On monoidal (co)nuclei and their applications

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Outline

Monoidal topology

- *Monoidal topology (MT)* is a branch of categorical topology.
- MT is based in a *monad* ($\mathbb{T}$) and a *quantale* (V).
- MT studies the category $(\mathbb{T}, V)$-**Cat** of generalized topological structures and their respective structure-preserving maps.
- Examples of $(\mathbb{T}, V)$-**Cat** include the categories
  - **Set** (sets),
  - **Ord** (preordered sets),
  - **Met** (premetric spaces),
  - **ProbMet** (probabilistic metric spaces),
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Change-of-base functors

- Given a lax homomorphism of quantales $V_1 \xrightarrow{\varphi} V_2$, there exists the *change-of-base functor* $(\mathbb{T}, V_1)\text{-}\mathbf{Cat} \xrightarrow{B_\varphi} (\mathbb{T}, V_2)\text{-}\mathbf{Cat}$.
- This technique gives rise to the following pairs of functors:
  - $\mathbf{Ord} \to \mathbf{Set} \to \mathbf{Ord}$,
  - $\mathbf{Met} \to \mathbf{Ord} \to \mathbf{Met}$,
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Quantic (co)nuclei

Quantic (co)nuclei provide a convenient technique to obtain quotients of quantales (subquantales).

Theorem 1

Every quantic (co)nucleus $V \xrightarrow{h} V$ gives rise to a quantale $V_h = \{u \in V \mid h(u) = u\}$ and also a quantale homomorphism $V \xrightarrow{h} V_h$ ($V_h \xrightarrow{h} V$). Every surjective (injective) quantale homomorphism can be represented in this form.

Theorem 2 (Quantale representation theorem)

Every (unital) quantale V has a semigroup (monoid) S and a quantic nucleus j on the free quantale $\mathcal{P}(S)$ over S such that $V \cong \mathcal{P}(S)_j$.

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- A (unital) quantic (co)nucleus h, compatible with the monad $\mathbb{T}$, gives the change-of-base functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{B_h}$ $(\mathbb{T}, V)$-**Cat**.
- This talk presents the monoidal analogue of Theorem 1, replacing the quantale V with the category $(\mathbb{T}, V)$-**Cat**, and calling a compatible quantic (co)nucleus *monoidal (co)nucleus*.
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Quantales

Definition 3

A *quantale* V is a $\bigvee$-semilattice, equipped with an associative binary operation $V \times V \xrightarrow{\otimes} V$ (*multiplication*) such that $v \otimes (\bigvee S) = \bigvee_{s \in S}(v \otimes s)$ and $(\bigvee S) \otimes v = \bigvee_{s \in S}(s \otimes v)$ for every $v \in V$, $S \subseteq V$.

Definition 4

A quantale V is *unital* provided that its multiplication has a unit k.

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Quantale-valued relations

Definition 5

Given a unital quantale V, V-**Rel** is the category, whose objects are sets, and whose morphisms are *V-relations* $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, which are maps $X \times Y \xrightarrow{r} V$. The composition with a V-relation $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$ is defined by $(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$. Given a set X, the identity morphism 1_X on X is provided by the V-relation

$$1_X(x, y) = \begin{cases} k, & x = y \\ \bot_V := \bigvee \varnothing, & \text{otherwise.} \end{cases}$$

A V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ has the converse V-relation $Y \overset{r^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$, which is defined by $r^\circ(y, x) = r(x, y)$.

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Maps as relations

Proposition 6

1. *There exists a functor* **Set** $\xrightarrow{(-)_\circ}$ V-**Rel**, *which takes a map* $X \xrightarrow{f} Y$ *to the* V*-relation* $X \overset{f_\circ}{\longrightarrow\!\!\!\!\!+\,} Y$ *given by*

$$f_\circ(x,y) = \begin{cases} k, & f(x) = y \\ \bot_V, & \textit{otherwise.} \end{cases}$$

2. *If* $k \neq \bot_V$*, then* $(-)_\circ$ *is a non-full embedding.*

For the sake of simplicity, we will identify a map $X \xrightarrow{f} Y$ and its respective relation $X \overset{f_\circ}{\longrightarrow\!\!\!\!\!+\,} Y$, employing the notation f for both.

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Lax extensions of functors

V-**Rel** is a quantaloid, in which $\bigvee$ on hom-sets are given by the pointwise evaluation of maps.

Definition 7

Given a functor $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$, a *lax extension* $\hat{T}$ of T to V-**Rel** takes a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+} Y$ to a V-relation $TX \overset{\hat{T}r}{\longrightarrow\!\!\!\!\!\!\!+} TY$ such that

1. $r \leqslant s$ implies $\hat{T}r \leqslant \hat{T}s$ for every V-relations $X \underset{s}{\overset{r}{\rightrightarrows}} Y$;
2. $\hat{T}s \cdot \hat{T}r \leqslant \hat{T}(s \cdot r)$ for every V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+} Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+} Z$;
3. $Tf \leqslant \hat{T}f$ and $(Tf)^{\circ} \leqslant \hat{T}(f^{\circ})$ for every map $X \xrightarrow{f} Y$.

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3. $Tf \leqslant \hat{T}f$ and $(Tf)^\circ \leqslant \hat{T}(f^\circ)$ for every map $X \xrightarrow{f} Y$.

Lax extensions of monads

Definition 8

Given a monad $\mathbb{T} = (T, m, e)$ on **Set**, a *lax extension* $\hat{\mathbb{T}} = (\hat{T}, m, e)$ of $\mathbb{T}$ to V-**Rel** consists of a lax extension $\hat{T}$ of T to V-**Rel** such that $\hat{T}\hat{T} \xrightarrow{m} \hat{T}$ and $1_{V\text{-}\mathbf{Rel}} \xrightarrow{e} \hat{T}$ are *lax natural transformations*, i.e.,

$$\begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "\hat{T}\hat{T}r"', "|" marking] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "\hat{T}r", "|"' marking] \\ TTY \arrow[r, "m_Y"'] & TY \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"', "|" marking] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "\hat{T}r", "|"' marking] \\ Y \arrow[r, "e_Y"'] & TY \end{tikzcd}$$

for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$.

$(\mathbb{T}, V)$-categories

Let $\hat{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$.

Definition 9

A *$(\mathbb{T}, V)$-category* is a pair (X, a), comprising a set X and a V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} X$ such that

$$\begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "\hat{T}a"'] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] \arrow[r, phantom, shift right=3, "\leqslant"] & TX \arrow[d, "a"] \\ & X. \end{tikzcd}$$

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A *$(\mathbb{T}, V)$-functor* $(X, a) \xrightarrow{f} (Y, b)$ is a map $X \xrightarrow{f} Y$ such that

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- $(\mathbb{T}, V)$-**Cat** is the construct of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors (skipping $\mathbb{T}$ in the notations for the identity monad).
- Examples of $(\mathbb{T}, V)$-**Cat** are the categories **Ord** (preordered sets), **Met** (premetric spaces), **ProbMet** (probabilistic metric spaces), **Top** (topological spaces), **App** (approach spaces).

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Lax homomorphisms of quantales

Definition 11

A *lax homomorphism of unital quantales* $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, l)$ is a map $V \xrightarrow{\varphi} W$ such that

1. $\bigvee \varphi(S) \leqslant \varphi(\bigvee S)$ for every $S \subseteq V$;
2. $\varphi(u) \otimes \varphi(v) \leqslant \varphi(u \otimes v)$ for every $u, v \in V$;
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The first condition is equivalent to φ being order-preserving.

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Lax homomorphisms and relations

Theorem 12

*Every lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ provides a lax functor V-**Rel** $\xrightarrow{\varphi}$ W-**Rel** defined by $\varphi(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y) = X \overset{\varphi r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, where φr is the composition of the maps $X \times Y \xrightarrow{r} V$ and $V \xrightarrow{\varphi} W$.*

Lemma 13

Given a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$, maps $X \xrightarrow{f} Y$, $W \xrightarrow{g} Z$, and V-relations $Y \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$, $U \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$,

$$f \leqslant \varphi f, \;\; f^{\circ} \leqslant \varphi(f^{\circ}), \;\; \varphi(g^{\circ} \cdot r \cdot f) = g^{\circ} \cdot \varphi r \cdot f, \;\; f \cdot \varphi s \leqslant \varphi(f \cdot s).$$

If φ is a homomorphism, then the three inequalities are equalities.

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If φ is a homomorphism, then the three inequalities are equalities.

Compatible lax homomorphisms

Definition 14

Given lax extensions $\hat{T}$ and $\check{T}$ of a functor T on **Set** to the categories V-**Rel** and W-**Rel**, respectively, a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ is said to be *compatible* with $\hat{T}$ and $\check{T}$ provided that $\check{T}(\varphi r) \leqslant \varphi(\hat{T} r)$ for every V-relation r, which means

$$\begin{tikzcd} V\text{-}\mathbf{Rel} \arrow[r, "\hat{T}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\leqslant"] & V\text{-}\mathbf{Rel} \arrow[d, "\varphi"] \\ W\text{-}\mathbf{Rel} \arrow[r, "\check{T}"'] & W\text{-}\mathbf{Rel}. \end{tikzcd}$$

φ is *strictly compatible* if the above inequalities are equalities.

Change-of-base functors

Theorem 15

Let $\hat{\mathbb{T}}$, $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on **Set** *to V-***Rel**, *W-***Rel**.

1. *A compatible lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ induces a functor $(\mathbb{T}, V)$-***Cat** $\xrightarrow{B_\varphi}$ *$(\mathbb{T}, W)$-***Cat** *defined by $B_\varphi((X, a) \xrightarrow{f} (Y, b)) = (X, \varphi a) \xrightarrow{f} (Y, \varphi b)$.*
2. *If φ is injective (resp. a $\bigvee$-preserving order-embedding), then B_φ is a (resp. full) embedding.*

B_φ is called the *change-of-base functor* associated to φ.

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Change-of-base functor adjunctions

Definition 16

Given partially ordered sets $(X, \leqslant)$, $(Y, \leqslant)$ and order-preserving maps $(X, \leqslant) \underset{g}{\overset{f}{\rightleftarrows}} (Y, \leqslant)$, g is said to be *right adjoint* to f (denoted $f \dashv g$) provided that $1_X \leqslant gf$ and $fg \leqslant 1_Y$ (pointwise).

Theorem 17

Let $\hat{\mathbb{T}}$, $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on **Set** *to V-***Rel***, W-***Rel***, and let $V \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W$ be compatible lax homomorphisms of unital quantales. If $\varphi \dashv \psi$, then B_ψ is a right adjoint functor to B_φ.*

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Nuclei in ordered categories

Definitions 18

Given an ordered category $\mathbf{C}$, $\mathbf{C}_{\triangleright}$ is a subcategory of $\mathbf{C}$, with the same objects, and whose morphisms $V \xrightarrow{\varphi} W$ are such that there is a $\mathbf{C}$-morphism $W \xrightarrow{\psi} V$ with $\varphi \dashv \psi$ in $\mathbf{C}$, i.e., $1_V \leqslant \psi \cdot \varphi$ and $\varphi \cdot \psi \leqslant 1_W$. The right adjoint of a $\mathbf{C}_{\triangleright}$-morphism φ is denoted $\varphi^{\triangleright}$.

Definitions 19

A morphism $V \xrightarrow{j} V$ of an ordered category $\mathbf{C}$ is called a *$\mathbf{C}$-nucleus on V* provided that j is idempotent ($j \cdot j = j$) and expanding ($1_V \leqslant j$).

Nuclei in ordered categories

Definitions 18

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A morphism $V \xrightarrow{j} V$ of an ordered category $\mathbf{C}$ is called a $\mathbf{C}$-*nucleus on* V provided that j is idempotent ($j \cdot j = j$) and expanding ($1_V \leqslant j$).

Factorizations of nuclei

Definition 20

An ordered category **C** *has equalizers of nuclei* provided that for every **C**-nucleus $V \xrightarrow{j} V$, there exists an equalizer of the pair $(j, 1_V)$.

Theorem 21

Let $\mathbf{C}$ *be an ordered category with equalizers of nuclei, and let* j *be a* $\mathbf{C}$*-nucleus on* V*. There exists a* $\mathbf{C}_{\triangleright}$*-morphism* $V \xrightarrow{j^*} V_j$ *such that*

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "j^*"'] & & V \\ & V_j \arrow[ur, "j^{*\triangleright}"'] & \end{tikzcd}$$

commutes, and, moreover, $j^* \cdot j^{*\triangleright} = 1_{V_j}$ *(i.e.,* j^* *is a* $\mathbf{C}$*-retraction).*

Factorizations of nuclei

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An ordered category $\mathbf{C}$ *has equalizers of nuclei* provided that for every $\mathbf{C}$-nucleus $V \xrightarrow{j} V$, there exists an equalizer of the pair $(j, 1_V)$.

Theorem 21

Let $\mathbf{C}$ be an ordered category with equalizers of nuclei, and let j be a $\mathbf{C}$-nucleus on V. There exists a $\mathbf{C}_{\triangleright}$-morphism $V \xrightarrow{j^*} V_j$ such that

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "j^*"'] & & V \\ & V_j \arrow[ur, "j^{*\triangleright}"'] & \end{tikzcd}$$

commutes, and, moreover, $j^* \cdot j^{*\triangleright} = 1_{V_j}$ (i.e., j^* is a $\mathbf{C}$-retraction).

Nuclei versus epimorphisms

Theorem 22

Let $\mathbf{C}$ *be an ordered category, which has equalizers of nuclei, and let* $V \xrightarrow{\alpha} W$ *be a* $\mathbf{C}_{\triangleright}$*-morphism, which is a* $\mathbf{C}$*-epimorphism (and therefore,* α *is a* $\mathbf{C}$*-retraction).*

1. *For the adjunction* $\alpha \dashv \alpha^{\triangleright}$, $j := \alpha^{\triangleright} \cdot \alpha$ *is a* $\mathbf{C}$*-nucleus on* V.
2. *There exists a unique* $\mathbf{C}_{\triangleright}$*-isomorphism* $V_j \xrightarrow{\gamma} W$*, which makes the next diagram commute*

$$\begin{tikzcd} V \arrow[r, "j^*"] \arrow[d, "\alpha"'] & V_j \arrow[d, "j^{*\triangleright}"] \arrow[dl, dotted, "\gamma"'] \\ W \arrow[r, "\alpha^{\triangleright}"'] & V. \end{tikzcd}$$

Categorical conuclei and their factorizations

Definition 23

A morphism $V \xrightarrow{g} V$ of an ordered category $\mathbf{C}$ is called a $\mathbf{C}$*-conucleus on* V provided that g is idempotent and contracting ($g \leqslant 1_V$).

Theorem 24

Let $\mathbf{C}$ *be an ordered category with equalizers of conuclei, let* g *be a* $\mathbf{C}$*-conucleus on* V*. There is a* $\mathbf{C}_{\triangleright}$*-morphism* $V_g \xrightarrow{g^*} V$ *such that*

$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dr, "g^{*\triangleright}"'] & & V \\ & V_g \arrow[ur, "g^*"'] & \end{tikzcd}$$

commutes, and, moreover, $g^{*\triangleright} \cdot g^* = 1_{V_g}$ *(i.e.,* g^* *is a* $\mathbf{C}$*-section).*

Categorical conuclei and their factorizations

Definition 23

A morphism $V \xrightarrow{g} V$ of an ordered category $\mathbf{C}$ is called a **C**-*conucleus on* V provided that g is idempotent and contracting ($g \leqslant 1_V$).

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$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dr, "g^{*\triangleright}"'] & & V \\ & V_g \arrow[ur, "g^*"'] & \end{tikzcd}$$

commutes, and, moreover, $g^{*\triangleright} \cdot g^* = 1_{V_g}$ *(i.e.,* g^* *is a* $\mathbf{C}$*-section).*

Conuclei versus monomorphisms

Theorem 25

Let $\mathbf{C}$ be an ordered category, which has equalizers of conuclei, and let $V \xrightarrow{\alpha} W$ be a $\mathbf{C}_{\triangleright}$-morphism, which is a $\mathbf{C}$-monomorphism (and therefore, α is a $\mathbf{C}$-section).

1. *For the adjunction $\alpha \dashv \alpha^{\triangleright}$, $g := \alpha \cdot \alpha^{\triangleright}$ is a $\mathbf{C}$-conucleus on W.*
2. *There exists a unique $\mathbf{C}_{\triangleright}$-isomorphism $W_g \xrightarrow{\gamma} V$, which makes the next diagram commute*

$$\begin{tikzcd} W \arrow[r, "g^{*\triangleright}"] \arrow[d, "\alpha^{\triangleright}"'] & W_g \arrow[d, "g^*"] \arrow[dl, dotted, "\gamma"'] \\ V \arrow[r, "\alpha"'] & W. \end{tikzcd}$$

Folklore lemma

- Let **LQuant** (**LUQuant**) be the category of (unital) quantales and lax-homomorphisms of (unital) quantales.
- Lemma 26 shows that the category $\mathbf{LQuant}_{\triangleright}$ ($\mathbf{LUQuant}_{\triangleright}$) is precisely the category **Quant** (**UQuant**) of (unital) quantales and (unital) quantale homomorphisms.

Lemma 26

A lax homomorphism of (unital) quantales $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, I)$ has a right adjoint $\varphi^{\triangleright}$, which is, additionally, a lax homomorphism of (unital) quantales, iff φ is a (unital) quantale homomorphism.

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Quantic (co)nuclei

Definition 27

A *quantic nucleus* on a quantale V is a map $V \xrightarrow{j} V$ such that for every $u, v \in V$,

1. if $u \leqslant v$, then $j(u) \leqslant j(v)$;
2. $u \leqslant j(u)$;
3. $jj(u) = j(u)$;
4. $j(u) \otimes j(v) \leqslant j(u \otimes v)$.

Definition 28

A *quantic conucleus* on a quantale V is a map $V \xrightarrow{g} V$, which satisfies conditions (1), (3), (4) of Definition 27, and also the condition $g(u) \leqslant u$ for every $u \in V$. A quantic conucleus g on a unital quantale $(V, \otimes, k)$ is said to be *unital* provided that $k \leqslant g(k)$.

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Quantic (co)nuclei categorically

- Quantic nuclei are exactly **LQuant**-nuclei or **LUQuant**-nuclei.
- Quantic conuclei are exactly **LQuant**-conuclei.
- Every **LUQuant**-conucleus is a quantic conucleus. The converse implication though does not hold. As a counterexample, consider, e.g., the quantale $V = ([0,1], \wedge, 1)$ and the map $V \xrightarrow{g} V$ defined by $g(u) = u \wedge \frac{1}{2}$. Then g is a quantic conucleus, but it is not an **LUQuant**-conucleus, since $g(1) = \frac{1}{2} < 1$.
- Unital quantic conuclei are exactly **LUQuant**-conuclei.

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- Unital quantic conuclei are exactly **LUQuant**-conuclei.

Equalizers of (co)nuclei

Proposition 29

The category **LQuant** *has equalizers of (co)nuclei. The category* **LUQuant** *has equalizers of nuclei and unital conuclei.*

Proof.

Given a quantic nucleus $V \xrightarrow{j} V$, $V_j := \{u \in V \mid j(u) = u\}$ is a (unital) quantale, in which $\bigvee_j S = j(\bigvee S)$ for every $S \subseteq V_j$, and $u \otimes_j v = j(u \otimes v)$ for every $u, v \in V_j$ ($k_j = j(k)$). The inclusion $V_j \xhookrightarrow{e} V$ is an equalizer of $(j, 1_V)$ in **LQuant** (**LUQuant**).

Proposition 29 ensures the validity of the factorization properties of categorical (co)nuclei in the categories **LQuant** and **LUQuant**.

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Proposition 29 ensures the validity of the factorization properties of categorical (co)nuclei in the categories **LQuant** and **LUQuant**.

Monoidal nuclei

Definition 30

Given a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** to the category V-**Rel**, a quantic nucleus $V \xrightarrow{j} V$ is said to be *compatible* with $\hat{\mathbb{T}}$ provided that $\hat{T}(jr) \leqslant j(\hat{T}r)$ for every V-relation r.

(Strictly) compatible quantic nuclei are called *(strict)* $\mathbb{T}$*-nuclei* or *(strict) monoidal nuclei*.

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Lax extensions of monads through surjections

Proposition 31

Let $\hat{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on **Set** *to V-***Rel***, and let $V \xrightarrow{\varphi} W$ be a surjective unital quantale homomorphism.*

1. *If $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus, then the correspondence W-***Rel*** $\xrightarrow{\hat{T}_{\varphi}}$ W-***Rel*** defined by $\hat{T}_{\varphi}(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} Y) = TX \xrightarrow{\varphi \hat{T}(\varphi^{\triangleright} r)} TY$ provides a lax extension $\hat{\mathbb{T}}_{\varphi}$ of the monad $\mathbb{T}$ to W-***Rel***.*
2. *Let $\check{\mathbb{T}}$ be a lax extension of $\mathbb{T}$ to W-***Rel***. Then φ and $\varphi^{\triangleright}$ are compatible with $\hat{T}$ and $\check{T}$ iff $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus and $\check{T} = \hat{T}_{\varphi}$.*
3. *If $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus, then $(B_{\varphi}, B_{\varphi^{\triangleright}})$ is a Galois correspondence between $(\mathbb{T}, V)$-***Cat*** and $(\mathbb{T}, W)$-***Cat***, in which B_{φ} is surjective on morphisms.*

From nuclei to quotients

Theorem 32

Let j be a $\mathbb{T}$-nucleus on a unital quantale V.

1. *The correspondence V_j-**Rel** $\xrightarrow{\hat{T}_j}$ V_j-**Rel**, $\hat{T}_j(X \overset{r}{\longrightarrow\!\!\!\!\!\!+\;\;} Y) = TX \xrightarrow{j^* \hat{T}(j^{*\rhd} r)} TY$ is a lax extension $\hat{\mathbb{T}}_j$ of $\mathbb{T}$ to V_j-**Rel**.*
2. *Both $V \xrightarrow{j^*} V_j$ and $V_j \xrightarrow{j^{*\rhd}} V$ are compatible lax homomorphisms of unital quantales, and thus, there is a factorization*

$$\begin{tikzcd}
(\mathbb{T}, V)\text{-}\mathbf{Cat} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & (\mathbb{T}, V)\text{-}\mathbf{Cat} \\
& (\mathbb{T}, V_j)\text{-}\mathbf{Cat}. \arrow[ur, "B_{j^{*\rhd}}"'] &
\end{tikzcd}$$

3. *$(B_{j^*}, B_{j^{*\rhd}})$ is a Galois correspondence between $(\mathbb{T}, V)$-**Cat**, $(\mathbb{T}, V_j)$-**Cat**, in which B_{j^*} is surjective on morphisms.*

From quotients to nuclei

Theorem 33

Let $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on **Set** *to the categories V-***Rel** *and W-***Rel***, let $V \xrightarrow{\alpha} W$ be a surjective unital quantale homomorphism, and let $\alpha \dashv \alpha^{\triangleright}$ be the corresponding adjunction, in which both α and $\alpha^{\triangleright}$ are compatible with the lax extensions.*

1. *$j := \alpha^{\triangleright}\alpha$ is a $\mathbb{T}$-nucleus on V.*
2. *There exists a unique unital quantale isomorphism $V_j \xrightarrow{\gamma} W$, which makes the next diagram commute*

$$\begin{tikzcd}
(\mathbb{T}, V)\text{-}\mathbf{Cat} \arrow[r, "B_{j^*}"] \arrow[d, "B_\alpha"'] & (\mathbb{T}, V_j)\text{-}\mathbf{Cat} \arrow[d, "B_{j^{*\triangleright}}"] \arrow[dl, dotted, "B_\gamma"'] \\
(\mathbb{T}, W)\text{-}\mathbf{Cat} \arrow[r, "B_{\alpha^\triangleright}"'] & (\mathbb{T}, V)\text{-}\mathbf{Cat}.
\end{tikzcd}$$

Quantale representation theorem

- Given a semigroup $(S, \otimes)$ (monoid $(S, \otimes, k)$), the powerset $\mathcal{P}(S)$ is the free (unital) quantale over S, in which $\bigvee$ are given by the set-theoretic unions, and $A \otimes B = \{a \otimes b \mid a \in A,\ b \in B\}$ for every $A, B \in \mathcal{P}(S)$ (the unit is given by the singleton $\{k\}$).
- Given a (unital) quantale V, one has the underlying semigroup (monoid) of V. The quantic nucleus of the quantale representation theorem is given by the following commutative diagram

$$\begin{tikzcd}
\mathcal{P}(V) \arrow[rr, "j"] \arrow[dr, "\varphi:=\bigvee"'] & & \mathcal{P}(V) \\
& V, \arrow[ur, "\varphi^{\triangleright}:=\downarrow"'] &
\end{tikzcd}$$

with $\bigvee \dashv \downarrow$ the adjunction provided by arbitrary joins and lower sets, where $\bigvee$ is a surjective (unital) quantale homomorphism.

Quantale representation theorem

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Representation of the categories $(\mathbb{T}, V)$-**Cat**

Proposition 34

Let $\check{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on **Set** *to W-***Rel***, and let $V \xrightarrow{\varphi} W$ be a surjective unital quantale homomorphism.*

1. *The correspondence V-***Rel** $\xrightarrow{\check{T}^{\varphi}}$ V-**Rel**, $\check{T}^{\varphi}(X \xrightarrow{r} Y) = TX \xrightarrow{\varphi^{\triangleright}\check{T}(\varphi r)} TY$ *is a lax extension $\check{\mathbb{T}}^{\varphi}$ of $\mathbb{T}$ to V-***Rel**.
2. *Both φ and $\varphi^{\triangleright}$ are strictly compatible with $\check{T}^{\varphi}$ and $\check{T}$.*
3. *$\varphi^{\triangleright}\varphi$ is a strict $\check{\mathbb{T}}^{\varphi}$-nucleus.*

Theorem 35 (Representation theorem)

*Given a category $(\mathbb{T}, V)$-***Cat***, there exist a monoid S, a lax extension of $\mathbb{T}$ to $\mathcal{P}(S)$-***Rel***, and a strict $\mathbb{T}$-nucleus j on $\mathcal{P}(S)$ such that $(\mathbb{T}, V)$-***Cat** $\cong (\mathbb{T}, \mathcal{P}(S)_j)$-**Cat**.

Representation of the categories $(\mathbb{T}, V)$-**Cat**

Proposition 34

Let $\check{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on **Set** *to W-***Rel***, and let $V \xrightarrow{\varphi} W$ be a surjective unital quantale homomorphism.*

1. *The correspondence V-***Rel** $\xrightarrow{\check{T}^{\varphi}}$ *V-***Rel***, $\check{T}^{\varphi}(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\,} Y) = TX \xrightarrow{\varphi^{\triangleright}\check{T}(\varphi r)} TY$ is a lax extension $\check{\mathbb{T}}^{\varphi}$ of $\mathbb{T}$ to V-***Rel**.
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*Given a category $(\mathbb{T}, V)$-***Cat***, there exist a monoid S, a lax extension of $\mathbb{T}$ to $\mathcal{P}(S)$-***Rel***, and a strict $\mathbb{T}$-nucleus j on $\mathcal{P}(S)$ such that $(\mathbb{T}, V)$-***Cat** $\cong (\mathbb{T}, \mathcal{P}(S)_j)$-**Cat**.

Lax extensions of monads through injections

Proposition 36

Let $\check{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on **Set** *to W-***Rel***, and let $V \xrightarrow{\varphi} W$ be an injective unital quantale homomorphism.*

1. *If $\varphi\varphi^{\triangleright}$ is a $\mathbb{T}$-conucleus, then the correspondence V-***Rel** $\xrightarrow{\check{T}_{\varphi}}$ *V-***Rel** *defined by $\check{T}_{\varphi}(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y) = TX \xrightarrow{\varphi^{\triangleright}\check{T}(\varphi r)} TY$ provides a lax extension $\check{\mathbb{T}}_{\varphi}$ of the monad $\mathbb{T}$ to V-***Rel***.*
2. *Let $\hat{\mathbb{T}}$ be a lax extension of $\mathbb{T}$ to V-***Rel***. Then φ and $\varphi^{\triangleright}$ are compatible with $\hat{T}$ and $\check{T}$ iff $\varphi\varphi^{\triangleright}$ is a $\mathbb{T}$-conucleus and $\hat{T} = \check{T}_{\varphi}$.*
3. *If $\varphi\varphi^{\triangleright}$ is a $\mathbb{T}$-nucleus, then $(B_{\varphi}, B_{\varphi^{\triangleright}})$ is a Galois correspondence between $(\mathbb{T}, V)$-***Cat** *and $(\mathbb{T}, W)$-***Cat***, in which B_{φ} is a full embedding.*

From conuclei to subobjects

Theorem 37

Let g be a $\mathbb{T}$-conucleus on a unital quantale V.

1. *The correspondence* V_g-**Rel** $\xrightarrow{\check{T}_g}$ V_g-**Rel**, $\check{T}_g(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y) = TX \xrightarrow{g^{*\triangleright}\check{T}(g^* r)} TY$, *is a lax extension* $\check{\mathbb{T}}_j$ *of* $\mathbb{T}$ *to* V_g-**Rel**.
2. *Both* $V_g \xrightarrow{g^*} V$ *and* $V \xrightarrow{g^{*\triangleright}} V_g$ *are compatible lax homomorphisms of unital quantales, and thus, there is a factorization*

$$\begin{tikzcd}
(\mathbb{T}, V)\text{-}\mathbf{Cat} \arrow[rr, "B_g"] \arrow[dr, "B_{g^{*\triangleright}}"'] & & (\mathbb{T}, V)\text{-}\mathbf{Cat} \\
& (\mathbb{T}, V_g)\text{-}\mathbf{Cat}. \arrow[ur, "B_{g^*}"'] &
\end{tikzcd}$$

3. $(B_{g^*}, B_{g^{*\triangleright}})$ *is a Galois correspondence between* $(\mathbb{T}, V)$-**Cat**, $(\mathbb{T}, V_g)$-**Cat**, *and* B_{g^*} *is a full embedding.*

From subobjects to conuclei

Theorem 38

Let $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on **Set** *to the categories V-***Rel** *and W-***Rel***, let $V \xrightarrow{\alpha} W$ be an injective unital quantale homomorphism, and let $\alpha \dashv \alpha^{\triangleright}$ be the corresponding adjunction, in which both α and $\alpha^{\triangleright}$ are compatible with the lax extensions.*

1. *$g := \alpha\alpha^{\triangleright}$ is a $\mathbb{T}$-conucleus on W.*
2. *There exists a unique unital quantale isomorphism $W_g \xrightarrow{\gamma} V$, which makes the next diagram commute*

$$\begin{tikzcd}
(\mathbb{T}, W)\text{-}\mathbf{Cat} \arrow[r, "B_{g^{*\triangleright}}"] \arrow[d, "B_{\alpha^{\triangleright}}"'] & (\mathbb{T}, W_g)\text{-}\mathbf{Cat} \arrow[d, "B_{g^*}"] \arrow[dl, dotted, "B_\gamma"'] \\
(\mathbb{T}, V)\text{-}\mathbf{Cat} \arrow[r, "B_\alpha"'] & (\mathbb{T}, W)\text{-}\mathbf{Cat}.
\end{tikzcd}$$

Preordered sets

Example 39

- For $2 = \{\perp, \top\}$, 2-**Cat** is the category **Ord** of preordered sets.
- The quantic nucleus $2 \xrightarrow{j} 2$, where $j(\perp) = j(\top) = \top$, provides the commutative triangle

$$\begin{tikzcd} \mathbf{Ord} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & \mathbf{Ord} \\ & \mathbf{Set}. \arrow[ur, "B_\varphi"'] & \end{tikzcd}$$

Premetric spaces

Example 40

- For $\mathsf{P}_+ = ([0,\infty]^{op}, +, 0)$, P_+-**Cat** is the category **Met** of premetric spaces.
- The quantic nucleus $\mathsf{P}_+ \xrightarrow{j} \mathsf{P}_+$ given by

$$j(u) = \begin{cases} \infty, & u = \infty \\ 0, & \text{otherwise} \end{cases}$$

provides the commutative triangle

$$\begin{tikzcd} \mathbf{Met} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & \mathbf{Met} \\ & \mathbf{Ord}. \arrow[ur, "B_\varphi"'] & \end{tikzcd}$$

Probabilistic metric spaces

Example 41

- For $\Delta = \{[0,\infty] \xrightarrow{f} [0,1] \mid f$ is monotone and $f(x) = \bigvee_{y<x} f(y)\}$, Δ-**Cat** is the category **ProbMet** of probabilistic metric spaces.
- The quantic nucleus $\Delta \xrightarrow{j} \Delta$ given by

$$(j(f))(x) = \begin{cases} 0, & x \leqslant \sup\{y \in [0,\infty] \mid f(y) = 0\} \\ 1, & \sup\{y \in [0,\infty] \mid f(y) = 0\} < x \end{cases}$$

provides the commutative triangle

$$\begin{tikzcd} \mathbf{ProbMet} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & \mathbf{ProbMet} \\ & \mathbf{Met}. \arrow[ur, "B_\varphi"'] & \end{tikzcd}$$

Complete distributivity

Definition 42

A complete lattice V is *completely distributive* provided that for every family $\{S_i \mid i \in I\}$ of subsets of V, $\bigwedge_{i\in I} \bigvee S_i = \bigvee_{f\in F} \bigwedge_{i\in I} f(i)$, where F is the set of choice maps $I \xrightarrow{f} \bigcup_{i\in I} S_i$ with $f(i) \in S_i$.

Generalized approach spaces . . .

Let $\mathbb{U}$ be the ultrafilter monad on **Set**, and let V be a unital quantale with the following properties:

1. V is completely distributive;
2. $u \otimes v = \perp_V$ implies $u = \perp_V$ or $v = \perp_V$, for every $u, v \in V$;
3. $\perp_V < \bigwedge(V \backslash \{\perp_V\})$.

- The quantale P_+ satisfies (1) and (2), but not (3).

Example 43

- By (1), V-**Rel** $\xrightarrow{\hat{U}}$ V-**Rel** defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!+\!\!\!\!\!\longrightarrow} Y$ by $(\hat{U}r)(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x, y)$ is a lax extension of $\mathbb{U}$ to V-**Rel**. $(\mathbb{U}, 2)$-**Cat** ($(\mathbb{U}, \mathrm{P}_+)$-**Cat**) is isomorphic to the category **Top** of topological spaces (**App** of approach spaces).

Generalized approach spaces . . .

Let $\mathbb{U}$ be the ultrafilter monad on **Set**, and let V be a unital quantale with the following properties:

1. V is completely distributive;
2. $u \otimes v = \bot_V$ implies $u = \bot_V$ or $v = \bot_V$, for every $u, v \in V$;
3. $\bot_V < \bigwedge(V \backslash \{\bot_V\})$.

- The quantale P_+ satisfies (1) and (2), but not (3).

Example 43

- By (1), $V\text{-}\mathbf{Rel} \xrightarrow{\hat{U}} V\text{-}\mathbf{Rel}$ defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,} Y$ by $(\hat{U}r)(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x, y)$ is a lax extension of $\mathbb{U}$ to V-**Rel**. $(\mathbb{U}, 2)$-**Cat** ($(\mathbb{U}, \mathrm{P}_+)$-**Cat**) is isomorphic to the category **Top** of topological spaces (**App** of approach spaces).

Generalized approach spaces ...

Let $\mathbb{U}$ be the ultrafilter monad on **Set**, and let V be a unital quantale with the following properties:

1. V is completely distributive;
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. . . and their quotient

Example 43 (cont.)

- By (2), the map $V \xrightarrow{j} V$ defined by

$$j(u) = \begin{cases} \perp_V, & u = \perp_V \\ \top_V := \bigwedge \varnothing, & \text{otherwise} \end{cases}$$

  is a quantic nucleus on V.
- (3) provides compatibility of j and the commutative triangle

$$\begin{tikzcd} (\mathbb{U}, V)\text{-}\mathbf{Cat} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & (\mathbb{U}, V)\text{-}\mathbf{Cat} \\ & \mathbf{Top}. \arrow[ur, "B_\varphi"'] & \end{tikzcd}$$

Approach spaces

- We can not represent **Top** as a monoidal quotient of **App**.
- **Top** can be represented as a monoidal subobject of **App**.

Example 44

The unital quantic conucleus $\mathsf{P}_+ \xrightarrow{g} \mathsf{P}_+$ given by

$$g(u) = \begin{cases} 0, & u = 0 \\ \infty, & \text{otherwise} \end{cases}$$

provides the commutative triangle

$$\begin{tikzcd} \mathbf{App} \arrow[rr, "B_g"] \arrow[dr, "B_{g^{*\rhd}}"'] & & \mathbf{App} \\ & \mathbf{Top}. \arrow[ur, "B_{g^*}"'] & \end{tikzcd}$$

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V-closure spaces . . .

Let $\mathbb{P}$ be the powerset monad on **Set**, and let V be a unital quantale, which satisfies the properties from the previous example.

Example 45

- By (1), $V\text{-}\mathbf{Rel} \xrightarrow{\hat{P}} V\text{-}\mathbf{Rel}$ defined on a V-relation $X \overset{r}{\nrightarrow} Y$ by $(\hat{P}r)(A, B) = \bigwedge_{y\in B} \bigvee_{x\in A} r(x, y)$ is a lax extension of $\mathbb{P}$ to V-**Rel** (the canonical extension of G. Seal).
- $(\mathbb{P}, 2)$-**Cat** (resp. $(\mathbb{P}, \mathrm{P}_+)$-**Cat**) is isomorphic to the category **Cls** of closure spaces (resp. **Clns** of closeness spaces).
- $(\mathbb{P}, V)$-**Cat** $=$ V-**Cls** (*V-closure spaces* of G. Seal).

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Example 45 (cont.)

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$$\begin{tikzcd} V\text{-}\mathbf{Cls} \arrow[rr, "B_j"] \arrow[dr, "B_{j^*}"'] & & V\text{-}\mathbf{Cls} \\ & \mathbf{Cls}. \arrow[ur, "B_\varphi"'] & \end{tikzcd}$$

Closeness spaces

- We can not represent **Cls** as a monoidal quotient of **Clns**.
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Example 46

The unital quantic conucleus $\mathsf{P}_+ \xrightarrow{g} \mathsf{P}_+$ given by

$$g(u) = \begin{cases} 0, & u = 0 \\ \infty, & \text{otherwise} \end{cases}$$

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Conclusion

- Motivated by the convenient technique of obtaining quantic quotients (subobjects) with the help of quantic (co)nuclei, this talk presented the technique of obtaining monoidal quotients (subobjects) with the help of monoidal (co)nuclei.
- Given a category $(\mathbb{T}, V)$-**Cat** and a monoidal (co)nucleus on its underlying unital quantale V, one gets a category $(\mathbb{T}, W)$-**Cat**, which is a monoidal quotient (subobject) of $(\mathbb{T}, V)$-**Cat**.
- As the main consequence, one gets new categories from the already existing ones, saving thus the effort for their definition.
- With the technique of monoidal nuclei in hand, we provided a representation theorem for the categories $(\mathbb{T}, V)$-**Cat**.

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Open problem

Every surjective (injective) quantale homomorphism can be represented with the help of a quantic (co)nucleus.

Problem 47

What kind of concrete functors $(\mathbb{T}, V)$-$\mathbf{Cat} \xrightarrow{F} (\mathbb{T}, W)$-$\mathbf{Cat}$ *can be represented with the help of monoidal (co)nuclei?*

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References

J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications, 2009.

D. Hofmann and C. D. Reis, *Probabilistic metric spaces as enriched categories*, Fuzzy Sets Syst. **210** (2013), 1–21.

K. I. Rosenthal, *Quantales and Their Applications*, Addison Wesley Longman, 1990.

D. Hofmann, G. J. Seal, and W. Tholen (eds.), *Monoidal Topology: A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

G. J. Seal, *Canonical and op-canonical lax algebras*, Theory Appl. Categ. **14** (2005), 221–243.

Thank you for your attention!