Estimitavas no tempo (3/3)

Cross- Correlation and Cross- Covariance functions

Given N samples of a discrete time random process

\[ \tilde{R}_{XY} (m) = S \sum_{n=0}^{N-1-[m]} x[n]y[n+m] \]

\[ \tilde{C}_{XY} (m) = S \sum_{n=0}^{N-1-[m]} (x[n] - \tilde{m}_X)(y[n+m] - \tilde{m}_Y) \]

\[ S = 1 \quad S = \frac{1}{N} \quad S = \frac{1}{N-[m]} \quad S = \frac{1}{R_{XY} (0)} / S = \frac{1}{C_{XY} (0)} \]

Value 1.0 at m=0
Example: sinusoid2 (phase random variable)

Random process where $K$ is the random variable

$$X[n] = \cos\left(\frac{2\pi}{8} n + \frac{\pi}{3} k\right)$$

$$S_K = \{0, 1, 2, 3\}, \text{values of random variable with equal probability}$$

Example: autocorrelation function

```matlab
n=0:15
x=cos(2*pi/8*n)
subplot(511)
stem(n,x),grid on
[r1,d]=xcorr(x);
subplot(512)
stem(d,r1,'k'), grid on
[r2,d]=xcorr(x,'biased');
subplot(513)
stem(d, r2,'k'),grid on
[r3,d]=xcorr(x,'unbiased');
subplot(514)
stem(d,r3,'k'),grid on
[r4,d]=xcorr(x,'coeff')
subplot(515)
stem(d,r4,'k'),grid on
```
Estimates versus theoretical

Analytical result

\[ R_X(d) = 0.5 \cos\left(\frac{2\pi}{8} d\right) \]

Example: Gaussian Noise

```matlab
>> x = randn(1, 100);  mean(x)
0.0479
>> std(x)
0.8685
```
Example: Cross- correlation function

```matlab
>> x=randn(1,100);
>> subplot(311)
>> stem(x),grid on
>> y=filter([0.5 0.5],1,x);
>> subplot(312),
>> stem(y),grid on
>> [r,d]=xcorr(x,y,'biased');
>> subplot(313),
>> stem(d,r),grid on
```

Autocorrelation matrix and random processes

L elements of the autocorrelation function organized into L x L matrix

\[
R = \begin{bmatrix}
R_X(0) & R_X(1) & R_X(2) & \ldots & R_X(L-1) \\
R_X(1) & R_X(0) & R_X(1) & \ldots & R_X(L-2) \\
R_X(2) & R_X(1) & R_X(0) & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_X(L-1) & \ldots & \ldots & R_X(1) & R_X(0)
\end{bmatrix}
\]

Note: the autocorrelation function is an even function

\[
R_X(m) = R_X(-m)
\]
Convolution with matrix manipulations

Assuming two sequences
\[ h[n] \neq 0 \quad n = 0,1, \ldots, L-1 \]
\[ x[n] \neq 0 \quad n = 0,1, \ldots, N-1 \]
and \( N > L \)

The convolution is
\[ y[n] = x[n] \ast h[n] \]

\[
\begin{bmatrix}
  y[0] \\
  y[1] \\
  y[2] \\
  \vdots \\
  y[N-1]
\end{bmatrix} =
\begin{bmatrix}
  x[0] & 0 & \cdots & 0 \\
  x[1] & x[0] & \cdots & 0 \\
  x[2] & x[1] & \cdots & \vdots \\
  \vdots & \vdots & \ddots & x[N-L-1] \\
  x[N-1] & x[N-2] & \cdots & x[N-L]
\end{bmatrix}
\begin{bmatrix}
  h[0] \\
  h[1] \\
  \vdots \\
  h[L-1]
\end{bmatrix}
\]

\[ y = Xh \]

Lab. Assignement 6

\[ \tilde{h} = X^+ y = (X^T X)^{-1} X^T y \]

The impulse response can also be approximated by
\[ \tilde{h} = X^T y, \text{ if } X^T X = cI \]
Questions

I- Show that

\[ X^T X = \begin{bmatrix} R_x^{(N)}(0) & R_x(1) & \cdots \\ & \vdots & \vdots \\ & & R_x^{(N-2)}(0) & \cdots \end{bmatrix} \approx R \]

Estimation of the autocorrelation matrix, with the diagonal entries calculated with N, (N-1), (N-2)… Samples.

II- Show that

\[ X^T y = \begin{bmatrix} R_{xy}(0) \\ R_{xy}(1) \\ \vdots \\ R_{xy}(L-1) \end{bmatrix} \]

Appendix: Least Squares Solution

Given a system of equations

\[ Ax = b \]

If the columns of A are linearly independent, the system has an unique least squares solution given by

\[ \tilde{x} = (A^T A)^{-1} A^T b \]

The total squared error (TSE)

\[ A\tilde{x} = \tilde{b} \Rightarrow e = (b - \tilde{b})^T (b - \tilde{b}) \]
Appendix: example

\[
\begin{bmatrix}
3 & 2 \\
1 & 5 \\
2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix} \\
\Rightarrow 
\bar{x} = \frac{1}{251}\begin{bmatrix}
135 \\
67 \\
\end{bmatrix}
\]

Bibliography

- Paolo Prandoni, Martin Viterli, Signal Processing for Communication – Chapter 8 (sections 1, 2 and 3)
  Available online: (http://www.sp4comm.org/docs/sp4comm.pdf)