Regularity in relational algebras and some examples

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Eilenberg-Moore category

$\mathbb{T} = (\mathsf{T}, m, e)$ on Set

- Eilenberg-Moore algebras ($\mathbb{T}$-algebras)
  (X, a) with $a : \mathsf{T}X \to X$ a map satisfying

$$\begin{tikzcd} \mathsf{TT}X \arrow[r, "\mathsf{T}a"] \arrow[d, "m_X"'] & \mathsf{T}X \arrow[d, "a"] \\ \mathsf{T}X \arrow[r, "a"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & \mathsf{T}X \arrow[d, "a"] \\ & X \end{tikzcd}$$

- $\mathbb{T}$-homomorphisms
  $f : (X, a) \to (Y, b)$ satisfying

$$\begin{tikzcd} \mathsf{T}X \arrow[r, "\mathsf{T}f"] \arrow[d, "a"'] & \mathsf{T}Y \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$f \cdot a = b \cdot \mathsf{T}f$$

$(\mathbb{T}, 2) - \mathsf{Cat}$

$\mathbb{T} = (T, m, e)$ on Set, laxly extended to Rel by $\hat{\mathbb{T}} = (\hat{\mathsf{T}}, m, e)$

- relational $\mathbb{T}$-algebras
  (X, a) with $a : \mathsf{T}X \rightarrow\!\!\!\!\!\!+\; X$ a relation satisfying

$$\begin{tikzcd} \mathsf{TT}X \arrow[r, "\hat{\mathsf{T}}a"] \arrow[d, "m_X"'] \arrow[dr, phantom, "\geq"] & \mathsf{T}X \arrow[d, "a"] \\ \mathsf{T}X \arrow[r, "a"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & \mathsf{T}X \arrow[d, "a"] \arrow[dl, phantom, "\leq"] \\ & X \end{tikzcd}$$

  - Transitivity:
$$\mathfrak{X}\ (\hat{\mathsf{T}}a)\ \mathfrak{x}\ \&\ \mathfrak{x}\ a\ x \Rightarrow m_X(\mathfrak{X})\ a\ x \qquad \forall \mathfrak{X} \in \mathsf{TT}X, \forall \mathfrak{x} \in \mathsf{T}X, \forall x \in X$$
  - Reflexivity:
$$e_X(x)\ a\ x \qquad \forall x \in X$$

- morphisms
  $f : (X, a) \to (Y, b)$ satisfying
$$f \cdot a \leq b \cdot \mathsf{T}f$$

Some specific relational $\mathbb{T}$-algebras

- $(\beta, 2)-\mathsf{Cat} \cong \mathsf{Top}$ *(Barr, 1970)*
  - Ultrafiltermonad β + Barr-extension $\overline{\beta}$
  - Transition:

    $$\mathcal{U}\ a\ x \Leftrightarrow \mathcal{U} \to x$$

  - Topological spaces characterized by 2 axioms on ultrafilters
- $(\mathbb{F}, 2)-\mathsf{Cat} \cong \mathsf{Top}$ *(Seal, 2005)*
  - Filtermonad + Kleisli-extension (power-enriched)
  - Transition:

    $$\mathcal{F}\ a\ x \Leftrightarrow \mathcal{F} \to x$$

  - Topological spaces characterized by 2 axioms on filters

Regularity

- Consider relational $\mathbb{T}$-algebras as spaces

  $a : TX \relbar\joinrel\mapsto X$ describes a notion of convergence
- Introduced by Möbus

A. Möbus, relational-algebren, PhD thesis, Universität Düsseldorf, 1981.

- Monoidal topology
  - Definition for $\mathbb{T}$-regularity in $(\mathbb{T}, \mathcal{V}) - \mathsf{Cat}$
  - Results for β-regularity
- New results
  - Results for power-enriched monads ($\mathbb{I}$ and $\mathbb{F}$ as an example)
  - Results for $\mathbb{B}$

Definition $\mathbb{T}$-regularity

Definition

An object (X, a) in $(\mathbb{T}, 2) - \mathsf{Cat}$ is $\mathbb{T}$-regular if

$$\mathfrak{X}\ (\hat{\mathsf{T}}a)\ \mathfrak{x}\ \ \&\ \ m_X(\mathfrak{X})\ a\ z\ \Rightarrow \mathfrak{x}\ a\ z,$$

$\forall \mathfrak{X} \in \mathsf{TT}X, \forall \mathfrak{x} \in \mathsf{T}X, \forall z \in X.$

Note: comparison with formula of transitivity

$$\mathfrak{X}\ (\hat{\mathsf{T}}a)\ \mathfrak{x}\ \ \&\ \mathfrak{x}\ a\ z\ \ \Rightarrow m_X(\mathfrak{X})\ a\ z$$

Example: β-regularity (Monoidal Topology)

- (X, a) β-regular $\Leftrightarrow$ corresponding topological space is regular

Power-enriched monads

Power-enriched

$\mathbb{T} = (T, m, e)$ is power-enriched if there exists a monad morphism

$$\tau : \mathbb{P} = (\mathsf{P}, \bigcup, \{\}) \to \mathbb{T}$$

such that

$$f \leq_\tau g \Rightarrow m_Y \cdot \mathsf{T}f \leq_\tau m_Y \cdot \mathsf{T}g \qquad \forall f, g : X \to TY$$

$(\mathbb{T}, \tau)$ power-enriched monad:

- The order on $\mathsf{T}X$ associated to τ is defined as follows

$$\mathfrak{x} \leq \mathfrak{y} \Leftrightarrow m_X \cdot \tau_{\mathsf{T}X}(\{\mathfrak{x}, \mathfrak{y}\}) = \mathfrak{y}, \qquad \forall \mathfrak{x}, \mathfrak{y} \in \mathsf{T}X$$

- The order makes $\mathsf{T}X$ a complete lattice and m_X and $\mathsf{T}f$ become sup-maps.
- The improper element $\mathfrak{p}$ = the least element of $\mathsf{T}X$

Kleisli-extension

Given a power-enriched monad $(\mathbb{T}, \tau)$

The Kleisli-extension $\check{\mathsf{T}}$ to Rel

Given $a : \mathsf{T}X \relbar\joinrel\mapsto X$, then $\check{\mathsf{T}}a : \mathsf{TT}X \relbar\joinrel\mapsto \mathsf{T}X$ is defined by

$$\mathfrak{X}\ \check{\mathsf{T}}a\ \mathcal{X} \Leftrightarrow \mathfrak{X} \leq a^{\tau}(\mathcal{X}), \qquad \forall \mathfrak{X} \in \mathsf{TT}X, \forall \mathcal{X} \in \mathsf{T}X.$$

- $a^{\tau} := m_{\mathsf{T}X} \cdot \mathsf{T}(\tau_{\mathsf{T}X} \cdot a^{\flat})$
- $a^{\flat} : X \to \mathsf{PT}X$ defined as

$$\mathcal{X} \in a^{\flat}(x) \Leftrightarrow \mathcal{X}\ a\ x$$

Motivation for abandoning improper

Theorem

Let $(\mathbb{T}, \tau)$ *be a power-enriched monad with its Kleisli extesion to* Rel.

$$(X, a) \textit{ is } \mathbb{T}\textit{-regular} \Leftrightarrow (X{,}a) \textit{ indiscrete}$$

Proof.

(X, a) relational $\mathbb{T}$-algebra, $X \neq \emptyset$.

$\mathbb{T}$-regularity:

$$\boxed{\mathfrak{X}\ (\check{T}a)\ \mathfrak{x}\ \&\ m_X(\mathfrak{X})\ a\ z \Rightarrow \mathfrak{x}\ a\ z}$$

Take $\mathfrak{x} \in TX$ and $z \in X$ arbitrary.
Take $\mathfrak{X}$ the improper element of TTX.

$\square$

Restricting to proper elements

- $\mathsf{T}_p X = \mathsf{T}X \setminus \{\mathfrak{p}\}$ all proper elements of $\mathsf{T}X$
- $j_X : \mathsf{T}_p X \to \mathsf{T}X$ the canonical injection
- Under certain assumptions: T_p subfunctor of T
    - NOT necessarily a submonad!

T_p-regularity

Definition

An object (X, a) in $(\mathbb{T}, 2) - \mathsf{Cat}$ is T_p-regular if

$$\mathfrak{X}\ (\hat{\mathsf{T}}a)\ \mathfrak{x}\ \ \&\ \ m_X(\mathfrak{X})\ a\ z\ \Rightarrow \mathfrak{x}\ a\ z,$$

$\forall \mathfrak{X} \in \mathsf{T}j_X(\mathsf{T}\mathsf{T}_pX)$ with $m_X(\mathfrak{X}) \in \mathsf{T}_pX, \forall \mathfrak{x} \in \mathsf{T}_pX, \forall z \in X.$

T_p-regularity for power-enriched monads

Theorem

Assume $\mathfrak{p} \neq e_X(z), \forall z \in X$.

$$(X, a) \ \mathsf{T}_p\text{-}\mathit{regular} \Leftrightarrow (X, a) \ \mathit{indiscrete}$$

Sketch of the proof

(X, a) satisfies all conditions $(X \neq \emptyset)$.

Let $z \in X$ arbitrary.

$\mathfrak{q}$ = largest element of $\mathsf{T}X$

Define $a_p : \mathsf{T}_p X \relbar\joinrel\mapsto X$ by $a_p = a \cdot j_X$

Consider $\mathfrak{X} = \mathsf{T} j_X \big(a_p^\tau \cdot e_X(z) \big)$.

- $\mathfrak{X} \in \mathsf{T} j_X (\mathsf{T}\mathsf{T}_p X)$
- $m_X(\mathfrak{X}) \in \mathsf{T}_p X$

$$a_p^\tau \cdot e_X(z) = (\tau_{\mathsf{T}_p X} \cdot a_p^\flat)^{\mathbb{T}} \cdot e_X(z) = \tau_{\mathsf{T}_p X} \cdot a_p^\flat(z)$$

Application of several properties related to the power-enrichment

$$\mathfrak{X} \le a^\tau(e_X(z))$$

- Since $e_X(z) \le \mathfrak{q}$

$$\mathfrak{X}\ (\check{T}a)\ \mathfrak{q}$$

- Left-unitary:

$$\mathfrak{X}\ (\check{T}a)\ e_X(z)\ \Rightarrow m_X(\mathfrak{X})\ a\ z$$

T_p-Regularity:

$$\Rightarrow \mathfrak{q}\ a\ z$$

Take $\mathcal{X} \in \mathsf{T}X$ arbitrary
Right-unitary:

$$\mathcal{X} \le \mathfrak{q}\ \&\ \mathfrak{q}\ a\ z \Rightarrow \mathcal{X}\ a\ z$$

$\mathbb{F}$-regularity

$$(X,a)\ \mathbb{F}-\text{regular} \ \Leftrightarrow\ (X,a)\ \mathsf{F}_p-\text{regular} \ \Leftrightarrow\ (X,a)\ \text{indiscrete}$$

Solution: Restriction to pairs of selections

$$\begin{tikzcd}
 & & X & \psi(\mathcal{G}) = \mathcal{F} \\
\mathcal{G} & A \arrow[ur, "\psi"] \arrow[dr, "\sigma"'] & & \\
 & & \mathsf{F}_p X & \sigma(\mathcal{G}) = \mathfrak{F}
\end{tikzcd}$$

such that $\sigma(z) \to \psi(z), \forall z \in A$

$$\mathfrak{F}\ (\check{F}a)\ \mathcal{F}\ \&\ \Sigma\mathfrak{F} \to x \Rightarrow \mathcal{F} \to x$$

becomes

$$\Sigma\mathfrak{F} \to x \Rightarrow \mathcal{F} \to x$$

Approach spaces

$\mathcal{F} \to x$ is replaced by

$$\lambda : \mathsf{F}\, X \to [0,\infty]^X : \mathcal{F} \to \lambda\mathcal{F}$$

Axioms

1. $\lambda\dot{x}(x) = 0, \forall x \in X$
2. $\mathcal{G} \subseteq \mathcal{F} \Rightarrow \lambda\mathcal{F} \leq \lambda\mathcal{G}$
3. $\sigma : A \to \mathsf{F}\, X, \psi : A \to X, \mathcal{G} \in \mathsf{F}\, A :$

$$\lambda\Sigma\sigma(\mathcal{G}) \leq \lambda\psi(\mathcal{G}) + \sup_{z\in A} \lambda\sigma(z)\psi(z)$$

INTERPRETATION: $\lambda\mathcal{F}(x) =$ 'the distance that x is away from being a limit point of $\mathcal{F}$'

The category App

- Objects: (X, λ)
- Morphisms: $f : (X, \lambda) \to (Y, \lambda')$ with

$$\lambda' f(\mathcal{F}) \cdot f \leq \lambda \mathcal{F}$$

Functional ideals

Definition

An order-theoretic ideal $\mathcal{J}$ in $[0,\infty]^X$ is called a functional ideal if

1. all functions in $\mathcal{J}$ are bounded
2. $\mathcal{J}$ is saturated. For $\mu \in [0,\infty]^X_b$:

$$\forall \epsilon > 0 \exists \varphi \in \mathcal{J} : \mu \leq \varphi + \epsilon \Rightarrow \mu \in \mathcal{J}$$

$$\mathcal{J} \oplus \alpha := \begin{cases} \{\nu \in [0,\infty]^X_b \mid \exists \mu \in \mathcal{J} : \nu \leq \mu + \alpha\} & \alpha < \infty \\ [0,\infty]^X_b & \alpha = \infty \end{cases}$$

The functional ideal monad $\mathbb{I}$

- $\mathsf{I} : \mathsf{Set} \to \mathsf{Set}$
  - $\mathsf{I}X = \{\mathcal{J} \mid \mathcal{J} \text{ functional ideal on } X\}$
  - $\mathsf{I}f : \mathsf{I}X \to \mathsf{I}Y : \mathcal{J} \to \mathsf{I}f(\mathcal{J})$

    with

$$\nu \in \mathsf{I}f(\mathcal{J}) \Leftrightarrow \nu \cdot f \in \mathcal{J}$$

- $e_X(x) = \{\mu \in [0,\infty]_b^X \mid \mu(x) = 0\}$
- $m_X(\Phi) = \{\mu \in [0,\infty]_b^X \mid I_\mu \in \Phi\}$,

  with

$$\begin{array}{rrcl} I_\mu : & \mathsf{I}X & \to & [0,\infty] : \\ & \mathcal{J} & \mapsto & I_\mu(\mathcal{J}) := \inf\{\alpha \in [0,\infty] \mid \mu \in \mathcal{J} \oplus \alpha\} \end{array}$$

Approach spaces as relational algebras

- $\mathbb{I}$ is power-enriched by $\tau : \mathbb{P} \to \mathbb{I}$ with

$$\tau_X(A) = \{\mu \in [0,\infty]^X \mid \mu_{|A} = 0\}$$

- $\leq_\tau$ = reverse inclusion order
  $\to [0,\infty]^X_b$ = improper element
- Kleisli-extension
- $(\mathbb{I}, 2) - \mathsf{Cat} \cong \mathsf{App}$
  Transition:

$$\mathcal{J} \ a \ x \Leftrightarrow \forall \alpha \in [c(\mathcal{J}), \infty] : \lambda \mathfrak{f}_\alpha(\mathcal{J})(x) \leq \alpha$$

A new example

$$(X, a)\ \mathbb{T} - \text{regular} \Leftrightarrow (X, a)\ \mathsf{I}_p - \text{regular} \Leftrightarrow (X, a) \text{ indiscrete}$$

Restriction to pairs of functional ideals, generated by selections

$$\begin{tikzcd} & & X & \mathrm{I}\psi(\mathcal{J}) = \mathcal{K} \\ \mathcal{J} & A \arrow[ur, "\psi"] \arrow[dr, "s"'] & & \\ & & \mathrm{I}_p X & \mathrm{I}s(\mathcal{J}) = \Phi \end{tikzcd}$$

$$\text{such that } s(z) \to \psi(z), \forall z \in A$$

$$\Phi\ (\check{I}a)\ \mathcal{K}\ \&\ m_X(\Phi) \to x \Rightarrow \mathcal{K} \to x$$

becomes

$$m_X(\Phi) \to x \Rightarrow \mathcal{K} \to x$$

Still too strong

Extra conditions

- $c(s(z)) = 0, \forall z \in A$
- $\forall \delta > 0 : s(z) \oplus \delta \to \psi(z)$ then $\mathcal{K} \oplus \delta \to x$

$\Rightarrow$ regular approach spaces (Robeys, Brock & Kent)

$$\lambda\psi(\mathcal{G}) \leq \lambda\Sigma\sigma(\mathcal{G}) + \sup_{z \in A} \lambda\sigma(z)\psi(z)$$

The case for the prime functional ideal monad

Prime functional ideal

A functional ideal $\mathfrak{H}$ on X is prime if for all bounded functions μ, ν

$$\mu \wedge \nu \in \mathfrak{H} \Rightarrow \mu \in \mathfrak{H} \text{ or } \nu \in \mathfrak{H}$$

- $\mathbb{B}$ submonad $\mathbb{I}$
- not power-enriched
- initial extension to Rel
- $(\mathbb{B}, 2) - \mathsf{Cat} \cong \mathsf{App}$

Results

- BX is not a complete lattice, but has a least element (IMPROPER)

$$(X, a)\ \mathbb{B} - \text{regular} \Leftrightarrow (X, a) \text{ indiscrete}$$

- B_p-regularity gives interesting results

$$(X, a)\ \mathsf{B}_p\text{-regular} \Leftrightarrow (X, a) \text{ regular topological space}$$

- weakening the conditions gives regularity in App