Topoi generated by topological spaces

"A generalization of Johnstone´s topos"

"Topoi generated by topological monoids"

José Reinaldo Montañez Puentes
jrmontanezp@unal.edu.co

Departamento de Matemáticas
Universidad Nacional de Colombia

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Problem1

Given a topological space W , we consider the monoid $M = [W, W]$ which determines

$$E : Top \to MSets$$
$$E(X) = [W, X]$$
$$E_W(f) = \overline{f}$$

Problems:
1. Reduce Top in order to make the functor E full and faithful.

$$E : C \to MSets$$

2. Reduce $MSets$ so that all topological spaces are sheaves, and the functor remains full and faithful.

$$E : C \to \mathcal{E}$$

Solution to problem 1

Solution to problem 1: : Elevator Functors
Let W be a topological space. It determines the functor

$$E_W : Top \to Top$$
$$E_W(X) = \overline{X}$$
$$E_W(f) = f$$

$E_W(X)$ *is the final topology for the sink* $\{h : W \to \mathbf{X} | h \in [\mathbf{w}, \mathbf{x}]\}$

$$I : E_W(Top) \to Top$$

$E_W(Top)$ *is both a topological and correflective subcategory of* Top.

Topoi that extends topological subcategories of Top

Let W be a topological space and let $M = [W, W]$ the endomorfism monoid of W and $MSets$ the associated $MSets$ topos.
It determines the functor

$$E : E_W(Top) \longrightarrow MSets$$

Defined by:

$$E(X) := [W, X], \qquad \Sigma_W(f) := \overline{f}, \qquad con\ \overline{f}(h) = f \circ h$$

E is full and faithful and has left adjoint,:

$$L : MSets \longrightarrow E_W(Top)$$

The notion of the topological topos

Definition

Let C be a topological subcategory of Top and let $\mathcal{E}$ be a topos. It is said that $\mathcal{E}$ is a C-topological topos, if $\mathcal{E}$ contains an isomorphic reflective subcategory to C.

Example

Let W be a topological space and let $M = [W, W]$ the associated endomorfisms monoid. $MSets$ is a topological topos.

Solution to problem 2

Let W be a topological space and $M = [W, W]$ the associated endomorfisms monoid.

The topos of sheaves $Sh(W)$

$Sh(W)$ is the subtopos of $E_W(Top)$ formed by the M Sets that are sheaves by the Grotendieck topology determined by extensive ideals of M.

Theorem

let W be a topological space.

1. The topos $Sh(W)$ includes as sheaves all topological spaces.
2. The topos $Sh(W)$ includes $E_W(Top)$ like an isomorphic reflective subcategory of Top .
3. $Sh(W)$ is a topological topos.

Extensive topoi

Definition

It is said that $\mathcal{E}$ is Extensive Topos if it is equivalent to a topos $Sh(W)$ for some topological space W. Clearly $\mathcal{E}$ is a topological topos.

Examples

1. The Jhonstone's topos, is both topological topos and extensive topos and it is generated by $\mathbb{N}_\infty = \{0, 1, 1/2, \ldots, 1/n, \ldots\}$ as a subspace of the real numbers. $E_{\mathbb{N}_\infty}(Top)$ is the category of the sequential spaces.
2.The Bornological topos is both topological topos and extensive topos. This Topos was presented by F. W. Lawvere.

Remarks

1. Conjecture: The real numbers in the extensive topos generated by the topological space W are the continuous functions from W to the real numbers, $\mathbb{R}$, $[W, \mathbb{R}]_{Top}$.
2. The notion of topological topos can be presented beginning with a topological construct.

Other topoi for working in topology
"topoi generated by topological monoids"

1.Given a topological monoid M it determines the topos $MSets$ and the functor defined through the final topologies.

$$F : MSets \to Top$$
$$F(X) = [M, X]$$
$$F(f) = f$$

2. One topos equivalent to the form $MSets$ with M topological monoid is called Geometrical topos.
2. In particular, If M is abelian topological monoid, the actions remains continuous respect to the tensorial product topology.

3. Let (W, d) a compact metric space. The monoid $M = [W, W]$ having as elements the contractions. M is a topological monoid with the topology generated by the metric:
$D(f, g) = \sup\{d(f(x), g(x))\}$.
$MSets$ is a Geometrical topos.
4. It is said that the topological space W is A-Compact if their open subspaces are compacts. If $M = [W, W]$ has the open compact topology, then M is a topological monoid, generating both a geometrical and a topological topos.

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