STABILITY ANALYSIS OF NON-RECURSIVE PARALLEL CONCATENATED REAL NUMBER CODES

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ABSTRACT

Real number codes can be derived from most existing GF error correcting codes. This approach provides a fresh view on the fields of error correcting codes and signal processing. In this article we study the parallel concatenated codes (PCC), also known as ”Turbo Codes” (TC) on the real number field. We derive a known generalization of the coding matrix and a non-iterative decoding algorithm. We also present some simulation results comparing the decoding stability of real number Bose, Chaudhury and Hocquenghem (BCH) codes with real number PCC.

1. INTRODUCTION

Turbo codes represent one of the greatest development in many years in the field of coding theory. They deserved the attention of many researchers after their introduction in 1993 by Berou, Glavieux and Thitimajshima [1]. This effort, leaded to a great evolution, transforming the turbo coding in a mature topic [2, 3]. The coding gain achieved by the TC is very close to the Shannon limit but recently another coding scheme, called Low-Density Parity-Check Codes, unveiled the work of Galleger [4] introducing another improvement on the attainable coding gain.

In this article we show that the coding matrix of a complex number PCC is a frame robust to contiguous erasures. This kind of frame has several applications such as oversampled filter banks or systems with resilience to additive noise [5].

In this article we will study complex number PCC, in a effort to generalize some results and improve the knowledge in this topic. Some authors have studied in the past [6, 7, 8, 9, 10] complex number versions of BCH and other types of codes but most of them, did not address the problem of numerical stability of the decoding algorithms [8, 12].

The first section introduces the coding and decoding process of BCH codes on the complex field. The following section describes the PCC on the complex field as a generalization of the BCH codes. In section 4 we will compare the numerical stability of both types of codes. Some preliminary results and simulations concerning the stability of PCC will be presented.

2. BCH CODES ON THE COMPLEX FIELD $\mathbb{C}^N$

Consider the N order DFT matrix definition

$$F_{nk}^{(N)} = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}, \qquad n, k \in [0, \dots, N-1]$$

and the following partition

$$F = \left[\begin{array}{cc} G & H \end{array} \right],$$

where G is a $N \times K$ matrix and H is $N \times (N-K)$. This partition can be generalized and G can be formed with any K columns of F to obtain different symmetry properties. In a (N, K) BCH code, a codeword $x \in \mathbb{R}^K$ is generated by multiplication of a message $m \in \mathbb{R}^N$ by the generator matrix G

$$x = Gm.$$

If we want both vectors x and m, to belong to the real number field some symmetry properties have to be observed and the ODFT transform can also be used [11].

If the vector x is stored or transmitted and some samples are corrupted, then the received signal y would be

$$y = x + e,$$

where the error vector e has t samples different from zero at positions given by the complementary set $\bar{J}$, with J a subset of $\{0, 1, 2, \dots, N-1\}$. So, the correcting algorithm is able to reconstruct the message m from the known received samples y_J if $t < N - K$ [12]. There are several methods to obtain m from y_J [12] and a class of them are equivalent to find the solution in a minimum square sense of the following system

$$y_J = G_J m, \tag{1}$$

which can be found by

$$m = G_J^+ y_J. \tag{2}$$

The notation A_J stands for the lines of A with index given by the set J, and A^+ is the pseudo inverse of A.

Because we are doing the coding on the real field some small amplitude errors are introduced during the coding and decoding process due to roundoff or due to quantization applied to the coded vector x before transmission or storage. This noise affect all samples and can originate decoding errors if the system of equations (1) is ill posed. So, numerical stability is a problem of great importance and we will address it in section 4.

3. TURBO CODES ON THE COMPLEX FIELD $\mathbb{C}^N$

Parallel concatenated codes, as the BCH codes, can also be implemented using the complex number field $\mathbb{C}^N$. In figure 1 we can see a block diagram describing this type of coding, where a message signal m with K samples is coded in two different signals x_1 and x_2. The first signal is obtained multiplying the signal m by the matrix G of dimension $(N/2 \times K)$. The signal x_2 is obtained performing first a permutation P on the samples of the vector m, and multiplying the result by the same matrix G. At the end of this article we will show, that is possible to generalize this type of coding, by doing more than 2 partitions of the generator matrix.

The coding operation to obtain the signal $x_1$can be defined as

$$x_1 = F^{(N/2)} \begin{bmatrix} m \\ 0 \end{bmatrix},$$

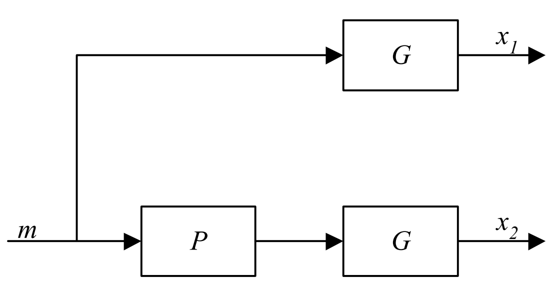

Fig. 1. Block diagram of a turbo coder, where a signal m with K samples is coded in two different signals x_1 and x_2, with $N/2$ samples each, and $N > K$.

where $N/2-K$ zeros where padded at the end of the vector m. If we consider a partition of $F^{(N/2)} = \begin{bmatrix} G & H \end{bmatrix}$ we get

$$x_1 = \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} m \\ 0 \end{bmatrix} = Gm. \tag{3}$$

On the other hand, the other codeword can be obtained by

$$x_2 = F^{(N/2)} \begin{bmatrix} Pm \\ 0 \end{bmatrix},$$

with P a given permutation matrix, and if we use the same partition of $F^{(N/2)}$ we get

$$x_2 = \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} Pm \\ 0 \end{bmatrix} = GPm. \tag{4}$$

In both codewords, errors may occur. Denoting the received signals by

$$\begin{aligned} y_1 &= x_1 + e_1 \\ y_2 &= x_2 + e_2 \end{aligned}$$

where e_1 and e_2 are the error vectors. Using equations (3) and (4), we can write the following equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} G \\ GP \end{bmatrix} m.$$

If we make the following change of variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad G' = \begin{bmatrix} G \\ GP \end{bmatrix}$$

we can finally write

$$x = G'm.$$

This coding equation transforms a message vector $m \in \mathbb{R}^K$ into a codeword $x \in \mathbb{R}^N$. Concatenating the error vectors we obtain the error vector e which can have a maximum of $t < N - K$ samples different from zero.

The decoding of a PCC code can be obtained as in the case of the BCH code by solving the following system of equations

$$y_J = G'_J m,$$

which can be solved by

$$m = (G'_J)^+ y_J.$$

It is possible to generalize the construction of a (N, K) PCC by observing that we have formed its coding matrix from a matrix G obtained from a $N/2$ order DFT instead of a N order, as used in the BCH code. It is possible to obtain a (N, K) PCC using for example matrices G of $N/4$ order and three permutation matrices

$$G' = \begin{bmatrix} G \\ GP_1 \\ GP_2 \\ GP_3 \end{bmatrix}.$$

4. STABILITY OF PARALLEL CONCATENATED CODES IN $\mathbb{C}^N$

In this section we will show preliminary results proving the robustness to burst errors of parallel concatenated codes (PCC, also designated by Turbo Codes), when compared to BCH codes in the complex field $\mathbb{C}$. Some simulation results will be presented and a first attempt of justification will be given.

4.1. Stability of BCH Codes in $\mathbb{C}^N$

Consider the set of vectors

$$Z = \{G_i\}_{i \in J}$$

where G_i are lines from the matrix G of a BCH code and J defined as before. The set Z forms an Harmonic tight frame and a subset of $k > K$ vectors of Z forms a frame [13]. The frame condition is defined by

$$a \|x\|^2 \leq \sum_{i \in J} |\langle x, G_i \rangle|^2 \leq b \|x\|^2 ,$$

where a and b are the frame bounds. Introducing the decimation matrix D defined as follow

$$D_{ij} = \begin{cases} 1, & \text{if} \quad i = j \quad \text{and} \quad i \in J, \\ 0, & otherwise \end{cases}$$

the frame condition can be written in algebraic form

$$aI \leq G^H D G \leq bI.$$

It is possible to show that the frame bounds a and b are given by

$$a = \lambda_{\min}, \qquad b = \lambda_{\max},$$

with $\lambda_{\min}$ and $\lambda_{\max}$ being the minimum and the largest eigenvalue of $G^H DG$.

The lines of the generator matrix G of a BCH code, can be interpreted as an Harmonic tight frame, and a set of $i > K$ lines of G is a frame. Defining the condition number [14] of a matrix A as

$$\kappa (A) = \frac{\lambda_{\max} (A)}{\lambda_{\min} (A)},$$

we can see that the condition number $\kappa \left(G^H DG\right)$ is only a function of D or, which is the same, of the erasures positions given by the set $\bar{J}$. Including the matrix D in the notation of equation (1), using the fact that $G_J = (DG)_J$ and $\mathrm{y}_J = (Dy)_J$, we obtain

$$m = (DG)^+ Dy,$$

but as

$$A^+ = \left(A^H A\right)^{-1} A^H,$$

we get

$$m = \left(G^H DG\right)^{-1} G^H Dy.$$

This equation shows the importance of the condition number of $G^H DG$ in solving this system of equations. To illustrate this problem we have evaluated $\kappa \left(G^H DG\right)$ for all possible patterns $\bar{J}$, for the following problem: $t = 7$, $N = 20$ and $K = 8$. In figure 3 we have κ in lexicographic order for all $\binom{N}{t}$ erasures patterns and in figure 2 we have sorted the values of κ. The peak values in figure 3 (BCH plot) corresponds to cases where the erasures positions are contiguous. In this case even for small dimension problems one can get a κ that exceeds the arithmetic machine precision resulting in large reconstruction errors.

4.2. Stability of PCC in $\mathbb{C}^N$

Consider now the set of vectors

$$X = \{G_i^{'}\}_{i \in J}$$

where G_i' are lines from the matrix G' of a PCC code and J defined as before. The frame bounds of this frame expansion is given by

$$a \|x\|^2 \leq \sum_{i \in J} |\langle x, G_i' \rangle|^2 \leq b \|x\|^2 , \tag{5}$$

and introducing the decimation matrix

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

we can write the expression

$$aI \leq G'^H DG' \leq bI,$$

or in expanded form

$$aI \leq \begin{bmatrix} G^H & (GP)^H \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} G \\ GP \end{bmatrix} \leq bI,$$

which can be written as

$$aI \leq G^H D_1 G + (GP)^H D_2 GP \leq bI. \tag{6}$$

Note that the matrix $G'^H DG'$ is Hermitian positive definite, because $D^2 = D$ and $D^H = D$ and then we can write

$$G'^H D^H DG' = (DG')^H DG'.$$

From equation (5) we can write the frame bounds as

$$\begin{aligned} a &= \lambda_{\min}\left(G'^H DG'\right) \\ b &= \lambda_{\max}\left(G'^H DG'\right) \end{aligned}$$

and using equation (6) we have for any eigenvalue of $G'^H DG'$

$$\begin{aligned} &\lambda_i\left(G^H D_1 G + P^H G^H D_2 G P\right) \geq \\ &\lambda_i\left(G^H D_1 G\right) + \lambda_i\left(P^H G^H D_2 G P\right). \end{aligned}$$

The previous result shows that the presence of the positive definite matrix $P^H G^H D_2 G P$ prevents $\lambda_{\min}\left(G'^H DG'\right)$ from being too small. The condition number of the matrix $G'^H DG'$ depends on the permutation P and on the decimation matrix D and is usually smaller than the condition number of the matrix $G^H DG$ for the BCH codes. In figure 3 we have the condition number of the PCC in lexicographic order for all possible $\binom{N}{t}$ error patterns. In figure 2 we have the same graphic but with the samples ordered by value of $\kappa\left(G'^H DG'\right)$, and we can see that the PCC have always a smaller condition number than the BCH code. However, this fact is not always true and depends on the permutation P.

In figures 4 and 5 we can see the distribution of the eigenvalues of matrices $G'^H DG'$ and $G^H DG$ for a reconstruction problem with 80 erasures at contiguous positions where $K = 20$ and $N = 256$. We can see that the quotient $\frac{\lambda_{\max}}{\lambda_{\min}}$ for the BCH matrix $G^H DG$ is larger than for the PCC case.

5. CONCLUSIONS

The PCC proved to be more numerically stable, since its decoding matrix has a better condition number regardless of the error pattern. They also have a better capability to correct burst errors. It also deserve some attention the fact that the PCC perform well despite of the arithmetic field used: Galoy field or Real/Complex field. This results are preliminary and deserve a more profound and systematic study.

6. REFERENCES

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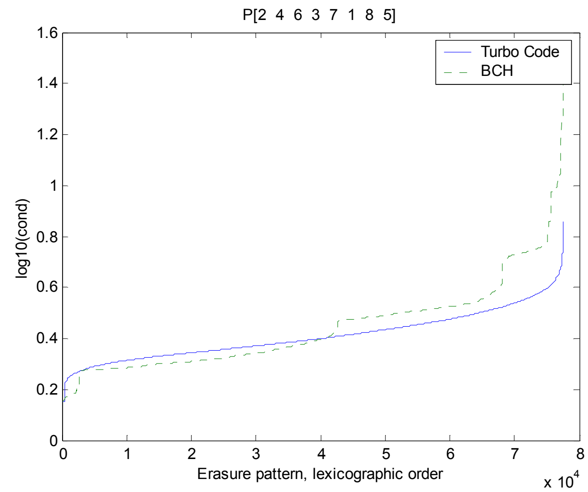

Fig. 2. To generate this plot, we evaluated the condition number of the decoding matrices of a BCH and TC codes, for all possible error patterns. Then, we sorted each condition number independently to guarantee that, for example, that the right most point corresponds to the worst error patterns for each code. For the simulation we used a vector m with 8 samples and a coded vector with 20 samples and 7 erasures.

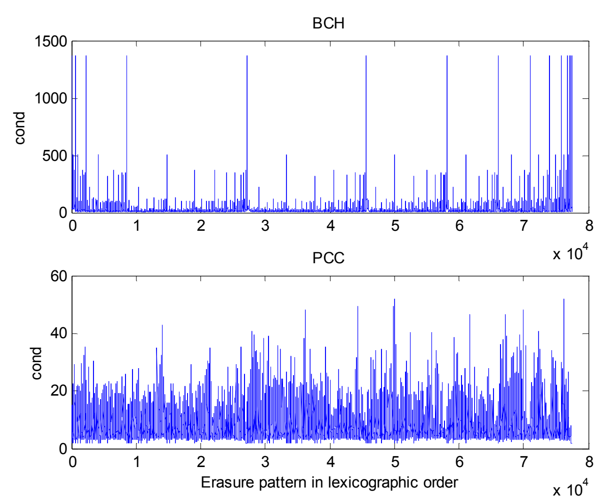

Fig. 3. In this figure we compare the condition number of matrices G' and G_B lexicographically ordered by the erasure pattern. It is possible to confirm that the TC in this case is more robust to burst errors.

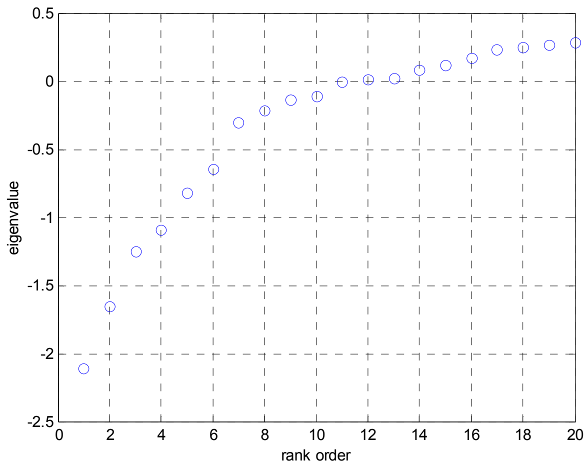

Fig. 4. PCC case: the sorted eigenvalues of $G'^H DG'$ for a block size of $N = 256$ and a message length of $K = 20$. The 60 erasures occured in a contigous way. The condition number is only 246.

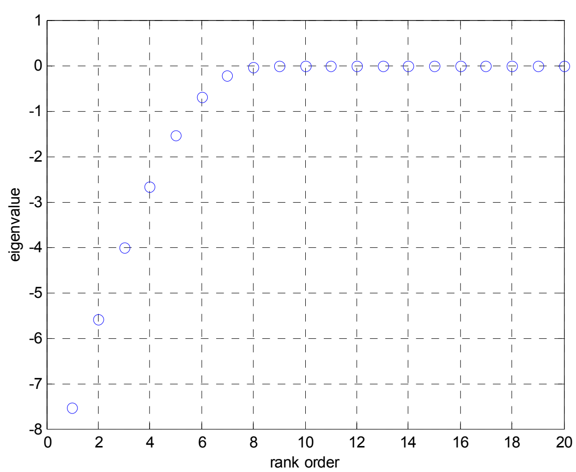

Fig. 5. BCH case: the sorted eigenvalues of $G^H DG$ for a block size of $N = 256$ and a message length of $K = 20$. The 60 erasures occured in a contigous way. The condition number of 3.5×10^7 is larger than the one obtained in figure (4). Note that in both figures we have the $\log_{10}(\lambda)$.

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