Categories of lax fractions

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Idempotent monads

Replete full reflective subcategories

Orthogonality

Categories of fractions

Ordinary categories	**Order-enriched categories**
Idempotent monads	Lax-idempotent monads (KZ-monads)
Replete full reflective subcategories	KZ-monadic subcategories
Orthogonality	Kan-injectivity
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$$\mathcal{X}(Y,A) \xrightarrow{\mathcal{X}(h,A)} \mathcal{X}(X,A) \quad \text{is a right adjoint retraction (in Pos).}$$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "a"'] & Y \arrow[dl, "a/h=(\mathcal{X}(h,A))^*(a)"] \\
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$k : A \to B$ is Kan-injective wrt $h : X \to Y$, if A and B are so, and

$$\begin{tikzcd} \hom(Y, A) \arrow[d, "\hom(Y,k)"'] & \hom(X, A) \arrow[l, "(\hom(h,A))^*"'] \arrow[d, "\hom(X,k)"] & A \arrow[d, "k"] \\ \hom(Y, B) & \hom(X, B) \arrow[l, "(\hom(h,B))^*"] & B \end{tikzcd}$$

i.e., $(ka)/h = k(a/h)$, for all $a : X \to A$. [CS, 2011]

For $\mathcal{H} \subseteq \mathsf{Mor}(\mathcal{X})$,

$\underbrace{\mathsf{KInj}(\mathcal{H})}_{\text{Kan-injective subcategory}}$:= subcategory of objs. and mors. Kan-injective wrt all $h \in \mathcal{H}$

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For a subcategory $\mathcal{A}$ of $\mathcal{X}$,

$\mathcal{A}^{\mathsf{KInj}}$:= class of all morphisms wrt to which $\mathcal{A}$ is Kan-injective

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1. A subcategory $\mathcal{A}$ of $\mathcal{X}$ is KZ-monadic, iff it is reflective

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, shift left=2] \arrow[r, phantom, "\top"] & \mathcal{X} \arrow[l, shift left=2, "R"] \end{tikzcd}$$

with $R\eta \leq \eta R$, and $\mathcal{A}$ is closed under left adjoint retractions

(i.e., if $\begin{tikzcd} A \arrow[r, "f\in\mathcal{A}"] \arrow[d, "e"'] & B \arrow[d, "e'"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$ with e and e' l. a. r. then $g \in \mathcal{A}$).

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2. If $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$, then:
   - $\mathcal{A}^{\mathsf{KInj}} = \{f \in \mathcal{X} \,|\, Rf \text{ left adj. section in } \mathcal{A}\}$
     $= \{f \in \mathcal{X} \,|\, Rf \text{ left adj. section in } \mathcal{X}\} = R\text{-embeddings}$
   - $\mathcal{A} = \mathsf{KInj}\,\{\eta_X \,|\, X \in \mathcal{X}\} = \mathsf{KInj}(\mathcal{A}^{\mathsf{KInj}})$

[CS, 2011]

$\mathcal{X}$	Objs. of a KZ-monadic subcategory $\mathcal{A}$	$\mathcal{A}^{\mathsf{KInj}}$	Injectivity of the objects of $\mathcal{A}$
Top_0	continuous lattices	embeddings	[Scott, 1972]
Top_0	continuous Scott domains	dense embeddings	[Scott, 1980]
Loc	stably locally compact locales (=retracts of coherent locales)	flat embeddings	[Johnstone, 1981]

In all three examples, $\mathcal{A}^{\mathsf{KInj}}$ may be replaced with $\mathcal{A}$ a finite subcategory.
[Carvalho, S., preprint]

$\mathcal{A}$ full reflective subcat. $\Rightarrow$ $\mathcal{A}$ is a category of fractions for $\mathcal{A}^{\mathsf{Orth}} = \{f \mid Rf \text{ is an iso}\}$, up to equiv.

$\Sigma \subseteq \mathsf{Mor}(\mathcal{X})$

Category of fractions: $\quad F : \mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$

Category of "lax fractions": $\quad F : \mathcal{X} \to \mathcal{X}[\Sigma_*]$

$$(Fs)_* \cdot Fs = \mathrm{id} \quad \text{and} \quad Fs \cdot (Fs)_* \leq \mathrm{id}$$

for all $s \in \Sigma$

$\mathcal{A}$ any subcat., $\mathcal{X}$ cocomplete $\Rightarrow \mathcal{A}^{\mathsf{Orth}}$ is closed under colimits in $\mathcal{X}^{\to}$
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For $\mathcal{A}^{\underline{\mathsf{KInj}}}$ with objects $\mathcal{A}^{\mathsf{KInj}}$ and convenient morphisms:

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Theorem

If $\mathcal{X}$ has weighted colimits, then $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is closed under weighted colimits in $\mathcal{X}^{\to}$.

$$\mathcal{A}^{\underline{\mathsf{KInj}}} \hookrightarrow \mathcal{X}^{\rightarrow}$$

The morphisms of $\mathcal{A}^{\underline{\mathsf{KInj}}}$: $\quad (f,g): h \to j \quad$ with $\quad (af)/h = (a/j)g$

for every

$$\begin{tikzcd}
 & X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[ddl, bend left, "(af)/h"] \\
 & Z \arrow[r, "j"'] \arrow[d, "a"'] & W \arrow[dl, "a/j"] \\
\mathcal{A} \ni & A &
\end{tikzcd}$$

Equivalently:

$$\begin{tikzcd}
\hom(X,A) \arrow[r, "(\hom(h,A))^*"] & \hom(Y,A) \\
\hom(Z,A) \arrow[u, "\hom(f,A)"] \arrow[r, "(\hom(j,A))^*"'] & \hom(W,A) \arrow[u, "\hom(g,A)"']
\end{tikzcd}$$

commutes.

Definition

A category of lax fractions for Σ
consists of a category $\mathcal{X}[\Sigma_*]$ and a functor $F : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ such that:

1. The functor F satisfies the conditions:
   (a) $F(s)$ is a left adjoint section, for all $s \in \Sigma$;

Definition

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1. The functor F satisfies the conditions:
   (a) $F(s)$ is a left adjoint section, for all $s \in \Sigma$;
   (b) For every square
   $$\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$
   with $(f, g) : r \to s$ a morphism of Σ, the following diagram is commutative:
   $$\begin{tikzcd} \bullet \arrow[d, "Ff"'] & \bullet \arrow[l, "(Fr)_*"'] \arrow[d, "Fg"] \\ \bullet & \bullet \arrow[l, "(Fs)_*"] \end{tikzcd}$$

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2. If $G : \mathcal{X} \to \mathcal{C}$ is another functor under the above conditions, then there is a unique fuctor $H : X[\Sigma_*] \to \mathcal{C}$ such that $HF = G$.

Theorem

Let $\mathcal{A}$ be a KZ-monadic subcategory of $\mathcal{X}$, with reflector $R : \mathcal{X} \to \mathcal{A}$.
$\mathcal{X}$ has a category of lax fractions $F : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ for $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$.
The unique functor H with $HF = R$ is full and faithful,

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] \arrow[d, "R"'] & \mathcal{X}[\Sigma_*] \arrow[dl, "H"] \\ \mathcal{A} & \end{tikzcd}$$

and, for every $f : A \to B$ of $\mathcal{A}$, there is some $g : X \to Y$ in $\mathcal{X}[\Sigma_*]$ and left adjoint retractions e and e' making the diagram

$$\begin{tikzcd} HX \arrow[r, "Hg"] \arrow[d, "e"'] & HY \arrow[d, "e'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutative.

objects of $\mathcal{X}[\Sigma_*]$: all of $\mathcal{X}$; morphisms of $\mathcal{X}[\Sigma_*](X, Y)$: all of $\mathcal{A}(FX, FY)$

Ordinary case:

$\Sigma \subseteq \mathcal{X}$ admits a left calculus of fractions if

1. *Identity.* Σ contains the identities.
2. *Composition.* Σ is closed under composition.
3. *Square.* Every span $\begin{tikzcd} \bullet \arrow[r, "s\in\Sigma"] \arrow[d, "f"'] & \bullet \\ \bullet \end{tikzcd}$ may be completed as $\begin{tikzcd} \bullet \arrow[r, "s\in\Sigma"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "s'\in\Sigma"'] & \bullet \end{tikzcd}$
4. *Coequalisation.* For every $\bullet \xrightarrow{r\in\Sigma} \bullet \overset{g}{\underset{f}{\rightrightarrows}} \bullet$ with $gr = hr$, there is

   $t \in \Sigma$ with $tf = tg$.

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4. *Coequalisation.* For every $\begin{tikzcd} \bullet \arrow[r, "r\in\Sigma"] & \bullet \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & \bullet \end{tikzcd}$ with $gr = hr$, there is $t \in \Sigma$ with $tf = tg$.

Description of $\mathcal{X}[\Sigma^{-1}]$ by means of cospans:

$A \xrightarrow{g} I \xleftarrow{s\in\Sigma} B$ as a formal representation of $s^{-1}g$

Let Σ be a subcategory of $\mathcal{X}^{\rightarrow}$.

A square of the form

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] & \bullet \end{tikzcd}$$

is going to mean that

$$s \text{ and } t \text{ belong to } \Sigma \text{ and } (f,g) : s \to t \text{ is a morphism of } \Sigma$$

and will be called a Σ-square.

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for $f \in \mathcal{X}$ and $s \in \Sigma$

2. *Composition.* For
$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "①"] & \bullet \arrow[r, "s'"] \arrow[d, "g"] \arrow[dr, phantom, "②"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "t"'] & \bullet \arrow[r, "t'"'] & \bullet \end{tikzcd}$$
if ① and ② are Σ-squares, so is the outside square ①+②

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3. *Square.* Every span $\begin{tikzcd} \bullet \arrow[r, "s \in \Sigma"] \arrow[d, "f"'] & \bullet \\ \bullet & \end{tikzcd}$ can be completed as $\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "s'"'] & \bullet \end{tikzcd}$

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4. *Coinsertion.* If
$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "g"'] \arrow[d, bend left, "h"] \\ \bullet \arrow[r, "t"'] & \bullet \end{tikzcd}$$
and $gs \le hs$, then there is
$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "u"] \\ \bullet \arrow[r, "ut"'] & \bullet \end{tikzcd}$$
with $ug \le uh$

Remark

For $\mathcal{A}$ a subcategory of $\mathcal{X}$, $\mathcal{X}$ with weighted colimits,
$\mathcal{A}^{\underline{\mathsf{KInj}}}$ admits a left calculus of lax fractions.

For
$$\begin{tikzcd} \bullet \arrow[r, "s\in\Sigma"] \arrow[d, "f"'] & \bullet \\ \bullet & \end{tikzcd}$$
form the pushout:
$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "s'"'] & \bullet \end{tikzcd}$$

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with $gs \leq hs$, take $u = \mathrm{coins}(g, h)$:
$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "u"] \\ \bullet \arrow[r, "ut"'] & \bullet \end{tikzcd}$$

Objects of $\mathcal{X}[\Sigma_*]$: all of $\mathcal{X}$

$\mathcal{X}[\Sigma_*](A,B)$: equivalence classes of Σ-cospans $A \xrightarrow{g} I \xleftarrow{s\in\Sigma} B$

(representing s_*g)

obtained as follows:

A category of lax fractions for Σ admitting a left calculus of lax fractions

Define a relation $\lesssim$ between Σ-cospans from A to B by

$$(f, I, s) \lesssim (g, J, t)$$

if there is a diagram of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "x"'] \arrow[dr, phantom, "\Sigma"] & B \arrow[l, "s"'] \arrow[d, equal] \\
\arrow[r, phantom, "\Nwarrow"] & X & B \arrow[l, "xs=yt"'] \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[u, "y"] \arrow[ur, phantom, "\Sigma"] & B \arrow[l, "t"]
\end{tikzcd}$$

$\lesssim$ is reflexive and transitive, then determines an equivalence relation $\sim$, and

$$[(f, I, s)] \leq [(g, J, t)] \text{ if } (f, I, s) \lesssim (g, J, t)$$

Composition: Given two Σ-cospans

$$(f, I, s) : A \to B \qquad \text{and} \qquad (g, J, t) : B \to C,$$

$$[(g, J, t)] \cdot [(f, I, s)] = [(\bar{g}f, K, \bar{s}t)]$$

where $\bar{g}$ and $\bar{s}$ are obtained using a Σ-square:

$$\begin{tikzcd}
 & & B \arrow[dl, "s"'] \arrow[dr, "g"] & & \\
A \arrow[r, "f"] & I \arrow[dr, "\bar{g}"'] & & J \arrow[dl, "\bar{s}"] & C \arrow[l, "t"'] \\
 & & K & &
\end{tikzcd}$$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad F \quad} & \mathcal{X}[\Sigma_*] \\ \left(A \xrightarrow{f} B\right) & \longmapsto & \left(A \xrightarrow{f} B \xleftarrow{\mathrm{id}} B\right) \end{array}$$

Theorem

If Σ admits a left calculus of lax fractions, then

$F : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ defines a category of lax fractions for Σ.