

Categories of lax fractions

Lurdes Sousa

IPV / CMUC

June 18, 2015

Idempotent monads

Replete full reflective subcategories

Orthogonality

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Ordinary categories

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Order-enriched categories

Lax-idempotent monads
(KZ-monads)

KZ-monadic subcategories

Kan-injectivity

[Carvalho, S. , 2011]

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A is **Kan-injective** wrt $h : X \rightarrow Y$ if

$\mathcal{X}(Y, A) \xrightarrow{\mathcal{X}(h, A)} \mathcal{X}(X, A)$ is a right adjoint retraction (in Pos).

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ a \downarrow & \swarrow & \\ A & & \end{array} \quad a/h = (\mathcal{X}(h, A))^*(a)$$

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$k : A \rightarrow B$ is **Kan-injective** wrt $h : X \rightarrow Y$, if A and B are so, and

$$\begin{array}{ccccc} \text{hom}(Y, A) & \xleftarrow{(\text{hom}(h, A))^*} & \text{hom}(X, A) & & A \\ \text{hom}(Y, k) \downarrow & & \downarrow \text{hom}(X, k) & & \downarrow k \\ \text{hom}(Y, B) & \xleftarrow{(\text{hom}(h, B))^*} & \text{hom}(X, B) & & B \end{array}$$

i.e., $(ka)/h = k(a/h)$, for all $a : X \rightarrow A$.

[CS, 2011]

For $\mathcal{H} \subseteq \text{Mor}(\mathcal{X})$,

$\underbrace{\text{KInj}(\mathcal{H})}_{\text{Kan-injective subcategory}} :=$ subcategory of objs. and mors. Kan-injective wrt all $h \in \mathcal{H}$

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For a subcategory \mathcal{A} of \mathcal{X} ,

$\mathcal{A}^{\text{KInj}} :=$ class of all morphisms wrt to which \mathcal{A} is Kan-injective

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- ① A subcategory \mathcal{A} of \mathcal{X} is KZ-monadic, iff it is reflective

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad \top \quad} & \mathcal{X} \\ & \xleftarrow{\quad R \quad} & \end{array}$$

with $R\eta \leq \eta R$, and \mathcal{A} is closed under left adjoint retractions

(i.e., if $A \xrightarrow{f \in \mathcal{A}} B$ with e and e' l. a. r. then $g \in \mathcal{A}$).

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- ② If \mathcal{A} is a KZ-monadic subcategory of \mathcal{X} , then:

- $\mathcal{A}^{\text{KInj}} = \{f \in \mathcal{X} \mid Rf \text{ left adj. section in } \mathcal{A}\}$
 $= \{f \in \mathcal{X} \mid Rf \text{ left adj. section in } \mathcal{X}\} = R\text{-embeddings}$
- $\mathcal{A} = \text{KInj} \{ \eta_X \mid X \in \mathcal{X} \} = \text{KInj}(\mathcal{A}^{\text{KInj}})$

[CS, 2011]

Examples

\mathcal{X}	Objs. of a KZ-monadic subcategory \mathcal{A}	$\mathcal{A}^{\text{KInj}}$	Injectivity of the objects of \mathcal{A}
Top_0	continuous lattices	embeddings	[Scott, 1972]
Top_0	continuous Scott domains	dense embeddings	[Scott, 1980]
Loc	stably locally compact locales (=retracts of coherent locales)	flat embeddings	[Johnstone, 1981]

In all three examples, $\mathcal{A}^{\text{KInj}}$ may be replaced with \mathcal{A} a finite subcategory.
[Carvalho, S., preprint]

\mathcal{A} full reflective subcat. $\Rightarrow \mathcal{A}$ is a category of fractions for
 $\mathcal{A}^{\text{Orth}} = \{f \mid Rf \text{ is an iso}\}, \text{ up to equiv.}$

$$\Sigma \subseteq \text{Mor}(\mathcal{X})$$

Category of fractions: $F : \mathcal{X} \rightarrow \mathcal{X}[\Sigma^{-1}]$

Category of “lax fractions”: $F : \mathcal{X} \rightarrow \mathcal{X}[\Sigma_*]$

$$(Fs)_* \cdot Fs = \text{id} \quad \text{and} \quad Fs \cdot (Fs)_* \leq \text{id}$$

for all $s \in \Sigma$

{ \mathcal{A} any subcat., \mathcal{X} cocomplete $\Rightarrow \mathcal{A}^{\text{Orth}}$ is closed under colimits in $\mathcal{X}^{\rightarrow}$
 $\Rightarrow \mathcal{A}^{\text{Orth}}$ admits a left calculus of fractions

$$\mathcal{A}^{\text{Orth}} \xrightarrow[\text{full}]{} \mathcal{X}^{\rightarrow}$$

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Theorem

If \mathcal{X} has weighted colimits, then $\mathcal{A}^{\text{KInj}}$ is closed under weighted colimits in $\mathcal{X}^{\rightarrow}$.

$$\mathcal{A}^{\text{Klnj}} \hookrightarrow \mathcal{X}^{\rightarrow}$$

The morphisms of $\mathcal{A}^{\text{Klnj}}$: $(f, g) : h \rightarrow j$ with $(af)/h = (a/j)g$

for every

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 f \downarrow & & \downarrow g \\
 Z & \xrightarrow{j} & W \\
 a \downarrow & \swarrow a/j & \\
 \mathcal{A} \ni A & & \xleftarrow{(af)/h}
 \end{array}$$

Equivalently:

$$\begin{array}{ccc}
 \text{hom}(X, A) & \xrightarrow{(\text{hom}(h, A))^*} & \text{hom}(Y, A) \\
 \uparrow \text{hom}(f, A) & & \uparrow \text{hom}(g, A) \\
 \text{hom}(Z, A) & \xrightarrow{(\text{hom}(j, A))^*} & \text{hom}(W, A)
 \end{array}
 \quad \text{commutes.}$$

Definition

A **category of lax fractions** for Σ

consists of a category $\mathcal{X}[\Sigma_*]$ and a functor $F : \mathcal{X} \rightarrow \mathcal{X}[\Sigma_*]$ such that:

- ① The functor F satisfies the conditions:
 - (a) $F(s)$ is a left adjoint section, for all $s \in \Sigma$;

Definition

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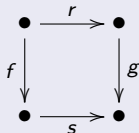
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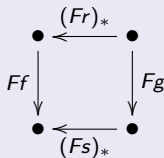
① The functor F satisfies the conditions:

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(b) For every square



Σ , the following diagram is commutative:



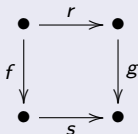
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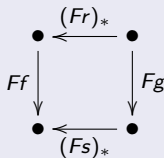
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(b) For every square



Σ , the following diagram is commutative:



② If $G : \mathcal{X} \rightarrow \mathcal{C}$ is another functor under the above conditions, then there is a unique functor $H : \mathcal{X}[\Sigma_*] \rightarrow \mathcal{C}$ such that $HF = G$.

Theorem

Let \mathcal{A} be a KZ-monadic subcategory of \mathcal{X} , with reflector $R : \mathcal{X} \rightarrow \mathcal{A}$.

\mathcal{X} has a category of lax fractions $F : \mathcal{X} \rightarrow \mathcal{X}[\Sigma_*]$ for $\Sigma = \mathcal{A}^{\text{Klnj}}$.

The unique functor H with $HF = R$ is full and faithful,

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{X}[\Sigma_*] \\ R \downarrow & \swarrow H & \\ \mathcal{A} & & \end{array}$$

and, for every $f : A \rightarrow B$ of \mathcal{A} , there is some $g : X \rightarrow Y$ in $\mathcal{X}[\Sigma_*]$ and left adjoint retractions e and e' making the diagram

$$\begin{array}{ccc} HX & \xrightarrow{Hg} & HY \\ e \downarrow & & \downarrow e' \\ A & \xrightarrow{f} & B \end{array}$$

commutative.

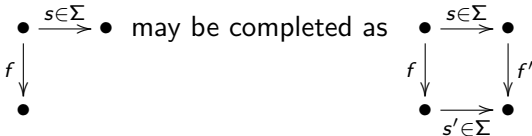
objects of $\mathcal{X}[\Sigma_*]$: all of \mathcal{X} ; morphisms of $\mathcal{X}[\Sigma_*)(X, Y)$: all of $\mathcal{A}(FX, FY)$

Ordinary case:

$\Sigma \subseteq \mathcal{X}$ admits a left calculus of fractions if

1. *Identity.* Σ contains the identities.
2. *Composition.* Σ is closed under composition.

3. *Square.* Every span $\bullet \xrightarrow{s \in \Sigma} \bullet$ may be completed as



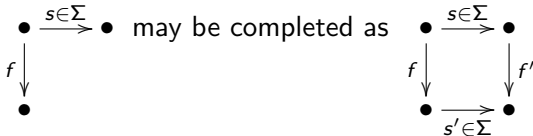
4. *Coequalisation.* For every $\bullet \xrightarrow{r \in \Sigma} \bullet \xrightleftharpoons[f]{g} \bullet$ with $gr = hr$, there is $t \in \Sigma$ with $tf = tg$.

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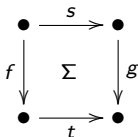
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Description of $\mathcal{X}[\Sigma^{-1}]$ by means of cospans:

$$A \xrightarrow{g} I \xleftarrow{s \in \Sigma} B \quad \text{as a formal representation of } s^{-1}g$$

Let Σ be a subcategory of \mathcal{X}^\rightarrow .

A square of the form



is going to mean that

s and t belong to Σ and $(f, g) : s \rightarrow t$ is a morphism of Σ

and will be called a Σ -square.

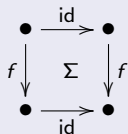
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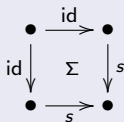
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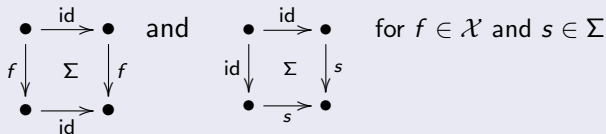


for $f \in \mathcal{X}$ and $s \in \Sigma$

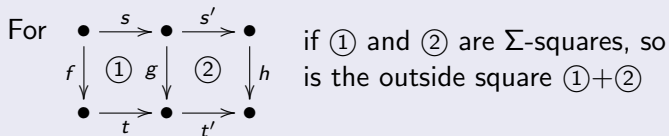
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2. Composition.



Definition

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1. *Identity.* $\begin{array}{ccc} \bullet & \xrightarrow{\text{id}} & \bullet \\ f \downarrow & \Sigma & \downarrow f \\ \bullet & \xrightarrow{\text{id}} & \bullet \end{array}$ and $\begin{array}{ccc} \bullet & \xrightarrow{\text{id}} & \bullet \\ \text{id} \downarrow & \Sigma & \downarrow s \\ \bullet & \xrightarrow{s} & \bullet \end{array}$ for $f \in \mathcal{X}$ and $s \in \Sigma$

2. *Composition.* For $\begin{array}{ccccc} \bullet & \xrightarrow{s} & \bullet & \xrightarrow{s'} & \bullet \\ f \downarrow & \textcircled{1} & g \downarrow & \textcircled{2} & \downarrow h \\ \bullet & \xrightarrow{t} & \bullet & \xrightarrow{t'} & \bullet \end{array}$ if $\textcircled{1}$ and $\textcircled{2}$ are Σ -squares, so is the outside square $\textcircled{1} + \textcircled{2}$

3. *Square.* Every span $\begin{array}{ccc} \bullet & \xrightarrow{s \in \Sigma} & \bullet \\ f \downarrow & & \\ \bullet & & \end{array}$ can be completed as $\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ f \downarrow & \Sigma & \downarrow f' \\ \bullet & \xrightarrow{s'} & \bullet \end{array}$

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4. *Coinsertion.* If $\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ f \downarrow & \Sigma & \downarrow g \\ \bullet & \xrightarrow{t} & \bullet \end{array}$ and $gs \leq hs$, then there is $\begin{array}{ccc} \bullet & \xrightarrow{t} & \bullet \\ \parallel & \Sigma & \downarrow u \\ \bullet & \xrightarrow{ut} & \bullet \end{array}$

with $ug \leq uh$

Remark

For \mathcal{A} a subcategory of \mathcal{X} , \mathcal{X} with weighted colimits,
 $\mathcal{A}^{\text{KInj}}$ admits a left calculus of lax fractions.

For $\bullet \xrightarrow{s \in \Sigma} \bullet$ form the pushout:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{s} & \bullet \\
 f \downarrow & \Sigma & \downarrow f' \\
 \bullet & \xrightarrow{s'} & \bullet
 \end{array}$$

For $\bullet \xrightarrow{s} \bullet$ with $gs \leq hs$, take $u = \text{coins}(g, h)$:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{s} & \bullet \\
 f \downarrow & \Sigma & \downarrow g \searrow h \\
 \bullet & \xrightarrow{t} & \bullet
 \end{array}
 \qquad
 \begin{array}{ccc}
 \bullet & \xrightarrow{t} & \bullet \\
 \parallel & \Sigma & \downarrow u \\
 \bullet & \xrightarrow{ut} & \bullet
 \end{array}$$

Objects of $\mathcal{X}[\Sigma_*]$: all of \mathcal{X}

$\mathcal{X}[\Sigma_*)(A, B)$: equivalence classes of Σ -cospans $A \xrightarrow{g} I \xleftarrow{s \in \Sigma} B$
(representing s_*g)

obtained as follows:

A category of lax fractions for Σ admitting a left calculus of lax fractions

Define a relation \lesssim between Σ -cospans from A to B by

$$(f, I, s) \lesssim (g, J, t)$$

if there is a diagram of the form

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & I & \xleftarrow{s} & B \\
 \parallel & & \downarrow x & \exists & \parallel \\
 & & X & \xleftarrow{xs=yt} & B \\
 \nearrow & & \uparrow y & \exists & \\
 A & \xrightarrow{g} & J & \xleftarrow{t} & B \\
 \parallel & & & & \parallel
 \end{array}$$

\lesssim is reflexive and transitive, then determines an equivalence relation \sim ,
and

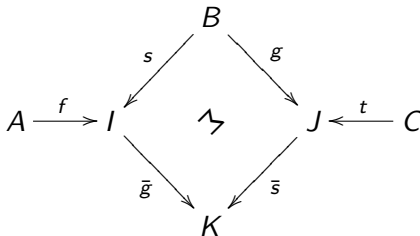
$$[(f, I, s)] \leq [(g, J, t)] \text{ if } (f, I, s) \lesssim (g, J, t)$$

Composition: Given two Σ -cospans

$$(f, I, s) : A \rightarrow B \quad \text{and} \quad (g, J, t) : B \rightarrow C,$$

$$[(g, J, t)] \cdot [(f, I, s)] = [(\bar{g}f, K, \bar{s}t)]$$

where \bar{g} and \bar{s} are obtained using a Σ -square:



$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{X}[\Sigma_*] \\ (A \xrightarrow{f} B) & \longmapsto & (A \xrightarrow{f} B \xleftarrow{\text{id}} B) \end{array}$$

Theorem

If Σ admits a left calculus of lax fractions, then

$F : \mathcal{X} \rightarrow \mathcal{X}[\Sigma_]$ defines a category of lax fractions for Σ .*