

Waves and Total Distributivity

RJ Wood

Mathematics, Dalhousie, Halifax, Canada

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Abstract

For **total** \mathcal{K} with defining adjunction $\bigvee \dashv Y: \mathcal{K} \rightarrow \widehat{\mathcal{K}} = \mathbf{set}^{\mathcal{K}^{\text{op}}}$ the **wave functor** $W: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ is well-defined by

$W(A)(K) = \widetilde{\mathcal{K}}(K, A) = \mathbf{set}^{\widetilde{\mathcal{K}}}(\mathcal{K}(A, \bigvee -), [K, -])$ where $[K, -]$ is evaluation at K . Also there are natural transformations $\beta: W \bigvee \rightarrow 1_{\widehat{\mathcal{K}}}$ and $\gamma: \bigvee W \rightarrow 1_{\mathcal{K}}$ satisfying $\bigvee \beta = \gamma \bigvee$ and $\beta W = W \gamma$. A total \mathcal{K} is **totally distributive** if \bigvee has a left adjoint. \mathcal{K} is totally distributive iff γ is invertible iff $W \dashv \bigvee$.

For total \mathcal{K} there is an associative composition of waves. Composition becomes an arrow $\bullet: \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}}$. Also there is an augmentation $(-): \widetilde{\mathcal{K}}(-, -) \rightarrow \mathcal{K}(-, -)$ corresponding to a natural $\delta: W \rightarrow Y$ constructed via β . We will show that if \mathcal{K} is totally distributive then $\bullet: \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}}$ is invertible and then $\widetilde{\mathcal{K}}$ supports an idempotent comonad structure. In fact, $\widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} = \widetilde{\mathcal{K}} \circ_{\widetilde{\mathcal{K}}} \widetilde{\mathcal{K}}$ so that \bullet is the coequalizer of $\bullet_{\mathcal{K}}$ and $\mathcal{K} \bullet$ making $\widetilde{\mathcal{K}}$ a **taxon** in the sense of Koslowski [KOS]. For a small taxon \mathcal{T} , the category of interpolative modules $\mathbf{iMod}(\mathbf{1}, \mathcal{T})$ is totally distributive [MRW]. Here we show, for totally distributive \mathcal{K} , there is an equivalence $\mathcal{K} \rightarrow \mathbf{iMod}(\mathbf{1}, \widetilde{\mathcal{K}})$.

References

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Total categories

Locally small \mathcal{K} is **total** [S&W] if $Y: \mathcal{K} \rightarrow \widehat{\mathcal{K}} = \mathbf{set}^{\mathcal{K}^{\text{op}}}$ has a left adjoint

- Write $\langle \eta, \epsilon: \bigvee \dashv Y: \mathcal{K} \rightarrow \widehat{\mathcal{K}} \rangle$ for such an adjunction
- For $P \in \widehat{\mathcal{K}}$, $\bigvee P = \text{colim}(\text{el} P \rightarrow \mathcal{K}) = \int^K [K, P] \cdot K$
- ($[K, P] = P(K)$) For $K \xrightarrow{x} P$, write $K \xrightarrow{i_x} \bigvee P$
- For $K \xrightarrow{x} P \xrightarrow{t} Q$,

$$\begin{array}{ccccc} K & \xrightarrow{i_x} & \bigvee P & \xrightarrow{\bigvee t} & \bigvee Q \\ & \searrow & & \nearrow & \\ & & i_{tx} & & \end{array}$$

- For $K \xrightarrow{f} A_* = \mathcal{K}(-, A)$, that is $K \xrightarrow{f} A$ in \mathcal{K}

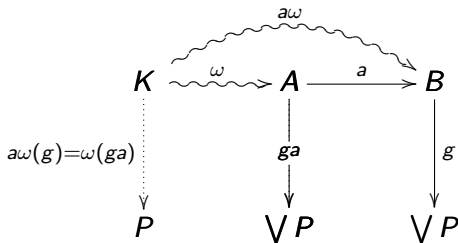
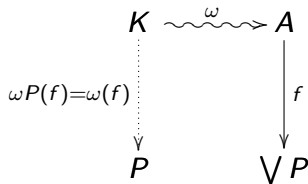
$$\begin{array}{ccccc} K & \xrightarrow{i_f} & \bigvee A_* & \xrightarrow{\epsilon A} & A \\ & \searrow & & \nearrow & \\ & & f & & \end{array}$$

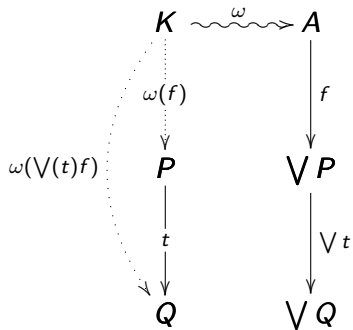
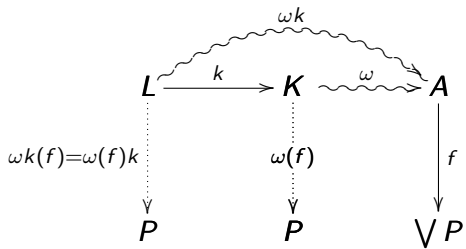
Total \mathcal{K} is **totally distributive** [R&W] if \bigvee has a left adjoint

Waves

For total \mathcal{K} , there is a $W: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ given by

$W(A)(K) = \mathbf{set}^{\widehat{\mathcal{K}}}(\mathcal{K}(A, \bigvee -), [K, -]) =: \widetilde{\mathcal{K}}(K, A)$ the **waves**
(wavy arrows[J&J]) from K to A





$$\begin{array}{ccc}
 K & \xrightarrow{\omega} & A \\
 \downarrow \bar{\omega} = \omega(\epsilon^{-1}A) & & \downarrow \epsilon^{-1}A \\
 A_* & & \bigvee A_*
 \end{array}$$

defines $\overline{(-)}: \widetilde{\mathcal{K}}(-, -) \rightarrow \mathcal{K}(-, -)$, equivalently $\delta: W \rightarrow Y$

LEMMA: For any wave $\omega: K \rightsquigarrow A$ in a total \mathcal{K} , any P in $\widehat{\mathcal{K}}$, and any $f: A \rightarrow \bigvee P$ in \mathcal{K} ,

$$\begin{array}{ccc}
 K & \xrightarrow{\bar{\omega}} & A \\
 \downarrow \omega(f) & & \downarrow f \\
 P & \xrightarrow{\eta P} & (\bigvee P)_*
 \end{array}$$

Proof:

$$\begin{array}{c}
 \begin{array}{ccccc}
 K & \xrightarrow{\omega} & A & & \\
 \downarrow \omega(f) & & \downarrow f & \searrow \epsilon^{-1}A & \\
 \omega(\bigvee \eta P.f)P & & \bigvee P & & \bigvee A_* \\
 \downarrow \eta P & & \downarrow \bigvee \eta P & \searrow \epsilon^{-1} \bigvee P & \\
 (\bigvee P)_* & & \bigvee (\bigvee P)_* & &
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{ccccc}
 K & \xrightarrow{\omega} & A & & \\
 \downarrow \omega & & \downarrow \epsilon^{-1}A & & \\
 A_* & & \bigvee A_* & & \\
 \downarrow f_* & & \downarrow \bigvee f_* & & \\
 (\bigvee P)_* & & \bigvee (\bigvee P)_* & &
 \end{array}
 \end{array}$$

$\omega(\bigvee \eta P.f)P$ $\omega(\bigvee f_*. \epsilon^{-1}A)$

□

LEMMA: For total \mathcal{K} , there are

$$\beta: W \vee \rightarrow 1_{\widehat{\mathcal{K}}} \quad \text{and} \quad \gamma: \vee W \rightarrow 1_{\mathcal{K}}$$

where $\beta P(K) = \beta(K, P): \widetilde{\mathcal{K}}(K, \vee P) \rightarrow [K, P]$ is given by

$$\beta(K, P)(\omega: K \rightsquigarrow \vee P) = \omega(1_{\vee P}): K \multimap P$$

and

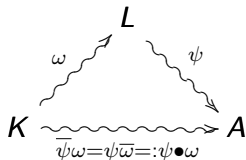
$$\begin{array}{ccc} \vee W & \xrightarrow{\gamma} & 1_{\mathcal{K}} \\ & \searrow \vee \delta & \nearrow \epsilon \\ & \vee Y & \end{array}$$

satisfying $\vee \beta = \gamma \vee$ and $\beta W = W \gamma$

THEOREM \mathcal{K} is totally distributive iff γ is invertible iff $W \dashv \vee$

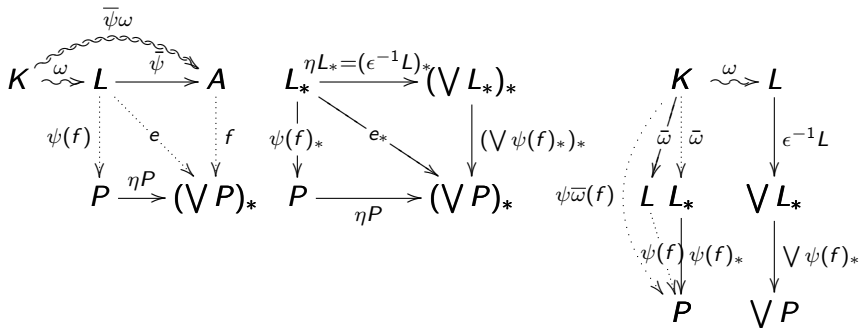
- For \mathcal{K} totally distributive, write $\alpha = \gamma^{-1}$

PROPOSITION: For total \mathcal{K} ,



provides an associative composition for $\widetilde{\mathcal{K}}(-, -)$

Proof



□

- For total \mathcal{K} , $\overline{(-)}: \widetilde{\mathcal{K}} \rightarrow \mathcal{K}$ is an identity-on-objects semifunctor
- For total \mathcal{K} , $\widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet} \widetilde{\mathcal{K}}$
(for $K \xrightarrow{\omega} L \xrightarrow{f} M \xrightarrow{\psi} A$, $\psi f \bullet \omega = \psi \bullet f \omega$)

THM For totally distributive \mathcal{K} , $\bullet: \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}}$ is invertible

Proof
$$W \bigvee WA \xrightarrow{W\gamma A} WA$$

$$\widetilde{\mathcal{K}}(-, \bigvee WA) \xrightarrow{\widetilde{\mathcal{K}}(\underset{\simeq}{-}, \gamma A)} \widetilde{\mathcal{K}}(-, A)$$

$$\widetilde{\mathcal{K}}(-, \int^L L \cdot \widetilde{\mathcal{K}}(L, A)) \xrightarrow{\widetilde{\mathcal{K}}(\underset{\simeq}{-}, \gamma A)} \widetilde{\mathcal{K}}(-, A)$$

$$\int^L \widetilde{\mathcal{K}}(-, L) \cdot \widetilde{\mathcal{K}}(L, A) \xrightarrow{\simeq} \widetilde{\mathcal{K}}(-, \bigvee WA) \xrightarrow{\widetilde{\mathcal{K}}(\underset{\simeq}{-}, \gamma A)} \widetilde{\mathcal{K}}(-, A)$$

$$\int^L \widetilde{\mathcal{K}}(K, L) \times \widetilde{\mathcal{K}}(L, A) \xrightarrow{\simeq} \widetilde{\mathcal{K}}(K, \bigvee WA) \xrightarrow{\widetilde{\mathcal{K}}(K, \epsilon A \cdot \bigvee \delta A)} \widetilde{\mathcal{K}}(K, A)$$

$$\widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \cong \widetilde{\mathcal{K}}$$

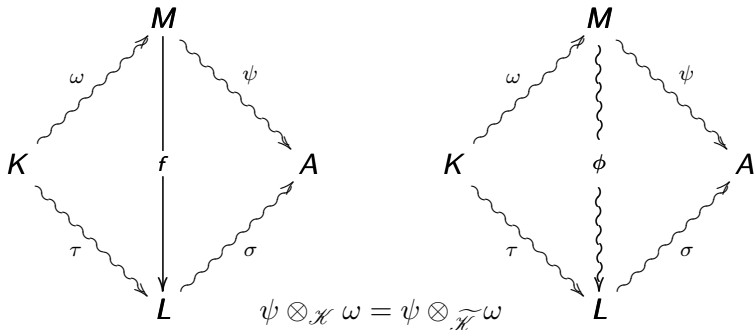
$$[K \overset{\psi}{\rightsquigarrow} L \overset{\chi}{\rightsquigarrow} A] \mapsto K \overset{\psi}{\rightsquigarrow} L \xrightarrow{i_{\chi}} \bigvee WA$$

$$\mapsto K \overset{\psi}{\rightsquigarrow} L \xrightarrow{i_{\overline{\chi}}} \bigvee YA \xrightarrow{\epsilon A} A = K \overset{\psi}{\rightsquigarrow} L \xrightarrow{\overline{\chi}} A = K \overset{\chi \bullet \psi}{\rightsquigarrow} A$$

$$\text{So } \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet} \widetilde{\mathcal{K}} \text{ and } (\widetilde{\mathcal{K}}, \bullet^{-1}, \overline{(-)}) \text{ idempotent comonad}$$

$$\begin{array}{ccc} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} & \begin{array}{c} \xrightarrow{\rho \widetilde{\mathcal{K}}} \\ \xrightarrow{\widetilde{\mathcal{K}} \lambda} \end{array} & \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \\ & \searrow \bullet & \downarrow \bullet \\ & & \widetilde{\mathcal{K}} \end{array}$$

$$\begin{array}{ccc} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} & \begin{array}{c} \xrightarrow{\rho \widetilde{\mathcal{K}}} \\ \xrightarrow{\widetilde{\mathcal{K}} \lambda} \end{array} & \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet} \widetilde{\mathcal{K}} \end{array}$$



“ \supseteq ” holds for total \mathcal{K} , “=” holds for totally distributive \mathcal{K}

$$\begin{array}{c} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet \widetilde{\mathcal{K}}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \longrightarrow \widetilde{\mathcal{K}} \circ_{\widetilde{\mathcal{K}}} \widetilde{\mathcal{K}} = \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \\ \xrightarrow{\widetilde{\mathcal{K}} \bullet} \end{array}$$

$$\begin{array}{c} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet \widetilde{\mathcal{K}}} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} \xrightarrow{\bullet} \widetilde{\mathcal{K}} \\ \xrightarrow{\widetilde{\mathcal{K}} \bullet} \end{array}$$

So $(|\mathcal{K}|, \widetilde{\mathcal{K}}, \bullet)$ **taxon** [KOS]

An *i*-module $P: \mathbf{1} \rightarrow \widetilde{\mathcal{K}}$ is a **set**-valued matrix (array) $P: \mathbf{1} \rightarrow |\mathcal{K}|$ together with an associative action $\lambda: \widetilde{\mathcal{K}} P \rightarrow P$ for which

$$\begin{array}{ccc} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} P & \xrightarrow[\widetilde{\mathcal{K}} \lambda]{\bullet P} & \widetilde{\mathcal{K}} P \xrightarrow{\lambda} P \end{array}$$

is a coequalizer

- Write $\mathbf{iMod}(\mathbf{1}, \widetilde{\mathcal{K}})$ for the category of such i-modules and equivariant (matrix) arrows
- Define $J: \mathbf{iMod}(\mathbf{1}, \widetilde{\mathcal{K}}) \rightarrow \mathbf{Prof}(\mathbf{1}, \mathcal{K}) = \widehat{\mathcal{K}}$ by $J(P, \lambda) = (P, \lambda_1)$ where

$$\begin{array}{ccccc} \mathcal{K} \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} P & \xrightarrow[\mathcal{K} \widetilde{\mathcal{K}} \lambda]{\mathcal{K} \bullet P} & \mathcal{K} \widetilde{\mathcal{K}} P & \xrightarrow{\mathcal{K} \lambda} & \mathcal{K} P \\ \downarrow @_{\mathcal{K} P} & & \downarrow @ P & & \downarrow \lambda_1 \\ \widetilde{\mathcal{K}} \widetilde{\mathcal{K}} P & \xrightarrow[\widetilde{\mathcal{K}} \lambda]{\bullet P} & \widetilde{\mathcal{K}} P & \xrightarrow{\lambda} & P \end{array}$$

THEOREM [MRW] For \mathcal{T} a small taxon, $\mathbf{iMod}(\mathbf{1}, \mathcal{T})$ is a totally distributive category □

THEOREM For \mathcal{K} totally distributive, the adjunction $\langle \alpha, \beta; W \dashv \bigvee : \mathbf{Prof}(\mathbf{1}, \mathcal{K}) \rightarrow \mathcal{K} \rangle$ gives rise to an adjoint equivalence $\langle \tilde{\alpha}, \tilde{\beta}; \tilde{W} \dashv \tilde{\bigvee} : \mathbf{iMod}(\mathbf{1}, \tilde{\mathcal{K}}) \rightarrow \mathcal{K} \rangle$:

$$\begin{array}{ccc}
 & \mathbf{iMod}(\mathbf{1}, \tilde{\mathcal{K}}) & \\
 \tilde{W} \nearrow & & \searrow \tilde{\bigvee} \\
 & \mathcal{K} & \\
 W \searrow & & \nearrow \bigvee \\
 & \mathbf{Prof}(\mathbf{1}, \mathcal{K}) = \widehat{\mathcal{K}} &
 \end{array}
 \quad
 \begin{array}{c}
 \downarrow J
 \end{array}$$

Proof: It is straightforward that W factors as $J\tilde{W}$ since each $WA = \tilde{\mathcal{K}}(-, A)$ is an i-module. We define $\tilde{\bigvee} = \bigvee J$. Clearly, $\tilde{\alpha} = \alpha : \mathbf{1}_{\mathcal{K}} \rightarrow \bigvee W = \bigvee J\tilde{W} = \tilde{\bigvee}\tilde{W}$ is an isomorphism. Next, we require a $\tilde{\beta} : \tilde{W}\tilde{\bigvee} \rightarrow \mathbf{1}_{\mathbf{iMod}(\mathbf{1}, \tilde{\mathcal{K}})}$ for which we require, for each i-module (P, λ) , a $\tilde{\mathcal{K}}$ -equivariant

$\widetilde{\beta}(K, (P, \lambda)) : \widetilde{\mathcal{K}}(K, \bigvee(P, \lambda_1)) \rightarrow [K, (P, \lambda)]$. The requisite functions are provided by the $\beta(K, (P, \lambda_1)) : \widetilde{\mathcal{K}}(K, \bigvee(P, \lambda_1)) \rightarrow [K, (P, \lambda_1)]$ but these must be shown to be $\widetilde{\mathcal{K}}$ -equivariant:

$$\begin{aligned}\beta(\omega \bullet \psi) &= \beta(\omega \bar{\psi}) = (\omega \bar{\psi})(1) = \omega(1) \bar{\psi} = \beta(\omega) \bar{\psi} = (p\chi) \bar{\psi} \\ &= p(\chi \bar{\psi}) = p(\chi \bullet \psi) = (p\chi) \psi = \beta(\omega) \psi\end{aligned}$$

where we have used that (P, λ) is an \mathbf{i} -module to express $\beta(\omega)$ as $p\chi$. A similar expression for a typical K -element p of P is used to show $\widetilde{\beta}$ invertible. □

REMARK $\widetilde{\mathcal{K}}$ does not have a small set of objects, however it is locally small and by the last theorem $\mathbf{iMod}(\mathbf{1}, \widetilde{\mathcal{K}})$, a power object for $\widetilde{\mathcal{K}}$ as a taxon, is locally small. By the [S&W] definition of smallness based on the [F&S] Theorem, it is reasonable to say that $\widetilde{\mathcal{K}}$ is *small as a taxon* and attempt to clarify this set-theoretically.

References

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