Fibrations of polynomial and analytic functors and monads

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Plan of the talk

1. Motivation (paradigmatic example)
2. (Co)completeness of and in 2-categories
3. Main theme: extensions of representations
4. Kleisli and Eilenberg-Moore objects in 2-category of monoidal objects and lax monoidal functors
5. Examples: fibrations over *Set*, *Cat*, $\omega - Gr$, *lccc*.

Motivating example

The category $Set_{/\omega}$ with the substitution tensor

$$\{A_n\}_n \otimes \{B_n\}_n = \{ \coprod_{k,m_1,\ldots,m_k \in \omega; \sum_{i=1}^{k} m_i = n} A_k \times B_{m_1} \times \ldots B_{m_k} \}_n$$

is a monoidal category. It acts on Set

$$Set_{/\omega} \times Set \longrightarrow Set$$

$$\langle \{A_n\}_n, X \rangle \mapsto \coprod_n A_n \times X^n$$

... and by exponential adjunction we get a strong monoidal functor

$$\mathbf{r} : Set_{/\omega} \longrightarrow End(Set)$$

that has a (lax monoidal) right adjoint.

Motivation

Fact

1. The closure under isomorphism in $End(Set)$ of the 'image' of
$$\mathbf{r} : Set_{/\omega} \longrightarrow End(Set)$$
is the category **Poly** of (finitary) polynomial endo-functors and cartesian natural transformations.
2. The closure under reflexive coequalizers **Poly** in $End(Set)$ is the category **An** of (finitary) analytic endo-functors and weakly cartesian natural transformations.

Some limits and colimits in 2-categories

Let $\mathcal{A}$ be a 2-category with finite products. We will consider the following weighted limits and colimits in $\mathcal{A}$

1. Kleisli and Eilenberg-Moore objects (for monads in $\mathcal{A}$);
2. objects of monoids (for monoidal objects in $\mathcal{A}$);

2-categories: global (co)completeness and exactness

...and moreover we ask for exactness properties:

1. Kleisli objects commute with finite products;
2. Comparison morphisms from Kleisli objects are full and faithful

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "F_{\mathcal{R}}"'] & & \mathcal{D} \\ & \mathcal{C}_{\mathcal{R}} \arrow[ur, "K"'] & \end{tikzcd} \qquad \begin{array}{l} F \dashv U \\ \mathcal{R} = UF \end{array}$$

We do not assume $\mathcal{A}$ has all of these (co)limits but just in some interesting cases.
Concerning the exactness properties, we expect them to hold whenever the constructions are available.

Reflexive coequalizers in 0-cells A 0-cell $\mathcal{C}$ in a 2-category $\mathcal{A}$ has *reflexive coequalizer* (or *is rc*) if for any 0-cell $\mathcal{X}$ of $\mathcal{A}$ the category $\mathcal{A}(\mathcal{X},\mathcal{C})$ has coequalizers of reflexive pairs of morphisms and for any 1-cell $H : \mathcal{Y} \to \mathcal{X}$ the functor

$$\mathcal{A}(H,\mathcal{C}) : \mathcal{A}(\mathcal{X},\mathcal{C}) \longrightarrow \mathcal{A}(\mathcal{Y},\mathcal{C})$$

preserves them.
A 1-cell $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{A}$ *preserves reflexive coequalizers* (or *is rc*) if for any $\mathcal{X}$ in $\mathcal{A}$ the functor

$$\mathcal{A}(\mathcal{X},F) : \mathcal{A}(\mathcal{X},\mathcal{C}) \longrightarrow \mathcal{A}(\mathcal{X},\mathcal{D})$$

preserves coequalizers of reflexive pairs of morphisms.

Extensions of representations

In a 2-category $\mathcal{A}$:

$\mathbf{r}$ representation 1-cell (faithful, conservative), i.e.

$\mathcal{C}$ - 'abstract', $\mathcal{M}$ - 'concrete' and rc.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\mathbf{r}"'] \\ \mathcal{M} \end{tikzcd}$$

1.
2.

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$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "\mathbf{r}"'] \\ \mathcal{M} \arrow[u, shift right, "U"'] \end{tikzcd} \qquad \mathbf{r} \dashv U$$

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$$\begin{tikzcd} \mathcal{C} \arrow[loop left, "\mathcal{R}"] \arrow[d, shift right, "\mathbf{r}"'] \\ \mathcal{M} \arrow[u, shift right, "U"'] \end{tikzcd}$$

$\mathbf{r} \dashv U$

$\mathcal{R} = U\mathbf{r}$

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$$\begin{tikzcd}
\mathcal{C}_{\mathcal{R}} \arrow[r, shift right, "U_{\mathcal{R}}"'] & \mathcal{C} \arrow[l, shift right, "F_{\mathcal{R}}"'] \arrow[loop below, "\mathcal{R}"'] \arrow[d, "\mathbf{r}"] \\
 & \mathcal{M}
\end{tikzcd}$$

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 & \mathcal{M}
\end{tikzcd}
$$

1. $\dot{\mathbf{r}}$ is full and faithful;
2.

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& & \mathcal{M} \arrow[urr, "K"] & &
\end{tikzcd}$$

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& & \mathcal{M} \arrow[urr, shift left, "K"] & &
\end{tikzcd}$$

$$\ddot{\mathbf{r}} \dashv K$$

1. $\dot{\mathbf{r}}$ is full and faithful;
2. $\ddot{\mathbf{r}}$ is also expected to be full and faithful with $\mathcal{C}^{\mathcal{R}}$ keeping some nice properties of $\mathcal{C}$.

Kleisli and Eilenberg-Moore monoidal objects
$\mathcal{A}$ - 2-category with finite products.

$\mathbf{Mon}_l(\mathcal{A})$ - 2-category:

- 0-cells: monoidal objects in $\mathcal{A}$;
- 1-cells: lax monoidal 1-cells in $\mathcal{A}$;
- 2-cells: monoidal 2-cells in $\mathcal{A}$.

Kleisli and Eilenberg-Moore monoidal objects

Theorem

Let $\mathcal{R}$ be an rc-lax monoidal monad on an rc-monoidal category $(C, \otimes)$ in $\mathcal{A}$ ($\mathcal{C}$ and $\otimes$ are rc). If $\mathcal{R}$ admits both Kleisli and Eilenberg-Moore objects as a monad in $\mathcal{A}$, then $\mathcal{R}$ admits both Kleisli and Eilenberg-Moore objects in $\mathbf{Mon}_l(\mathcal{A})$ and they are both standard. The tensor in $\mathcal{C}^{\mathcal{R}}$ is given by F. Linton formula.

$$
\begin{tikzcd}
\mathcal{C}_{\mathcal{R}} & \overset{(\otimes,I)}{\mathcal{C}} \arrow[l, "F_{\mathcal{R}}"'] \arrow[loop below, "\mathcal{R}"] & \mathcal{C}^{\mathcal{R}} \arrow[l, "U^{\mathcal{R}}"]
\end{tikzcd}
$$

Contributors to this result: F. Linton (1969), R. Guitar, I. Moerdijk, P. McCrudden, S. Szawiel, Z, G. Seal.

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$(\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho)$ rc-monoidal object in $\mathcal{A}$ ($\mathcal{C}$ and $\otimes$ are rc),

$\mathcal{X}$ an exponentiable rc-0-cell in $\mathcal{A}$, and

$$\mathcal{C} \times \mathcal{X} \xrightarrow{(\star, \psi)} \mathcal{X}$$

(strong) action of $\mathcal{C}$ on $\mathcal{X}$. By exponential adjunction we get a strong monoidal representation

$$\mathcal{C} \xrightarrow{(\mathbf{r}, \phi)} \mathcal{X}^{\mathcal{X}}$$

which can have a (lax) monoidal right adjoint ($\mathbf{r} \dashv U$)

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathbf{r}"'] & \mathcal{X}^{\mathcal{X}} \arrow[l, shift left, "U"'] \end{tikzcd}$$

inducing a lax monoidal monad $\mathcal{R} = U\mathbf{r}$ on $\mathcal{C}$.

Thus we have a situation

$$
\begin{tikzcd}
\underset{(\dot{\otimes}, \dot{I})}{\mathcal{C}_{\mathcal{R}}} \arrow[rr, shift right, "U_{\mathcal{R}}"'] \arrow[drr, "\dot{\mathbf{r}}"'] & & \overset{(\otimes, I)}{\mathcal{C}} \arrow[ll, shift right, "F_{\mathcal{R}}"'] \arrow[rr, shift left, "F^{\mathcal{R}}"] \arrow[d, shift right, "\mathbf{r}"'] \arrow[loop below, "\mathcal{R}"] & & \underset{(\ddot{\otimes}, \ddot{I})}{\mathcal{C}^{\mathcal{R}}} \arrow[ll, shift left, "U^{\mathcal{R}}"] \arrow[dll, shift right, "\ddot{\mathbf{r}}"'] \\
& & \mathcal{X}^{\mathcal{X}} \arrow[u, shift right, "U"'] \arrow[urr, shift right, "K"'] & &
\end{tikzcd}
$$

If free $\otimes$-monoids exist i.e. $\mathcal{U}^{\otimes}$ has a left adjoint, then the induced monad $\mathcal{T}^{\otimes}$ on $\mathcal{C}$ distributes over $\mathcal{R}$, i.e. we have a distributive law:

$$\kappa : \mathcal{T}^{\otimes}\mathcal{R} \longrightarrow \mathcal{R}\mathcal{T}^{\otimes}$$

Kleisli and Eilenberg-Moore monoidal objects

Thus we have a situation

$$
\begin{tikzcd}
\mathbf{mon}(\mathcal{C}_{\mathcal{R}}, \dot{\otimes}) \arrow[rr, shift right, "\widetilde{U_{\mathcal{R}}}"'] \arrow[dddd, "\mathcal{U}^{\dot{\otimes}}"'] \arrow[drrr, "\tilde{\dot{\mathbf{r}}}"] & & \mathbf{mon}(\mathcal{C}, \otimes) \arrow[ll, shift right, "\widetilde{F_{\mathcal{R}}}"'] \arrow[rr, shift left, "\widetilde{F^{\mathcal{R}}}"] \arrow[dr, shift right, "\tilde{\mathbf{r}}"'] \arrow[dddd, "\mathcal{U}^{\otimes}"'] & & \mathbf{mon}(\mathcal{C}^{\mathcal{R}}, \ddot{\otimes}) \arrow[ll, shift left, "\widetilde{U^{\mathcal{R}}}"] \arrow[dl, shift right, "\tilde{\ddot{\mathbf{r}}}"'] \arrow[dddd, "\mathcal{U}^{\ddot{\otimes}}"] \\
& & & \mathbf{mon}(\mathcal{X}^{\mathcal{X}}) \arrow[ul, shift right] \arrow[ur, shift right, "\widetilde{K}"'] \arrow[dddd, "\mathcal{U}"] & \\
& & & & \\
& & & & \\
\underset{(\dot{\otimes}, \dot{I})}{\mathcal{C}_{\mathcal{R}}} \arrow[rr, shift right, "U_{\mathcal{R}}"'] \arrow[drrr, "\dot{\mathbf{r}}"'] & & \overset{(\otimes, I)}{\mathcal{C}} \arrow[ll, shift right, "F_{\mathcal{R}}"'] \arrow[rr, shift left, "F^{\mathcal{R}}"] \arrow[loop below, "\mathcal{R}"] \arrow[dr, shift right, "\mathbf{r}"'] & & \underset{(\ddot{\otimes}, \ddot{I})}{\mathcal{C}^{\mathcal{R}}} \arrow[ll, shift left, "U^{\mathcal{R}}"] \arrow[dl, shift right, "\ddot{\mathbf{r}}"'] \\
& & & \mathcal{X}^{\mathcal{X}} \arrow[ul, shift right, "U"'] \arrow[ur, shift right, "K"'] &
\end{tikzcd}
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Examples

Analytic and polynomial endofunctors on slices of *Set*

2-category $\mathbf{Fib}_{/Set}$:

- 0-cells: fibrations over *Set*,
- 1-cells: functors commuting over the base (Fact of life: substitution tensor is NOT cartesian!),
- 2-cells: vertical natural transformations.

Burroni fibration of signatures

$$
\begin{tikzcd}
Gph(\mathbf{M}) \arrow[d, "p_{\mathbf{M}}"'] \\
Set
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
O \arrow[rr, phantom, ""] & \arrow[d, mapsto] & \mathbf{M}(O) \\
 & O &
\end{tikzcd}
$$

The tautologous action of Burroni fibration of signatures

$$\begin{tikzcd}
Gph(\mathbf{M}) \times_{Set} Set^{\to} \arrow[rr, "\star"] \arrow[dr] & & Set^{\to} \arrow[dl, "cod"] \\
& Set &
\end{tikzcd}$$

is defined on objects by

$$\begin{tikzcd}
& A \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & & X \arrow[d, "d"] & & A \star X \arrow[d] \\
O & & \mathbf{M}(O) & O & \longmapsto & O
\end{tikzcd}$$

where the right vertical arrow in the above diagram is the composite of the upper horizontal arrows in the following diagram

$$\begin{tikzcd}
O & A \arrow[l, "\gamma"'] \arrow[d, "\delta"'] & A \star X \arrow[l] \arrow[d] \\
& \mathbf{M}(O) & \mathbf{M}(X) \arrow[l, "\mathbf{M}(d)"]
\end{tikzcd}$$

in which the square is a pullback.

By the exponential adjunction, we get a strong monoidal morphism of (lax) monoidal fibrations

$$\begin{tikzcd}
Gph(\mathbf{M}) \arrow[rr, "\mathbf{r}"] \arrow[dr, "p_{\mathbf{M}}"'] & & Exp(Set^{\to}) \arrow[dl, "p_{exp}"] \\
& Set &
\end{tikzcd}$$

with **r** conservative but not full even on isomorphisms. **r** has a right adjoint U (in $\mathbf{Fib}_{/Set}$) and the induced monad $\mathcal{F}$ is

$$\mathcal{F}(A,\partial)_n = \coprod_{m\in\omega,\, a\in A_m} \mathbb{F}(\underline{m},\underline{n}) \times A_m$$

for a signature (A,∂) in $Grph(\mathbf{M})_O$, $n \in \omega$, $\underline{n} = \{1,\ldots,n\}$.

Examples: Joyal

The monad $\mathcal{F}$ has various submonads including symmetrization submonad $\mathcal{S}$ related to subcategory $\mathbb{B}$ (of finite sets and bijections) of $\mathbb{F}$.

$$\mathcal{S}(A,\partial)_n = \coprod_{m\in\omega,\, a\in A_m} \mathbb{B}(\underline{m},\underline{n}) \times A_m = S_n \times A_n$$

S_n -symmetric group.
This monad gives a finer extension of the representation on the category of signatures. It gives rise to polynomial (finitary) functors with cartesian natural transformations as Kleisli extension and analytic (finitary) functor with weakly cartesian natural transformations as Eilenberg-Moore fibration.

More Examples

1. Fiore-Gambino-Hyland-Winskel:
   1. need a modification of Burroni fibrations over *Cat*;
      span = two-sided discrete fibration;
   2. the monad for strict monoidal categories (preserves (discrete) fibrations and opfibrations);
   3. rest of the story similar to the above.
2. Batanin-like context: take the Burroni fibration for the strict ω-category monad over the category of ω-graphs.
   NB. Without additional modifications the notion of analytic functor does not add anything new as the representation is already full on isomorphisms.
3. Kock-Gambino: diagrams (defining polynomial functors) in a lcc category $\mathcal{C}$ form a fibration over $\mathcal{C}$ that acts on basic fibration over $\mathcal{C}$. We get a representation by an exponential adjoint...

Thank You for Your Attention!