

# **Exact completions as homotopical quotients**

Giuseppe Rosolini  
DIMA, Università di Genova

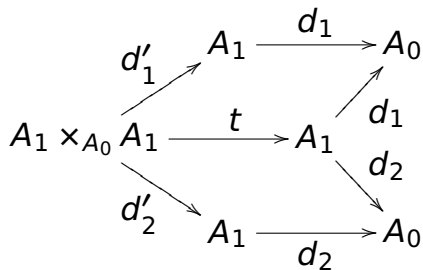
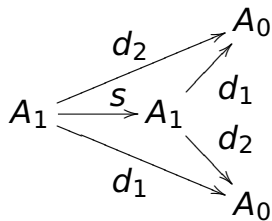
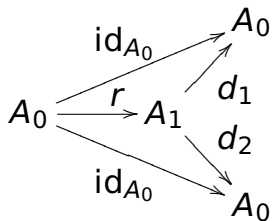
**CATEGORY THEORY 2015**  
Aveiro, 14-19 June

## Summary of the presentation

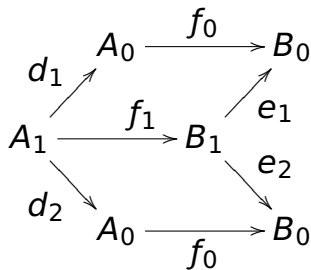
- Exact completion of a category with finite limits
- Exact completion from equivalence spans
- Exact completion from groupoids
- Exact completion from weak 2-groupoids

# Equivalence spans in a category $\mathcal{A}$ with finite limits

*Equivalence span* in  $\mathcal{A}$ :  $A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0$  with

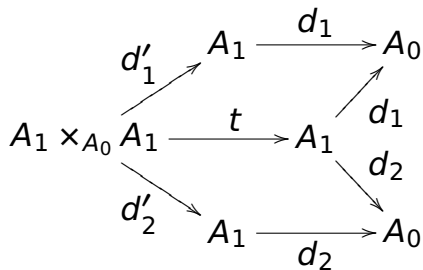
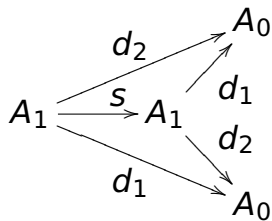
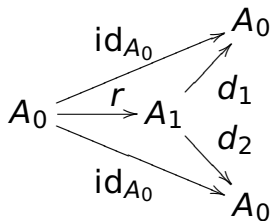


*Homomorphism* of equivalence spans in  $\mathcal{A}$ :

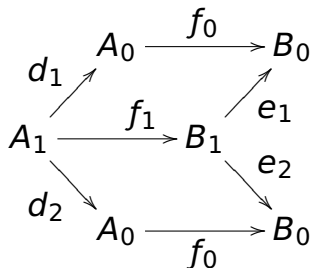


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*Equivalence span* in  $\mathcal{A}$ :  $A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0$  with



*Homomorphism* of equivalence spans in  $\mathcal{A}$ :



**Notation:**  $\text{EqSpan}(\mathcal{A})$  is the category of equivalence spans and homomorphisms.

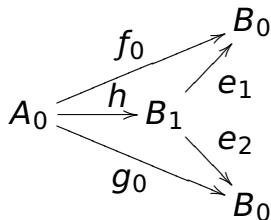
## The exact completion of $\mathcal{A}$

Recall that a category is *exact* when it has finite limits and stable effective coequalizers of equivalence relations.

The exact completion  $\mathcal{A}_{\text{ex}}$  is obtained by quotienting homsets between equivalence spans:

$$(A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0) \begin{smallmatrix} \xrightarrow{(f_1, f_0)} \\ \wr \\ \xrightarrow{(g_1, g_0)} \end{smallmatrix} (B_1 \begin{smallmatrix} \xrightarrow{e_1} \\ \xrightarrow{e_2} \end{smallmatrix} B_0)$$

if there is  $h: A_0 \longrightarrow B_1$  such that



Michael Barr

Exact categories Lecture Notes in Math. 236 (1971)

Aurelio Carboni, Rosa Celia Magno

The free exact category on a left exact one Journ. Austr. Math. Soc. (1982)

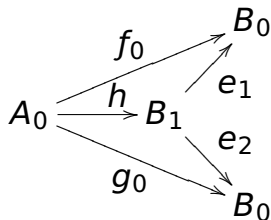
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$$\begin{array}{c} \text{EqSpan}(\mathcal{A}) \\ \downarrow Q \\ \mathcal{A}_{\text{ex}} \end{array}$$

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# Examples of exact completions

$$[C^{\text{op}}, \text{Set}] \equiv \equiv \equiv (\text{Fam}(C))_{\text{ex}} \quad C \text{ small with finite limits}$$

$$\text{Eff} \equiv \equiv \equiv \text{PAsm}_{\text{ex}}$$

$$\text{Equ} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad \perp \quad} \end{array} \text{Top}_{0\text{ex}}$$

Aurelio Carboni

Some free constructions in realizability and proof theory J. Pure Appl. Algebra (1995)

Aurelio Carboni, G.R.

Locally cartesian closed exact completions J. Pure Appl. Algebra (2000)

Matias Menni

A characterization of the left exact categories

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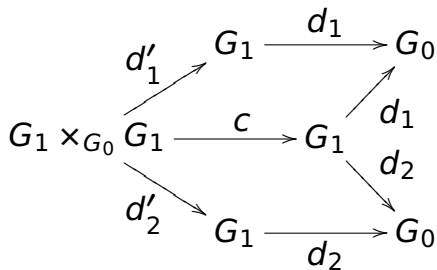
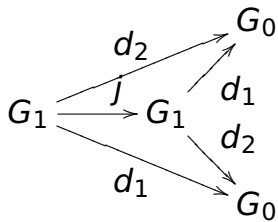
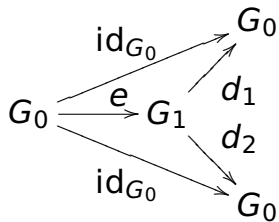
List-arithmetic distributive categories: Loco J. Pure Appl. Algebra (1990)



# Groupoids in $\mathcal{A}$

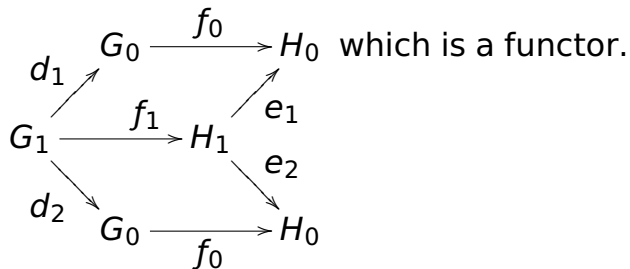
Groupoid in  $\mathcal{A}$ :

$$G_1 \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} G_0 \quad \text{together with}$$



such that  $(d_1, d_2, c, e)$  is a category and  $j$  is an inversion.

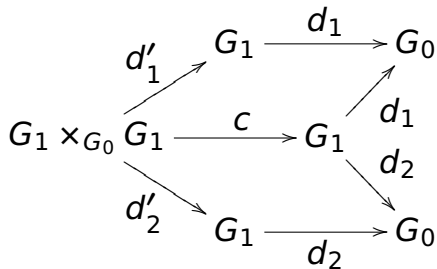
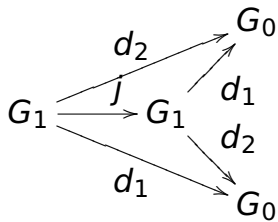
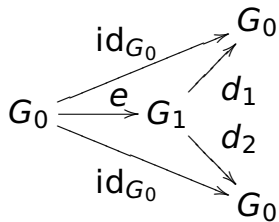
Homomorphism of groupoids:



# Groupoids in $\mathcal{A}$

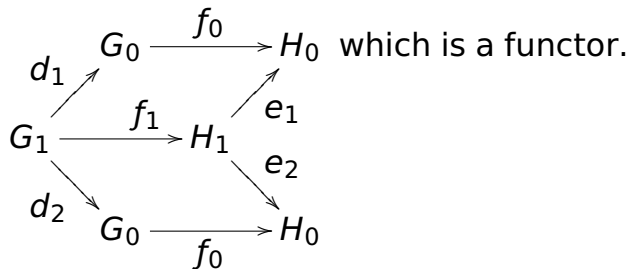
Groupoid in  $\mathcal{A}$ :

$$G_1 \begin{matrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{matrix} G_0 \quad \text{together with}$$



such that  $(d_1, d_2, c, e)$  is a category and  $j$  is an inversion.

Homomorphism of groupoids:



**Notation:**  $\text{Grpd}(\mathcal{A})$  is the category of groupoids and homomorphisms.

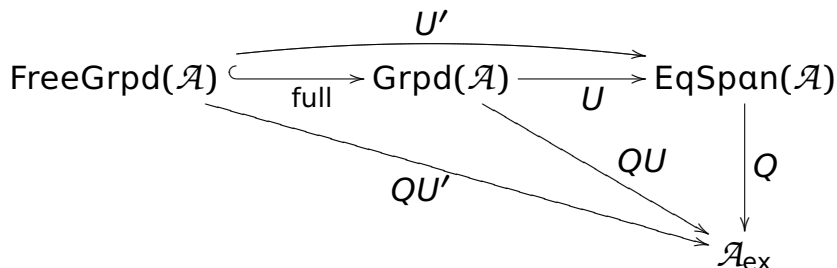
## Groupoids are equivalence spans

There is an obvious forgetful functor

$$\begin{array}{ccc} \mathbf{Grpd}(\mathcal{A}) & \xrightarrow{U} & \mathbf{EqSpan}(\mathcal{A}) \\ & \searrow QU & \downarrow Q \\ & & \mathcal{A}_{\text{ex}} \end{array}$$

# Groupoids are equivalence spans

There is an obvious forgetful functor



If  $\mathcal{A}$  is an arithmetic universe,  $QU'$  is a homotopical quotient.

The quotient is determined by the groupoid **I** on  $2 \times 2 \begin{matrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{matrix} 2$ .

André Joyal

The Gödel incompleteness theorem, a categorical approach

Cah. Top. Géom. Diff. Cat. (2005)

Maria Emilia Maietti

Joyal's arithmetic universe as list-arithmetic pretopos Theory Appl. Cat. (2010)

- $QU: \text{Grpd}(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is essentially surjective.

Given an equivalence span  $A_1 \times_{A_0} A_1 \xrightarrow{t} A_1 \xrightarrow[d_2]{d_1} A_0$

take the free groupoid on the graph  $A_1 \xrightarrow[d_2]{d_1} A_0$ .

- $QU': \text{FreeGrpd}(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is full.

$QU': \text{FreeGrpd}(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is a homotopical quotient

- For  $f, g: \mathbf{G} \rightarrow \mathbf{H}$  between free groupoids in  $\mathcal{A}$ ,

$$QU'(f) = QU'(g)$$

if and only if

there is  $\mathbf{G} \times \mathbf{I} \xrightarrow{k} \mathbf{H}$  such that

$$\begin{array}{ccc} \mathbf{G} \times \mathbf{1} & & \\ \text{id} \times l_1 \downarrow & \searrow f\pi_1 & \\ \mathbf{G} \times \mathbf{I} & \xrightarrow{k} & \mathbf{H} \\ \text{id} \times l_2 \uparrow & \nearrow g\pi_2 & \\ \mathbf{G} \times \mathbf{1} & & \end{array}$$

2-groupoid in  $\mathcal{A}$ :  $\mathbf{G} = \left( G_2 \begin{smallmatrix} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{smallmatrix} G_1 \begin{smallmatrix} \xrightarrow{d_{11}} \\ \xrightarrow{d_{12}} \end{smallmatrix} G_0, \dots \right)$  is a 2-category object in  $\mathcal{A}$  such that

- every 1-arrow is (part of) an equivalence
- every 2-arrow is iso.

Homomorphisms of 2-groupoids are weak 2-functors.

2-groupoid in  $\mathcal{A}$ :  $\mathbf{G} = \left( G_2 \begin{smallmatrix} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{smallmatrix} G_1 \begin{smallmatrix} \xrightarrow{d_{11}} \\ \xrightarrow{d_{12}} \end{smallmatrix} G_0, \dots \right)$  is a 2-category object in  $\mathcal{A}$  such that

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## Weak 2-groupoids in $\mathcal{A}$

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Homomorphisms of 2-groupoids are weak 2-functors.

Notation:  $2\text{-Grpd}_w(\mathcal{A})$  is the category of 2-groupoids and homomorphisms.

## Weak 2-groupoids in $\mathcal{A}$

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Homomorphisms of 2-groupoids are weak 2-functors.

There is an obvious forgetful functor

$$\begin{array}{ccc} 2\text{-Grpd}_w(\mathcal{A}) & \xrightarrow{V} & \text{EqSpan}(\mathcal{A}) \\ & \searrow QV & \downarrow Q \\ & & \mathcal{A}_{\text{ex}} \end{array}$$

Notation:  $2\text{-Grpd}_w(\mathcal{A})$  is the category of 2-groupoids and homomorphisms.

$QV: 2\text{-Grpd}_w(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is essentially surjective.

Given an equivalence span  $A_1 \times_{A_0} A_1 \xrightarrow{t} A_1 \xrightarrow{\begin{smallmatrix} \overset{s}{\curvearrowright} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix}} A_0$

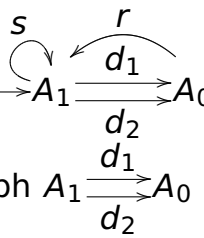
take the free dagger category on the graph  $A_1 \xrightleftharpoons[d_2]{d_1} A_0$

and turn it into a 2-groupoid  $\mathbf{D}(A_1 \xrightleftharpoons[d_2]{d_1} A_0)$

by making every 2-diagram commute.

$QV: 2\text{-Grpd}_w(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is essentially surjective.

Given an equivalence span  $A_1 \times_{A_0} A_1 \xrightarrow{t} A_1 \xrightarrow{\quad} A_0$



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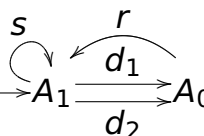
by making every 2-diagram commute.

Notation:  $\text{Esp}(\mathcal{A}) \xrightarrow[\text{full}]{\subset} 2\text{-Grpd}_w(\mathcal{A})$  is the full subcategory on retracts of such 2-groupoids.

# When $\mathcal{A}$ is a locus

$QV: 2\text{-Grpd}_w(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is essentially surjective.

Given an equivalence span  $A_1 \times_{A_0} A_1 \xrightarrow{t} A_1 \xrightarrow{\quad} A_0$




take the free dagger category on the graph  $A_1 \xrightleftharpoons[d_2]{d_1} A_0$

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by making every 2-diagram commute.

Notation:  $\text{Esp}(\mathcal{A}) \xrightarrow{\text{full}} 2\text{-Grpd}_w(\mathcal{A})$  is the full subcategory on retracts of such 2-groupoids.



$QV': \text{Esp}(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is a homotopical quotient.

- $QV': \text{Esp}(\mathcal{A}) \longrightarrow \mathcal{A}_{\text{ex}}$  is full.
- For  $f, g: \mathbf{G} \rightarrow \mathbf{H}$  between 2-groupoids in  $\text{Esp}(\mathcal{A})$ ,

$$QV'(f) = QV'(g)$$

if and only if

there is  $\mathbf{G} \times \mathbf{I} \xrightarrow{k} \mathbf{H}$  such that

$$\begin{array}{ccc}
 \mathbf{G} \times \mathbf{1} & \xrightarrow{f\pi_1} & \mathbf{H} \\
 \text{id} \times l_1 \downarrow & \nearrow k & \\
 \mathbf{G} \times \mathbf{I} & \xrightarrow{k} & \mathbf{H} \\
 \text{id} \times l_2 \uparrow & \nwarrow g\pi_2 & \\
 \mathbf{G} \times \mathbf{1} & \xrightarrow{g\pi_2} & \mathbf{H}
 \end{array}$$

# Examples of exact completions

$$[C^{\text{op}}, \text{Set}] \equiv \equiv \equiv (\text{Fam}(C))_{\text{ex}} \quad C \text{ small with finite limits}$$

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