Exact completions as homotopical quotients

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Summary of the presentation

- Exact completion of a category with finite limits
- Exact completion from equivalence spans
- Exact completion from groupoids
- Exact completion from weak 2-groupoids

Equivalence spans in a category $\mathscr{A}$ with finite limits

Equivalence span in $\mathscr{A}$: $A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0$ with

$$
\begin{tikzcd}
 & & A_0 \\
A_0 \arrow[urr, "\mathrm{id}_{A_0}"] \arrow[r, "r"] \arrow[drr, "\mathrm{id}_{A_0}"'] & A_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A_0 \\
A_1 \arrow[urr, "d_2"] \arrow[r, "s"] \arrow[drr, "d_1"'] & A_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_1 \arrow[r, "d_1"] & A_0 \\
A_1 \times_{A_0} A_1 \arrow[ur, "d_1'"] \arrow[rr, "t"] \arrow[dr, "d_2'"'] & & A_1 \arrow[u, "d_1"'] \arrow[d, "d_2"] \\
 & A_1 \arrow[r, "d_2"'] & A_0
\end{tikzcd}
$$

Homomorphism of equivalence spans in $\mathscr{A}$:

$$
\begin{tikzcd}
 & A_0 \arrow[rr, "f_0"] & & B_0 \\
A_1 \arrow[ur, "d_1"] \arrow[rr, "f_1"] \arrow[dr, "d_2"'] & & B_1 \arrow[ur, "e_1"'] \arrow[dr, "e_2"] & \\
 & A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}
$$

Equivalence spans in a category $\mathscr{A}$ with finite limits

Equivalence span in $\mathscr{A}$: $A_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} A_0$ with

$$
\begin{tikzcd}
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 & & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A_0 \\
A_1 \arrow[urr, "d_2"] \arrow[r, "s"] \arrow[drr, "d_1"'] & A_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_1 \arrow[r, "d_1"] & A_0 \\
A_1 \times_{A_0} A_1 \arrow[ur, "d'_1"] \arrow[r, "t"] \arrow[dr, "d'_2"'] & A_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & A_1 \arrow[r, "d_2"'] & A_0
\end{tikzcd}
$$

Homomorphism of equivalence spans in $\mathscr{A}$:

$$
\begin{tikzcd}
 & A_0 \arrow[r, "f_0"] & B_0 \\
A_1 \arrow[ur, "d_1"] \arrow[r, "f_1"] \arrow[dr, "d_2"'] & B_1 \arrow[ur, "e_1"'] \arrow[dr, "e_2"] & \\
 & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}
$$

Notation: $\mathrm{EqSpan}(\mathscr{A})$ is the category of equivalence spans and homomorphisms.

The exact completion of $\mathcal{A}$

Recall that a category is *exact* when it has finite limits and stable effective coequalizers of equivalence relations.

The exact completion $\mathcal{A}_{\mathrm{ex}}$ is obtained by quotienting homsets between equivalence spans:

$$\begin{tikzcd}
(A_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & A_0) \arrow[rr, bend left, "(f_1, f_0)"] \arrow[rr, bend right, "(g_1, g_0)"'] \arrow[rr, phantom, "\wr"] & & (B_1 \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & B_0)
\end{tikzcd}$$

if there is $h\colon A_0 \longrightarrow B_1$ such that

$$\begin{tikzcd}
 & & B_0 \\
A_0 \arrow[urr, "f_0"] \arrow[r, "h"] \arrow[drr, "g_0"'] & B_1 \arrow[ur, "e_1"'] \arrow[dr, "e_2"] & \\
 & & B_0
\end{tikzcd}$$

Michael Barr
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Aurelio Carboni, Rosa Celia Magno
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\end{tikzcd}$$

if there is $h \colon A_0 \longrightarrow B_1$ such that

$$\begin{tikzcd}
 & & B_0 \\
A_0 \arrow[urr, "f_0"] \arrow[r, "h"] \arrow[drr, "g_0"'] & B_1 \arrow[ur, "e_1"'] \arrow[dr, "e_2"] & \\
 & & B_0
\end{tikzcd}$$

$$\boxed{\begin{tikzcd}
\mathrm{EqSpan}(\mathcal{A}) \arrow[d, "Q"] \\
\mathcal{A}_{\mathrm{ex}}
\end{tikzcd}}$$

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Examples of exact completions

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{S}et] \equiv (\mathrm{Fam}(\mathcal{C}))_{\mathrm{ex}} \qquad \mathcal{C} \text{ small with finite limits}$$

$$\mathcal{E}ff \equiv \mathcal{PA}sm_{\mathrm{ex}}$$

$$\begin{tikzcd} \mathcal{E}qu \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathcal{T}op_{0\,\mathrm{ex}} \arrow[l, bend right, shift right] \end{tikzcd}$$

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A characterization of the left exact categories whose exact completions are toposes J. Pure Appl. Algebra (2003)

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Groupoid in $\mathcal{A}$: $G_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} G_0$ together with

$$
\begin{tikzcd}
 & & G_0 \\
G_0 \arrow[urr, "\mathrm{id}_{G_0}"] \arrow[r, "e"] \arrow[drr, "\mathrm{id}_{G_0}"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & G_0 \\
G_1 \arrow[urr, "d_2"] \arrow[r, "j"] \arrow[drr, "d_1"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G_1 \arrow[r, "d_1"] & G_0 \\
G_1 \times_{G_0} G_1 \arrow[ur, "d_1'"] \arrow[r, "c"] \arrow[dr, "d_2'"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & G_1 \arrow[r, "d_2"'] & G_0
\end{tikzcd}
$$

such that (d_1, d_2, c, e) is a category and j is an inversion.

Homomorphism of groupoids:

$$
\begin{tikzcd}
 & G_0 \arrow[r, "f_0"] & H_0 \\
G_1 \arrow[ur, "d_1"] \arrow[r, "f_1"] \arrow[dr, "d_2"'] & H_1 \arrow[ur, "e_1"'] \arrow[dr, "e_2"] & \\
 & G_0 \arrow[r, "f_0"'] & H_0
\end{tikzcd}
$$

which is a functor.

Groupoid in $\mathcal{A}$: $\quad G_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} G_0 \quad$ together with

$$
\begin{tikzcd}
 & & G_0 \\
G_0 \arrow[urr, "\mathrm{id}_{G_0}"] \arrow[r, "e"] \arrow[drr, "\mathrm{id}_{G_0}"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & & G_0
\end{tikzcd}
\qquad
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 & & G_0 \\
G_1 \arrow[urr, "d_2"] \arrow[r, "j"] \arrow[drr, "d_1"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
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 & G_1 \arrow[r, "d_1"] & G_0 \\
G_1 \times_{G_0} G_1 \arrow[ur, "d_1'"] \arrow[r, "c"] \arrow[dr, "d_2'"'] & G_1 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & \\
 & G_1 \arrow[r, "d_2"'] & G_0
\end{tikzcd}
$$

such that (d_1, d_2, c, e) is a category and j is an inversion.

Homomorphism of groupoids:

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 & G_0 \arrow[r, "f_0"'] & H_0
\end{tikzcd}
$$

which is a functor.

Notation: Grpd($\mathcal{A}$) is the category of groupoids and homomorphisms.

Groupoids are equivalence spans

There is an obvious forgetful functor

$$\begin{tikzcd} \mathrm{Grpd}(\mathcal{A}) \arrow[r, "U"'] \arrow[dr, "QU"] & \mathrm{EqSpan}(\mathcal{A}) \arrow[d, "Q"] \\ & \mathcal{A}_{\mathrm{ex}} \end{tikzcd}$$

Groupoids are equivalence spans

There is an obvious forgetful functor

$$\begin{tikzcd}
\mathrm{FreeGrpd}(\mathcal{A}) \arrow[r, hook, "\text{full}"'] \arrow[rr, bend left, "U'"] \arrow[drr, "QU'"'] & \mathrm{Grpd}(\mathcal{A}) \arrow[r, "U"'] \arrow[dr, "QU"] & \mathrm{EqSpan}(\mathcal{A}) \arrow[d, "Q"] \\
& & \mathcal{A}_{\mathrm{ex}}
\end{tikzcd}$$

If $\mathcal{A}$ is an arithmetic universe, QU' is a homotopical quotient.

The quotient is determined by the groupoid $\mathbf{I}$ on $2 \times 2 \overset{d_1}{\underset{d_2}{\rightrightarrows}} 2$.

André Joyal
The Gödel incompleteness theorem, a categorical approach
Cah. Top. Géom. Diff. Cat. (2005)

Maria Emilia Maietti
Joyal's arithmetic universe as list-arithmetic pretopos Theory Appl. Cat. (2010)

$QU'\colon \mathrm{FreeGrpd}(\mathscr{A}) \longrightarrow \mathscr{A}_{\mathrm{ex}}$ is a homotopical quotient

- $QU\colon \mathrm{Grpd}(\mathscr{A}) \twoheadrightarrow \mathscr{A}_{\mathrm{ex}}$ is essentially surjective.

Given an equivalence span
$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "t"] & A_1 \arrow[loop above, "s"] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & A_0 \arrow[l, bend right, "r"'] \end{tikzcd}$$

take the free groupoid on the graph $A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0$.

- $QU'\colon \mathrm{FreeGrpd}(\mathscr{A}) \twoheadrightarrow \mathscr{A}_{\mathrm{ex}}$ is full.

$QU'\colon \mathrm{FreeGrpd}(\mathcal{A}) \longrightarrow \mathcal{A}_{\mathrm{ex}}$ is a homotopical quotient

- For $f, g\colon \mathbf{G} \to \mathbf{H}$ between free groupoids in $\mathcal{A}$,

  $QU'(f) = QU'(g)$

  if and only if

  there is $\mathbf{G} \times \mathbf{I} \xrightarrow{k} \mathbf{H}$ such that

$$
\begin{tikzcd}
\mathbf{G} \times \mathbf{1} \arrow[d, "\mathrm{id} \times \iota_1"'] \arrow[dr, "f\pi_1"] & \\
\mathbf{G} \times \mathbf{I} \arrow[r, "k"] & \mathbf{H} \\
\mathbf{G} \times \mathbf{1} \arrow[u, "\mathrm{id} \times \iota_2"] \arrow[ur, "g\pi_2"'] &
\end{tikzcd}
$$

2-groupoids in $\mathscr{A}$

2-groupoid in $\mathscr{A}$: $\mathbf{G} = \left(G_2 \overset{d_{21}}{\underset{d_{22}}{\rightrightarrows}} G_1 \overset{d_{11}}{\underset{d_{12}}{\rightrightarrows}} G_0, \ldots \right)$ is a 2-category object in $\mathscr{A}$ such that

every 1-arrow is (part of) an equivalence
every 2-arrow is iso.

Homomorphisms of 2-groupoids are weak 2-functors.

Weak 2-groupoids in $\mathcal{A}$

2-groupoid in $\mathcal{A}$: $\mathbf{G} = \left(G_2 \overset{d_{21}}{\underset{d_{22}}{\rightrightarrows}} G_1 \overset{d_{11}}{\underset{d_{12}}{\rightrightarrows}} G_0, \ldots \right)$ is a 2-category object in $\mathcal{A}$ such that

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Homomorphisms of 2-groupoids are weak 2-functors.

<ins>Notation</ins>: 2-Grpd$_w(\mathcal{A})$ is the category of 2-groupoids and homomorphisms.

Weak 2-groupoids in $\mathscr{A}$

2-groupoid in $\mathscr{A}$: $\mathbf{G} = \left(G_2 \overset{d_{21}}{\underset{d_{22}}{\rightrightarrows}} G_1 \overset{d_{11}}{\underset{d_{12}}{\rightrightarrows}} G_0, \ldots \right)$ is a 2-category object in $\mathscr{A}$ such that

every 1-arrow is (part of) an equivalence
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Homomorphisms of 2-groupoids are weak 2-functors.

There is an obvious forgetful functor

$$\begin{tikzcd} \text{2-Grpd}_w(\mathscr{A}) \arrow[r, "V"] \arrow[dr, "QV"'] & \mathrm{EqSpan}(\mathscr{A}) \arrow[d, "Q"] \\ & \mathscr{A}_{\mathrm{ex}} \end{tikzcd}$$

<u>Notation</u>: 2-Grpd$_w(\mathscr{A})$ is the category of 2-groupoids and homomorphisms.

When $\mathcal{A}$ is a locos

$QV\colon \text{2-Grpd}_w(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$ is essentially surjective.

Given an equivalence span $A_1 \times_{A_0} A_1 \xrightarrow{t} A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0$, with $s\colon A_1 \to A_1$ and $r\colon A_0 \to A_1$,

take the free dagger category on the graph $A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0$

and turn it into a 2-groupoid $\mathbf{D}(A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0)$

by making every 2-diagram commute.

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take the free dagger category on the graph $A_1 \overset{d_1}{\underset{d_2}{\rightrightarrows}} A_0$

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by making every 2-diagram commute.

Notation: $\text{Esp}(\mathcal{A}) \underset{\text{full}}{\hookrightarrow} \text{2-Grpd}_w(\mathcal{A})$ is the full subcategory on retracts of such 2-groupoids.

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by making every 2-diagram commute.

Notation:
$$\begin{tikzcd} \text{Esp}(\mathcal{A}) \arrow[r, hook, "\text{full}"'] \arrow[dr, "V'"'] & \text{2-Grpd}_w(\mathcal{A}) \arrow[d, "V"] \\ & \text{EqSpan}(\mathcal{A}) \end{tikzcd}$$
is the full subcategory on retracts of such 2-groupoids.

When $\mathcal{A}$ is a locos

$QV'\colon \mathrm{Esp}(\mathcal{A}) \longrightarrow \mathcal{A}_{\mathrm{ex}}$ is a homotopical quotient.

- $QV'\colon \mathrm{Esp}(\mathcal{A}) \longrightarrow \mathcal{A}_{\mathrm{ex}}$ is full.
- For $f, g\colon \mathbf{G} \to \mathbf{H}$ between 2-groupoids in $\mathrm{Esp}(\mathcal{A})$,

  $QV'(f) = QV'(g)$

  if and only if

  there is $\mathbf{G} \times \mathbf{I} \xrightarrow{k} \mathbf{H}$ such that

$$\begin{tikzcd} \mathbf{G} \times \mathbf{1} \arrow[d, "\mathrm{id} \times \iota_1"'] \arrow[dr, "f\pi_1"] & \\ \mathbf{G} \times \mathbf{I} \arrow[r, "k"] & \mathbf{H} \\ \mathbf{G} \times \mathbf{1} \arrow[u, "\mathrm{id} \times \iota_2"] \arrow[ur, "g\pi_2"'] & \end{tikzcd}$$

Examples of exact completions

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$$\mathcal{E}ff \equiv \mathcal{PA}sm_{\mathrm{ex}}$$

$$\begin{tikzcd} \mathcal{E}qu \arrow[r, hook, bend right=0] \arrow[r, phantom, "\perp", shift left=2] & \mathcal{T}op_{0\,\mathrm{ex}} \arrow[l, bend right] \end{tikzcd}$$

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