Partial Mal’tsevness and category of quandles

Dominique Bourn

Lab. Math. Pures Appliquées J. Liouville, CNRS (FR.2956)
Université du Littoral
Calais - France

CT 2015, Aveiro, 14-19 june 2015
Outline

Monoids and partial pointed protomodularity

Mal’tsev and $\Sigma$-Mal’tsev category

Quandles

Naturally Mal’tsev and $\Sigma$-naturally Mal’tsev category
Outline

Monoids and partial pointed protomodularity

Mal’tsev and Σ-Mal’tsev category

Quandles

Naturally Mal’tsev and Σ-naturally Mal’tsev category
**Definition (B. 1990)**

A pointed category $$\mathbb{C}$$ is protomodular when, for any split epimorphism $$(f, s)$$, the following pullback,:

$$
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\uparrow & & \uparrow \\
1 & \xrightarrow{\alpha_Y} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
f & \xrightarrow{s} & X
\end{array}
$$

is such that the pair $$(k_f, s)$$ is jointly extremally epic, or in other words $1_X = \text{sup}(k_f, s)$.

**Examples:** Groups, non-unital Rings, $$K$$-algebras of any non-unitary kind, Lie$$_K$$-algebras; dual of pointed objects in any topos ....
Definition (B. 1990)

A pointed category $\mathbb{C}$ is protomodular when, for any split epimorphism $(f, s)$, the following pullback:

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow f \\
1 & \xrightarrow{\alpha_Y} & Y
\end{array}
\]

is such that the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \text{sup}(k_f, s)$.

Examples: Groups, non-unital Rings, $K$-algebras of any non-unitary kind, Lie$_K$-algebras; dual of pointed objects in any topos ....
protomodularity is the right context to deal with exact sequences and homological lemmas in a non-abelian setting.

on the other hand, any protomodular category is a Mal’tsev one.

Definition (Carboni, Lambek, Pedicchio 1990)
A Mal’tsev category is such that any reflexive relation is an equivalence relation.

a first simple consequence:
in a Mal’tsev category, on any reflexive graph there is at most one structure of internal category which is necessarily a groupoid structure.
protomodularity is the right context to deal with exact sequences and homological lemmas in a non-abelian setting.

on the other hand, any protomodular category is a Mal’tsev one.

Definition (Carboni, Lambek, Pedicchio 1990)
A Mal’tsev category is such that any reflexive relation is an equivalence relation.

a first simple consequence:
in a Mal’tsev category, on any reflexive graph there is at most one structure of internal category which is necessarily a groupoid structure.
protomodularity is the right context to deal with exact sequences and homological lemmas in a non-abelian setting.

on the other hand, any protomodular category is a Mal’tsev one.

**Definition (Carboni, Lambek, Pedicchio 1990)**

*A Mal’tsev category is such that any reflexive relation is an equivalence relation.*

a first simple consequence:
in a Mal’tsev category, on any reflexive graph there is at most one structure of internal category which is necessarily a groupoid structure.
protomodularity is the right context to deal with exact sequences and homological lemmas in a **non-abelian setting**.

on the other hand, any protomodular category is a Mal’tsev one.

**Definition (Carboni, Lambek, Pedicchio 1990)**

A Mal’tsev category is such that any reflexive relation is an equivalence relation.

a first simple consequence:
in a Mal’tsev category, on any reflexive graph there is at most one structure of internal category which is necessarily a groupoid structure.
More importantly, there are two major structural facts for a Mal’tsev category $\mathcal{D}$:

1) when, in addition, $\mathcal{D}$ is regular, the reflexive relations can be composed and do permute; i.e. $R \circ S = S \circ R$; [Carboni, Lambek, Pedicchio]

2) Mal’tsevness is the right context to deal with the notion of centralization of equivalence relations [Pedicchio 1995; B.+ Gran 2002]
More importantly, there are two major structural facts for a Mal’tsev category $\mathcal{D}$:

1) when, in addition, $\mathcal{D}$ is regular, the reflexive relations can be composed and do permute; i.e. $R \circ S = S \circ R$; [Carboni, Lambek, Pedicchio]

2) Mal’tsevness is the right context to deal with the notion of centralization of equivalence relations [Pedicchio 1995; B.+ Gran 2002]
More importantly, there are two major structural facts for a Mal’tsev category $\mathbb{D}$:

1) when, in addition, $\mathbb{D}$ is regular, the reflexive relations can be composed and do permute; i.e. $R \circ S = S \circ R$; [Carboni, Lambek, Pedicchio]

2) Mal’tsevness is the right context to deal with the notion of centralization of equivalence relations [Pedicchio 1995; B.+ Gran 2002]
The idea of partial protomodularity only relative to a class \( \Sigma \) of split epimorphisms.

The category \( \text{Mon} \) of monoids.

**Definition (Martins-Ferreira, Montoli, Sobral 2013)**

A split monoid homomorphism is a Schreier one when the application \( \mu_y : \text{Ker}f \to f^{-1}(y) \) defined by \( \mu_y(k) = s(y) \cdot k \) is bijective.

Any Schreier split homomorphism is such that in the following diagram:

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow f \\
1 & \xrightarrow{\alpha_Y} & Y
\end{array}
\]

the pair \((k_f, s)\) is jointly extremally epic, or in other words \(1_X = \text{sup}(k_f, s)\).

The class \( \Sigma \) of Schreier split epimorphisms is:
- stable under composition and pullback
- contains the isomorphisms.
- stable under finite limits inside the split epimorphisms.
The idea of partial protomodularity only relative to a class $\Sigma$ of split epimorphims. 

The category $\text{Mon}$ of monoids.

**Definition (Martins-Ferreira, Montoli, Sobral 2013)**

A split monoid homomorphism is a Schreier one when the application $\mu_y : \text{Ker} f \to f^{-1}(y)$ defined by $\mu_y(k) = s(y) \cdot k$ is bijective.

Any Schreier split homomorphism is such that in the following diagram:

$$
\begin{array}{c}
K[f] \\ k_f \\
\Rightarrow \\
\downarrow \\
\Rightarrow \\
\downarrow \\
X \\ f \\
\downarrow \\
\uparrow \\
Y \\
\downarrow \\
\Rightarrow \\
\uparrow \\
1 \\
\Rightarrow \\
\\alpha_Y \\
\end{array}
$$

the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \sup(k_f, s)$.

The class $\Sigma$ of Schreier split epimorphisms is:
- stable under composition and pullback
- contains the isomorphisms.
- stable under finite limits inside the split epimorphisms.
The idea of partial protomodularity only relative to a class $\Sigma$ of split epimorphisms.

The category $Mon$ of monoids.

**Definition (Martins-Ferreira, Montoli, Sobral 2013)**

A *split monoid homomorphism* is a Schreier one when the application $\mu_y : \text{Ker}f \to f^{-1}(y)$ defined by $\mu_y(k) = s(y) \cdot k$ is bijective.

Any Schreier split homomorphism is such that in the following diagram:

$$
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow f \\
1 & \xrightarrow{\alpha_Y} & Y
\end{array}
$$

the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = sup(k_f, s)$.

The class $\Sigma$ of Schreier split epimorphisms is:
- stable under composition and pullback
- contains the isomorphisms.
- stable under finite limits inside the split epimorphisms.
The idea of partial protomodularity only relative to a class \( \Sigma \) of split epimorphisms.

The category \( \text{Mon} \) of monoids.

**Definition (Martins-Ferreira, Montoli, Sobral 2013)**

A split monoid homomorphism is a Schreier one when the application \( \mu_y : \text{Ker} f \rightarrow f^{-1}(y) \) defined by \( \mu_y(k) = s(y) \cdot k \) is bijective.

Any Schreier split homomorphism is such that in the following diagram:

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\uparrow & & \downarrow \\
1 & \xrightarrow{\alpha \gamma} & Y
\end{array}
\]

the pair \((k_f, s)\) is jointly extremally epic, or in other words \(1_X = \text{sup}(k_f, s)\).

The class \( \Sigma \) of Schreier split epimorphisms is:
- stable under composition and pullback
- contains the isomorphisms.
- stable under finite limits inside the split epimorphisms.
The idea of partial protomodularity only relative to a class $\Sigma$ of split epimorphisms.

The category $\text{Mon}$ of monoids.

Definition (Martins-Ferreira, Montoli, Sobral 2013)

A split monoid homomorphism is a Schreier one when the application $\mu_y : \text{Ker}f \rightarrow f^{-1}(y)$ defined by $\mu_y(k) = s(y) \cdot k$ is bijective.

Any Schreier split homomorphism is such that in the following diagram:

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow f \\
1 & \xrightarrow{\alpha_Y} & Y
\end{array}
\]

the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \text{sup}(k_f, s)$.

The class $\Sigma$ of Schreier split epimorphisms is:
- stable under composition and pullback
- contains the isomorphisms.
- stable under finite limits inside the split epimorphisms.
Definition (B., Martins-Ferreira, Montoli, Sobral 2014)

A pointed category $\mathbb{C}$ is said to be $\Sigma$-protomodular provided:

- the class $\Sigma$ is point-congruous: i.e. is stable under pullback, contains the isomorphisms and is stable under finite limits inside the class of all split epimorphisms.

- any split epimorphism $(f, s) \in \Sigma$ is strongly split: i.e. such that in the following diagram:

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow s \\
1 & \xrightarrow{\alpha_Y} & Y
\end{array}
\]

the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \sup(k_f, s)$.

- Examples: $Mon$, and on strictly the same model as $Mon$, the category $SRg$ of semi-rings by means of $U : SRg \to CoM$ with the class $U^{-1}(\Sigma)$. 
Definition (B., Martins-Ferreira, Montoli, Sobral 2014)

A pointed category $\mathbb{C}$ is said to be $\Sigma$-protomodular provided:

- The class $\Sigma$ is point-congruous: i.e. is stable under pullback, contains the isomorphisms and is stable under finite limits inside the class of all split epimorphisms.

- Any split epimorphism $(f, s) \in \Sigma$ is strongly split: i.e. such that in the following diagram:

$$
\begin{array}{ccc}
K[f] & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\alpha_Y} & Y \\
\end{array}
\quad
\begin{array}{ccc}
& & f \\
\downarrow & & \downarrow \\
1 & \xrightarrow{s} & Y \\
\end{array}
$$

The pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \sup(k_f, s)$.

- Examples: $\text{Mon}$, and on strictly the same model as $\text{Mon}$, the category $\text{SRg}$ of semi-rings by means of $U : \text{SRg} \to \text{CoM}$ with the class $U^{-1}(\Sigma)$. 
Definition (B., Martins-Ferreira, Montoli, Sobral 2014)

A pointed category $\mathbb{C}$ is said to be $\Sigma$-protomodular provided:

- the class $\Sigma$ is point-congruous: i.e. is stable under pullback, contains the isomorphisms and is stable under finite limits inside the class of all split epimorphisms.

- any split epimorphism $(f, s) \in \Sigma$ is strongly split: i.e. such that in the following diagram:

\[
\begin{array}{c}
K[f] \xrightarrow{k_f} X \\
\downarrow \quad \downarrow f \\
1 \xrightarrow{\alpha_Y} Y
\end{array}
\]

the pair $(k_f, s)$ is jointly extremally epic, or in other words $1_X = \sup(k_f, s)$.

- Examples: $Mon$, and on strictly the same model as $Mon$, the category $SRg$ of semi-rings by means of $U : SRg \to CoM$ with the class $U^{-1}(\Sigma)$. 
Main tools:

- **Σ-relation**: a relation which is reflexive and such that \((d_0, s_0)\) belongs to Σ:

  \[
  R \xleftarrow{s_0} X \xrightarrow{d_0} \xrightarrow{d_1}
  \]

- **Σ-special morphism** \(f : X \to Y\): when the kernel relation \(R[f]\) is a Σ-relation

- **Σ-special object** \(X\): when the terminal map \(X \to 1\) is Σ-special.
Main tools:

- **Σ-relation**: a relation which is reflexive and such that \((d_0, s_0)\) belongs to \(\Sigma\):

\[
\begin{array}{c}
\text{R} \\
\downarrow \quad \downarrow \quad \downarrow \\
\downarrow \quad \quad \quad \downarrow \\
d_1 \\
\end{array}
\]

- **Σ-special morphism** \(f : X \to Y\): when the kernel relation \(R[f]\) is a Σ-relation

- **Σ-special object** \(X\): when the terminal map \(X \to 1\) is Σ-special.
Main tools:

- **Σ-relations**: a relation which is reflexive and such that \((d_0, s_0)\) belongs to \(\Sigma\):

\[
\begin{array}{c}
\xrightarrow{d_0} \\
R \xleftarrow{s_0} X \\
\xrightarrow{d_1}
\end{array}
\]

- **Σ-special morphism** \(f : X \to Y\): when the kernel relation \(R[f]\) is a \(\Sigma\)-relation

- **Σ-special object** \(X\): when the terminal map \(X \to 1\) is Σ-special.
Main results: aspects of partial pointed protomodularity:

- the $\Sigma$-exact sequences, where $f$ is a $\Sigma$-special regular epimorphism

\[ 1 \to K[f] \to X \to Y \to 1 \]

satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

\[ 1 \to A \to X \to Y \to 1 \]

- the full subcategory $\Sigma C_\sharp$ of $\Sigma$-special objects (called the core of the pointed $\Sigma$-protomodular category) is protomodular

- the core of $Mon$ is the category $Gp$ of goups
- the core of $SRg$ is the category $Rg$ of rings
Main results= aspects of partial pointed protomodularity:

- the $\Sigma$-exact sequences, where $f$ is a $\Sigma$-special regular epimorphism

$$1 \to K[f] \hookrightarrow X \to Y \to 1$$

satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

$$1 \to A \hookrightarrow X \to Y \to 1$$

- the full subcategory $\Sigma C_\#$ of $\Sigma$-special objects (called the core of the pointed $\Sigma$-protomodular category) is protomodular

- the core of $\text{Mon}$ is the category $\text{Gp}$ of groups
- the core of $\text{SRg}$ is the category $\text{Rg}$ of rings
Main results: aspects of partial pointed protomodularity:

- the $\Sigma$-exact sequences, where $f$ is a $\Sigma$-special regular epimorphism
  
  $1 \to K[f] \hookrightarrow X \twoheadrightarrow Y \to 1$

  satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

  $1 \to A \hookrightarrow X \twoheadrightarrow Y \to 1$

- the full subcategory $\Sigma \mathcal{C}_\#$ of $\Sigma$-special objects
  (called the core of the pointed $\Sigma$-protomodular category)
  is protomodular

- the core of $\text{Mon}$ is the category $\text{Gp}$ of goups

- the core of $\text{SRg}$ is the category $\text{Rg}$ of rings
Main results = aspects of partial pointed protomodularity:

- the $\Sigma$-exact sequences, where $f$ is a $\Sigma$-special regular epimorphism

$$1 \to K[f] \hookrightarrow X \twoheadrightarrow Y \twoheadrightarrow 1$$

satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

$$1 \to A \hookrightarrow X \twoheadrightarrow Y \twoheadrightarrow 1$$

- the full subcategory $\Sigma \mathcal{C}_\#$ of $\Sigma$-special objects (called the core of the pointed $\Sigma$-protomodular category) is protomodular

- the core of $\text{Mon}$ is the category $\text{Gp}$ of goups

- the core of $\text{SRg}$ is the category $\text{Rg}$ of rings
Main results: aspects of partial pointed protomodularity:

- the $\Sigma$-exact sequences, where $f$ is a $\Sigma$-special regular epimorphism

$$1 \to K[f] \hookrightarrow X \twoheadrightarrow Y \to 1$$

satisfy some homological lemmas

- there is a Baer sum on the abelian special extensions:

$$1 \to A \hookrightarrow X \twoheadrightarrow Y \to 1$$

- the full subcategory $\Sigma C\#_\Sigma$ of $\Sigma$-special objects (called the core of the pointed $\Sigma$-protomodular category) is protomodular

- the core of $Mon$ is the category $Gp$ of goups

- the core of $SRg$ is the category $Rg$ of rings
we noticed not-unexpected partial aspects of Mal’tsevness:

1) any $\Sigma$-relation is transitive

2) an intrinsic notion of centralization for $\Sigma$-relations
we noticed not-unexpected partial aspects of Mal’tsevness:

1) any $\Sigma$-relation is transitive

2) an intrinsic notion of centralization for $\Sigma$-relations
we noticed not-unexpected partial aspects of Mal’tsevness:

1) any \( \Sigma \)-relation is transitive

2) an intrinsic notion of centralization for \( \Sigma \)-relations
From some limitations of this example to some questions:

1) only one kind of example; how distinguish what is important from what is incidental for the class $\Sigma$ concerning this question of partial pointed protomodularity

2) only pointed case, although protomodularity is not a pointed concept

3) how to unknot what comes from partial Mal’tsevness and what comes from partial protomodularity

4) to produce a discriminating example: here comes the notion of quandle.
From some limitations of this example to some questions:

1) only one kind of example; how distinguish what is important from what is incidental for the class $\Sigma$ concerning this question of partial pointed protomodularity

2) only pointed case, although protomodularity is not a pointed concept

3) how to unknot what comes from partial Mal’tsevness and what comes from partial protomodularity

4) to produce a discriminating example: here comes the notion of quandle.
From some limitations of this example to some questions:

1) only one kind of example; how distinguish what is important from what is incidental for the class $\Sigma$ concerning this question of partial pointed protomodularity

2) only pointed case, although protomodularity is not a pointed concept

3) how to unknot what comes from partial Mal’tsevness and what comes from partial protomodularity

4) to produce a discriminating example: here comes the notion of quandle.
From some limitations of this example to some questions:

1) only one kind of example; how distinguish what is important from what is incidental for the class $\Sigma$
   concerning this question of partial pointed protomodularity

2) only pointed case, although protomodularity is not a pointed concept

3) how to unknot what comes from partial Mal’tsevness and what comes from partial protomodularity

4) to produce a discriminating example: here comes the notion of quandle.
From some limitations of this example to some questions:

1) only one kind of example; how distinguish what is important from what is incidental for the class $\Sigma$ concerning this question of partial pointed protomodularity

2) only pointed case, although protomodularity is not a pointed concept

3) how to unknot what comes from partial Mal’tsevness and what comes from partial protomodularity

4) to produce a discriminating example: here comes the notion of quandle.
Outline

Monoids and partial pointed protomodularity

Mal’tsev and $\Sigma$-Mal’tsev category

Quandles

Naturally Mal’tsev and $\Sigma$-naturally Mal’tsev category
Back to Mal’tsev categories. We have the following characterization:

▶ **Proposition (B. 1996)**

A category $\mathcal{D}$ is a Mal’tsev one if and only if any fibre of the fibration of points $\mathcal{P}_\mathcal{D}$ is unital.

▶ which means that in the following rightward pullback of split epimorphisms:

\[
\begin{array}{ccc}
X \times_Y Z & \xleftarrow{i_X} & X \\
\downarrow{p_Z} & & \downarrow{p_X} \\
Z & \xleftarrow{i_Z} & Y \\
\end{array}
\]

the pair of sections $(i_Z, i_X)$ is jointly extremal epic.
Back to Mal’tsev categories. We have the following characterization:

- **Proposition (B. 1996)**
  
  A category \( \mathcal{D} \) is a Mal’tsev one if and only if any fibre of the fibration of points \( \mathcal{P}_\mathcal{D} \) is unital.

  which means that in the following rightward pullback of split epimorphisms:

  \[
  \begin{array}{ccc}
  X \times_Y Z & \xleftarrow{\iota_X} & X \\
  \downarrow{p_Z} & & \downarrow{p_X} \\
  Z & \xleftarrow{\iota_Z} & Y \\
  \downarrow{t} & & \downarrow{s} \\
  Z & \xrightarrow{g} & Y
  \end{array}
  \]

  the pair of sections \((i_Z, i_X)\) is jointly extremal epic.
So the natural notion of $\Sigma$-Mal’tsev category should be:

**Definition**

- A category $\mathbb{D}$ is a $\Sigma$-Mal’tsev category Mal’tsev when, in the following rightward pullback of split epimorphisms:

\[
\begin{array}{ccc}
X & \times & Y \\
\downarrow & & \downarrow \\
Z & \leftarrow & \leftarrow X \\
p_Z & & p_X \\
\downarrow & & \downarrow \\
Z & \leftarrow & \leftarrow Y \\
g & & t \\
\end{array}
\]

the pair $(i_Z, i_X)$ is jointly extremal epic, provided that the split epimorphism $(f, s)$ belongs to $\Sigma$.

- + some condition on the class $\Sigma$ to be precised

- actually we shall see that there is an important distinction between two levels.
So the natural notion of $\Sigma$-Mal’tsev category should be:

**Definition**

- A category $\mathbb{D}$ is a $\Sigma$-Mal’tsev category Mal’tsev when, in the following rightward pullback of split epimorphisms:

$$
\begin{array}{c}
X \times_Y Z \\
\downarrow \downarrow \\
X \\
\downarrow \downarrow \\
Z
\end{array}
$$

the pair $(i_Z, i_X)$ is jointly extremal epic, **provided that** the split epimorphism $(f, s)$ belongs to $\Sigma$.

- + some condition on the class $\Sigma$ to be precised

- actually we shall see that there is an important distinction between two levels.
So the natural notion of $\Sigma$-Mal’tsev category should be:

**Definition**

A category $\mathcal{D}$ is a $\Sigma$-Mal’tsev category Mal’tsev when, in the following rightward pullback of split epimorphisms:

\[
\begin{array}{ccc}
X \times_Y Z & \xrightarrow{i_X} & X \\
\downarrow p_Z & & \downarrow p_X \\
Z & \xleftarrow{i_Z} & Y \\
\end{array}
\]

the pair $(i_Z, i_X)$ is jointly extremal epic, **provided that** the split epimorphism $(f, s)$ belongs to $\Sigma$.

+ some condition on the class $\Sigma$ to be precised

+ actually we shall see that there is an important distinction between two levels.
Outline

Monoids and partial pointed protomodularity

Mal’tsev and \(\Sigma\)-Mal’tsev category

Quandles

Naturally Mal’tsev and \(\Sigma\)-naturally Mal’tsev category
Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.

A quandle is a set $X$ endowed with a binary idempotent operation: $\triangleright : X \times X \to X$ such that for any object $x$ the translation $-\triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $-\triangleright^{-1} x$.

A homomorphism of quandles is an application $f : (X,\triangleright) \to (Y,\triangleright)$ which respects the binary operation. This defines the category $Qnd$ of quandles.

Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G,.)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.

Since $\emptyset$ belongs to $Qnd$, no hope for any kind of $\Sigma$-protomodularity, so it would be the desired discriminating example.
Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.

A quandle is a set $X$ endowed with a binary idempotent operation: $\triangleright : X \times X \to X$ such that for any object $x$ the translation $- \triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $- \triangleright^{-1} x$.

A homomorphism of quandles is an application $f : (X,\triangleright) \to (Y,\triangleright)$ which respects the binary operation. This defines the category $Qnd$ of quandles.

Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G,.)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.

Since $\emptyset$ belongs to $Qnd$, no hope for any kind of $\Sigma$-protomodularity, so it would be the desired discriminating example.
Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.

A quandle is a set $X$ endowed with a binary idempotent operation: $\triangleright : X \times X \rightarrow X$ such that for any object $x$ the translation $-\triangleright x : X \rightarrow X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $-\triangleright^{-1}x$.

A homomorphism of quandles is an application $f : (X,\triangleright) \rightarrow (Y,\triangleright)$ which respects the binary operation. This defines the category $Qnd$ of quandles.

Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G,.)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.

Since $\emptyset$ belongs to $Qnd$, no hope for any kind of $\Sigma$-protomodularity, so it would be the desired discriminating example.
Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.

A quandle is a set $X$ endowed with a binary idempotent operation: $\triangleright : X \times X \rightarrow X$ such that for any object $x$ the translation $-\triangleright x : X \rightarrow X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $-\triangleright^{-1} x$.

A homomorphism of quandles is an application $f : (X,\triangleright) \rightarrow (Y,\triangleright)$ which respects the binary operation. This defines the category $Qnd$ of quandles.

Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G,.)$, the binary operation $x \triangleright_G y = y.x.y^{-1}$ is a quandle operation.

Since $\emptyset$ belongs to $Qnd$, no hope for any kind of $\Sigma$-protomodularity, so it would be the desired discriminating example.
Attending a talk on a work of [Even+Gran 2014] on quandles, I learnt that there were a certain class of equivalence relations which does permute with any equivalence relation.

A quandle is a set $X$ endowed with a binary idempotent operation: $\triangleright : X \times X \to X$ such that for any object $x$ the translation $- \triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $- \triangleright^{-1} x$.

A homomorphism of quandles is an application $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary operation. This defines the category $Qnd$ of quandles.

Example: the quandles recapture the formal aspects of group conjugation: starting with any group $(G, \cdot)$, the binary operation $x \triangleright_G y = y \cdot x \cdot y^{-1}$ is a quandle operation.

Since $\emptyset$ belongs to $Qnd$, no hope for any kind of $\Sigma$-protomodularity, so it would be the desired discriminating example.
Let us introduce the following definitions:

- A split epimorphism \((f, s) : X \leftrightarrow Y\) in \(Qnd\) is called: *puncturing* when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is surjective (the class \(\Sigma\)).

- The class \(\Sigma\) is only stable under pullback and contains the isomorphisms (i.e. fibrational: -first level of left exactness).

- A split epimorphism is called *acupuncturing* when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is bijective (the subclass \(\Sigma' \subset \Sigma\)).

- The class \(\Sigma' \subset \Sigma\) is point-congruous (-second level of left exactness: in addition, \(\Sigma'\) is stable under finite limits inside the class of all split epimorphisms).

- Both classes satisfy the desired condition on pullback of split epimorphisms detailed above.
Let us introduce the following definitions:

- A split epimorphism \((f, s) : X \sqsupseteq Y\) in \(Qnd\) is called: **puncturing** when, for any element \(y \in Y\), the application \(s(y) \mapsto f^{-1}(y) \rightarrow f^{-1}(y)\) is surjective (the class \(\Sigma\)).

- The class \(\Sigma\) is only stable under pullback and contains the isomorphisms (i.e. fibrational: -first level of left exactness).

- A split epimorphism is called **acupuncturing** when, for any element \(y \in Y\), the application \(s(y) \mapsto f^{-1}(y) \rightarrow f^{-1}(y)\) is bijective (the subclass \(\Sigma' \subset \Sigma\)).

- The class \(\Sigma' \subset \Sigma\) is point-congruous (-second level of left exactness: in addition, \(\Sigma'\) is stable under finite limits inside the class of all split epimorphisms).

- Both classes satisfy the desired condition on pullback of split epimorphisms detailed above.
Let us introduce the following definitions:

- A split epimorphism $(f, s) : X \rightleftarrows Y$ in $Qnd$ is called: **puncturing** when, for any element $y \in Y$, the application $s(y) \triangleleft - : f^{-1}(y) \rightarrow f^{-1}(y)$ is surjective (the class $\Sigma$).

- The class $\Sigma$ is only stable under pullback and contains the isomorphisms (i.e. fibrational: -first level of left exactness).

- A split epimorphism is called **acupuncturing** when, for any element $y \in Y$, the application $s(y) \triangleleft - : f^{-1}(y) \rightarrow f^{-1}(y)$ is bijective (the subclass $\Sigma' \subset \Sigma$).

- The class $\Sigma' \subset \Sigma$ is point-congruous (-second level of left exactness: in addition, $\Sigma'$ is stable under finite limits inside the class of all split epimorphisms).

- Both classes satisfy the desired condition on pullback of split epimorphisms detailed above.
Let us introduce the following definitions:

- A split epimorphism \((f, s) : X \rightleftharpoons Y\) in \(Qnd\) is called: **puncturing** when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is surjective (the class \(\Sigma\)).

- The class \(\Sigma\) is only stable under pullback and contains the isomorphisms (i.e. fibrational: -first level of left exactness).

- A split epimorphism is called **acupuncturing** when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is bijective (the subclass \(\Sigma' \subset \Sigma\)).

- The class \(\Sigma' \subset \Sigma\) is point-congruous (-second level of left exactness: in addition, \(\Sigma'\) is stable under finite limits inside the class of all split epimorphisms).

- Both classes satisfy the desired condition on pullback of split epimorphisms detailed above.
Let us introduce the following definitions:

- A split epimorphism \((f, s) : X \leftrightarrow Y\) in \(Qnd\) is called: **puncturing** when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is surjective (the class \(\Sigma\)).

- The class \(\Sigma\) is only stable under pullback and contains the isomorphisms (i.e. fibrational: -first level of left exactness).

- A split epimorphism is called **acupuncture** when, for any element \(y \in Y\), the application \(s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)\) is bijective (the subclass \(\Sigma' \subset \Sigma\)).

- The class \(\Sigma' \subset \Sigma\) is point-congruous (-second level of left exactness: in addition, \(\Sigma'\) is stable under finite limits inside the class of all split epimorphisms).

- Both classes satisfy the desired condition on pullback of split epimorphisms detailed above.
Definition
A category $\mathcal{C}$ is a $\Sigma$-Mal’tsev category when $\Sigma$ is a class of split epimorphisms stable under pullback, containing the isomorphisms and such the previous condition on pullback is satisfied.

- Main tools are the same as for $S$-protomodularity:
  - $\Sigma$-relation: a relation which is reflexive and such that $(d_0, s_0)$ belongs to $\Sigma$: 
    \[
    \begin{array}{c}
    d_0 \\
    R \\
    \downarrow \\
    X
    \end{array}
    \]
    \[
    \begin{array}{c}
    s_0 \\
    \leftarrow \\
    \downarrow \\
    d_1
    \end{array}
    \]
  - $\Sigma$-special morphism $f : X \to Y$: when the kernel relation $R[f]$ is a $\Sigma$-relation
  - $\Sigma$-special object $X$: when the terminal map $X \to 1$ is $\Sigma$-special.
Definition
A category $\mathbb{C}$ is a $\Sigma$-Mal’tsev category when $\Sigma$ is a class of split epimorphims stable under pullback, containing the isomorphisms and such the previous condition on pullback is satisfied.

Main tools are the same as for $S$-protomodularity:
$\Sigma$-relation: a relation which is reflexive and such that $(d_0, s_0)$ belongs to $\Sigma$:

$\Sigma$-special morphism $f : X \rightarrow Y$: when the kernel relation $R[f]$ is a $\Sigma$-relation

$\Sigma$-special object $X$: when the terminal map $X \rightarrow 1$ is $\Sigma$-special.
Definition

A category $C$ is a $\Sigma$-Mal’tsev category when $\Sigma$ is a class of split epimorphisms stable under pullback, containing the isomorphisms and such the previous condition on pullback is satisfied.

- Main tools are the same as for $S$-protomodularity:
  - $\Sigma$-relation: a relation which is reflexive and such that $(d_0, s_0)$ belongs to $\Sigma$:
    
    \[
    \begin{array}{c}
    d_0 \\
    R \\
    d_1
    \end{array}
    \leftarrow
    \begin{array}{c}
    s_0 \\
    \downarrow
    \\
    \rightarrow
    \end{array}
    \begin{array}{c}
    X
    \end{array}
    \]

- $\Sigma$-special morphism $f : X \rightarrow Y$: when the kernel relation $R[f]$ is a $\Sigma$-relation

- $\Sigma$-special object $X$: when the terminal map $X \rightarrow 1$ is $\Sigma$-special.
Definition
A category $\mathcal{C}$ is a $\Sigma$-Mal’tsev category when $\Sigma$ is a class of split epimorphims stable under pullback, containing the isomorphisms and such the previous condition on pullback is satisfied.

- Main tools are the same as for $S$-protomodularity:
  $\Sigma$-relation: a relation which is reflexive and such that $(d_0, s_0)$ belongs to $\Sigma$:

  \[
  \begin{array}{c}
  \xymatrix{
  R \ar[r]<1.5ex>^{d_0} & X \\
  R \ar[r]<-1.5ex>_{d_1} \ar[u]<1.5ex>^{s_0} & \}
  \end{array}
  \]

- $\Sigma$-special morphism $f : X \to Y$: when the kernel relation $R[f]$ is a $\Sigma$-relation

- $\Sigma$-special object $X$: when the terminal map $X \to 1$ is $\Sigma$-special.
Main results= partial aspects of Mal’tsevness:

- any $\Sigma$-relation is transitive; on a $\Sigma$-graph there is at most one structure of internal category

- and similarly to the global Mal’tsev context, we have the structural facts:

  - 1) in the regular context:
    given any pair of a reflexive relation $R$ and a symmetric $\Sigma$-relation $S$ (and so an equivalence relation) on an object $X$, the two relations do permute, i.e. $R \circ S = S \circ R$.

  - 2) there an intrinsic notion of centralization for $\Sigma$-relations

  - 3) subtle partial variations on these facts.
Main results: partial aspects of Mal’tsevness:

- any $\Sigma$-relation is transitive; on a $\Sigma$-graph there is at most one structure of internal category
- and similarly to the global Mal’tsev context, we have the structural facts:
  1) in the regular context: given any pair of a reflexive relation $R$ and a symmetric $\Sigma$-relation $S$ (and so an equivalence relation) on an object $X$, the two relations do permute, i.e. $R \circ S = S \circ R$.
  2) there an intrinsic notion of centralization for $\Sigma$-relations
  3) subtle partial variations on these facts.
Main results= partial aspects of Mal’tsevness:

- any \( \Sigma \)-relation is transitive; on a \( \Sigma \)-graph there is at most one structure of internal category

- and similarly to the global Mal’tsev context, we have the structural facts:

  1) in the regular context:
     given any pair of a reflexive relation \( R \) and a symmetric \( \Sigma \)-relation \( S \) (and so an equivalence relation) on an object \( X \), the two relations do permute, i.e. \( R \circ S = S \circ R \).

  2) there an intrinsic notion of centralization for \( \Sigma \)-relations

  3) + subtle partial variations on these facts.
Main results = partial aspects of Mal’tsevness:

- any $\Sigma$-relation is transitive; on a $\Sigma$-graph there is at most one structure of internal category

- and similarly to the global Mal’tsev context, we have the structural facts:

- 1) in the regular context:
  given any pair of a reflexive relation $R$ and a symmetric $\Sigma$-relation $S$ (and so an equivalence relation) on an object $X$, the two relations do permute, i.e. $R \circ S = S \circ R$.

- 2) there an intrinsic notion of centralization for $\Sigma$-relations

- 3) + subtle partial variations on these facts.
Main results= partial aspects of Mal’tsevness:

- any $\Sigma$-relation is transitive; on a $\Sigma$-graph there is at most one structure of internal category

- and similarly to the global Mal’tsev context, we have the structural facts:

- 1) in the regular context:
  given any pair of a reflexive relation $R$ and a symmetric $\Sigma$-relation $S$ (and so an equivalence relation) on an object $X$, the two relations do permute, i.e. $R \circ S = S \circ R$.

- 2) there an intrinsic notion of centralization for $\Sigma$-relations

- 3) + subtle partial variations on these facts.
more important (second level of left exactness): when, in addition, the class $\Sigma$ is point-congruous, the full subcategory $\Sigma C_\#$ of $\Sigma$-special objects is a Mal’tsev one (called the core of the $\Sigma$-Mal’tsev category).

the core of $Qnd$ is the category $LQd$ of latin quandles (when $\chi \triangleright -$ is bijective as well).

even more generally any full subcategory $Spl_Y \subset C/Y$ of the slice category whose objects are the $\Sigma$-special maps is a Mal’tsev one.
more important (second level of left exactness): when, in addition, the class $\Sigma$ is point-congruous, the full subcategory $\Sigma C_{\#}$ of $\Sigma$-special objects is a Mal’tsev one (called the core of the $\Sigma$-Mal’tsev category).

the core of $Qnd$ is the category $LQd$ of latin quandles (when $x \triangleright -$ is bijective as well).

even more generally any full subcategory $Spl_Y \subset C/Y$ of the slice category whose objects are the $\Sigma$-special maps is a Mal’tsev one.
more important (second level of left exactness): when, in addition, the class $\Sigma$ is point-congruous, the full subcategory $\Sigma C_\#$ of $\Sigma$-special objects is a Mal’tsev one (called the core of the $\Sigma$-Mal’tsev category).

the core of $Qnd$ is the category $LQd$ of latin quandles (when $x \triangleright -$ is bijective as well).

even more generally, any full subcategory $Spl_Y \subset C/Y$ of the slice category whose objects are the $\Sigma$-special maps is a Mal’tsev one.
Back to $Mon$ and $SRg$, new observations:

**Definition**

A split monoid homomorphism is a weakly Schreier one when the application $\mu_y: \text{Ker} f \to f^{-1}(y)$ defined by $\mu_y(k) = s(y) \cdot k$ is surjective.

This class $\bar{\Sigma}$ is stable under pullback and contain the isomorphisms; it is not point-congruous.

the category $Mon$ (resp. $SRg$) is a $\bar{\Sigma}$-Mal’tsev one (resp. $U^{-1}(\bar{\Sigma})$-Mal’tsev one).

So that the previous results are already valid for the class of weakly Schreier split homomorphisms.
Outline

Monoids and partial pointed protomodularity

Mal’tsev and $\Sigma$-Mal’tsev category

Quandles

Naturally Mal’tsev and $\Sigma$-naturally Mal’tsev category
Recall that the Mal’tsev context possesses an additive heart:

**Definition (P.T. Johnstone 1989)**

A naturally Mal’tsev category is such that any object $X$ is endowed with a natural Mal’tsev operation $p : X \times X \times X \to X$.

- examples: a pointed category $\mathbb{A}$ is additive if and only if it is a naturally Mal’tsev one
- any slice category $\mathbb{A}/Y$ of an additive category is a non-pointed naturally Mal’tsev one

**Proposition (B.2008)**

When $\mathcal{C}$ is a Mal’tsev category, then any fibre $\text{Grd}_Y \mathcal{C}$ of the fibration of groupoids $\text{Grd}\mathcal{C} \to \mathcal{C}$ is a naturally Mal’tsev category.
Recall that the Mal’tsev context possesses an additive heart:

**Definition (P.T. Johnstone 1989)**

A naturally Mal’tsev category is such that any object $X$ is endowed with a natural Mal’tsev operation $p : X \times X \times X \to X$.

- examples: a pointed category $\mathbb{A}$ is additive if and only if it is a naturally Mal’tsev one
- any slice category $\mathbb{A}/Y$ of an additive category is a non-pointed naturally Mal’tsev one

**Proposition (B.2008)**

When $\mathcal{C}$ is a Mal’tsev category, then any fibre $\text{Grd}_Y \mathcal{C}$ of the fibration of groupoids $\text{Grd} \mathcal{C} \to \mathcal{C}$ is a naturally Mal’tsev category.
Recall that the Mal’tsev context possesses an additive heart:

**Definition (P.T. Johnstone 1989)**

A naturally Mal’tsev category is such that any object $X$ is endowed with a natural Mal’tsev operation $p : X \times X \times X \to X$.

- examples: a pointed category $\mathbb{A}$ is additive if and only if it is a naturally Mal’tsev one
- any slice category $\mathbb{A}/Y$ of an additive category is a non-pointed naturally Mal’tsev one

**Proposition (B.2008)**

When $\mathcal{C}$ is a Mal’tsev category, then any fibre $\text{Grd}_Y \mathcal{C}$ of the fibration of groupoids $\text{Grd}\mathcal{C} \to \mathcal{C}$ is a naturally Mal’tsev category.
Recall that the Mal’tsev context possesses an additive heart:

**Definition (P.T. Johnstone 1989)**

A naturally Mal’tsev category is such that any object $X$ is endowed with a natural Mal’tsev operation $p : X \times X \times X \to X$.

- examples: a pointed category $\mathbb{A}$ is additive if and only if it is a naturally Mal’tsev one
- any slice category $\mathbb{A}/Y$ of an additive category is a non-pointed naturally Mal’tsev one

**Proposition (B.2008)**

When $\mathbb{C}$ is a Mal’tsev category, then any fibre $\text{Grd}_Y \mathbb{C}$ of the fibration of groupoids $\text{Grd} \mathbb{C} \to \mathbb{C}$ is a naturally Mal’tsev category.
we have a list of characterization

**Proposition (B. 1996)**

A category $\mathcal{D}$ is a naturally Mal’tsev one if and only if any of the following conditions is satisfied:

- 1) any fibre of the fibration of points $\mathcal{D}$ is linear
- 1’) any fibre of the fibration of points $\mathcal{D}$ is additive
- 2) it is a Mal’tsev category in which any pair of equivalence relations centralizes each other
- 3) any internal reflexive graph is a groupoid (the Lawvere condition)
- 4) any base change along any split epimorphism with respect to the fibration of points $\mathcal{D}$ is an equivalence of categories.
we have a list of characterization

**Proposition (B. 1996)**

A category $\mathcal{D}$ is a naturally Mal’tsev one if and only if any of the following conditions is satisfied:

- 1) any fibre of the fibration of points $\mathcal{D}$ is linear
- 1’) any fibre of the fibration of points $\mathcal{D}$ is additive
- 2) it is a Mal’tsev category in which any pair of equivalence relations centralizes each other
- 3) any internal reflexive graph is a groupoid (the Lawvere condition)
- 4) any base change along any split epimorphism with respect to the fibration of points $\mathcal{D}$ is an equivalence of categories.
we have a list of characterization

**Proposition (B. 1996)**

A category $\mathcal{D}$ is a naturally Mal’tsev one if and only if any of the following conditions is satisfied:

- 1) any fibre of the fibration of points $\mathcal{D}$ is linear
- 1’) any fibre of the fibration of points $\mathcal{D}$ is additive
- 2) it is a Mal’tsev category in which any pair of equivalence relations centralizes each other
- 3) any internal reflexive graph is a groupoid (the Lawvere condition)
- 4) any base change along any split epimorphism with respect to the fibration of points $\mathcal{D}$ is an equivalence of categories.
we have a list of characterization

**Proposition (B. 1996)**

A category $\mathbb{D}$ is a naturally Mal’tsev one if and only if any of the following conditions is satisfied:

- 1) any fibre of the fibration of points $\mathbb{D}$ is linear
- 1’) any fibre of the fibration of points $\mathbb{D}$ is additive
- 2) it is a Mal’tsev category in which any pair of equivalence relations centralizes each other
- 3) any internal reflexive graph is a groupoid (the Lawvere condition)
- 4) any base change along any split epimorphism with respect to the fibration of points $\mathbb{D}$ is an equivalence of categories.
we have a list of characterization

**Proposition (B. 1996)**

A category $\mathcal{D}$ is a naturally Mal’tsev one if and only if any of the following conditions is satisfied:

- 1) any fibre of the fibration of points $\mathcal{D}$ is linear
- 1’) any fibre of the fibration of points $\mathcal{D}$ is additive
- 2) it is a Mal’tsev category in which any pair of equivalence relations centralizes each other
- 3) any internal reflexive graph is a groupoid (the Lawvere condition)
- 4) any base change along any split epimorphism with respect to the fibration of points $\mathcal{D}$ is an equivalence of categories.
Condition 1) means that in the following rightward pullback of split epimorphisms:

\[ \begin{array}{ccc}
X \times_Y Z & \xrightarrow{\iota_X} & X \\
\downarrow & & \downarrow \\
Z & \xleftarrow{\iota_Z} & Y \\
p_Z & & g \\
\downarrow & & \downarrow \\
Z & \xleftarrow{\iota_Z} & Y \\
\end{array} \]

the rightward and upward square is a pushout.
whence the idea for a notion of partial natural Mal’tsevness:

**Definition**

A $\Sigma$-naturally Mal’tsev category is a category such that, given the following pullback of split epimorphisms:

\[
\begin{array}{ccc}
X \times_Y Z & \xrightarrow{i_X} & X \\
\downarrow p_X & & \downarrow s \\
Z & \xleftarrow{\iota_Z} & Y
\end{array}
\]

the rightward and upward square is a pushout provided that the split epimorphism $(f, s)$ belongs to the class $\Sigma$. 

whence the idea for a notion of partial natural Mal’tsevness:

**Definition**

A $\Sigma$-naturally Mal’tsev category is a category such that, given the following pullback of split epimorphisms:

\[
\begin{array}{ccc}
X \times_Y Z & \xrightarrow{i_X} & X \\
p_Z \downarrow & & \downarrow p_X \\
Z & \xleftarrow{i_Z} & X \\
g \downarrow & & \downarrow s \\
Y & \xrightarrow{t} & Z \\
f \downarrow & & \downarrow \end{array}
\]

the rightward and upward square is a pushout provided that the split epimorphism $(f, s)$ belongs to the class $\Sigma$. 
examples:

- the category $CoM$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms

- the category $AQd$ of autonomous quandles with $\Sigma$ the class of acupuncturing split epimorphisms

where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.

expected first results:

- any $\Sigma$-graph is endowed with a (unique) internal category structure

- any $\Sigma$-equivalence relation centralizes every reflexive relation

- when $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $Grd_{\gamma}\mathcal{C}$ of the fibration of groupoids $Grd\mathcal{C} \to \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
examples:

- the category $\text{CoM}$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms

- the category $\text{AQd}$ of autonomous quandles with $\Sigma$ the class of acupuncturing split epimorphisms where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.

expected first results:

- any $\Sigma$-graph is endowed with a (unique) internal category structure

- any $\Sigma$-equivalence relation centralizes every reflexive relation

- when $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $Grd_\gamma\mathcal{C}$ of the fibration of groupoids $Grd\mathcal{C} \to \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
examples:

- The category $CoM$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms.

- The category $AQd$ of autonomous quandles with $\Sigma$ the class of acupuncturing split epimorphisms where autonomous means that the law $\rhd$ is itself a quandle homomorphism.

expected first results:

- Any $\Sigma$-graph is endowed with a (unique) internal category structure.

- Any $\Sigma$-equivalence relation centralizes every reflexive relation.

- When $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $Grd_\gamma \mathcal{C}$ of the fibration of groupoids $Grd\mathcal{C} \to \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
examples:

- the category $CoM$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms

- the category $AQd$ of autonomous quandles with $\Sigma$ the class of acupunctureing split epimorphisms where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.

expected first results:

- any $\Sigma$-graph is endowed with a (unique) internal category structure

- any $\Sigma$-equivalence relation centralizes every reflexive relation

- when $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $Grd_{\gamma}\mathcal{C}$ of the fibration of groupoids $Grd\mathcal{C} \to \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
examples:

- the category $CoM$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms

- the category $AQd$ of autonomous quandles with $\Sigma$ the class of acupuncturing split epimorphisms where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.

expected first results:

- any $\Sigma$-graph is endowed with a (unique) internal category structure

- any $\Sigma$-equivalence relation centralizes every reflexive relation

- when $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $Grd_{Y}\mathcal{C}$ of the fibration of groupoids $Grd\mathcal{C} \rightarrow \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
examples:

- the category $\text{CoM}$ of commutative rings with $\Sigma$ the class of Schreier split epimorphisms

- the category $\text{AQd}$ of autonomous quandles with $\Sigma$ the class of acupunctureting split epimorphisms where autonomous means that the law $\triangleright$ is itself a quandle homomorphism.

expected first results:

- any $\Sigma$-graph is endowed with a (unique) internal category structure

- any $\Sigma$-equivalence relation centralizes every reflexive relation

- when $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, then any fibre $\text{Grd}_\gamma \mathcal{C}$ of the fibration of groupoids $\text{Grd} \mathcal{C} \to \mathcal{C}$ is a $\Sigma$-naturally Mal’tsev category.
Proposition

When $\mathcal{C}$ is a point congruous $\Sigma$-naturally Mal’tsev category, then the full subcategory $\text{Spl}_Y$ of the slices category $\mathcal{C}/Y$ are naturally Mal’tsev ones.

In particular its Mal’tsev core $\Sigma\mathcal{C}_\#$ is a naturally Mal’tsev category.

- the Mal’tsev core of $\text{CoM}$ is the category $\text{Ab}$ of abelian groups.

- the Mal’tsev core of $\text{AQd}$ is the category $\text{LAQd}$ of latin autonomous quandles.
Proposition

When $\mathbb{C}$ is a point congruous $\Sigma$-naturally Mal’tsev category, then the full subcategory $\text{Spl}_Y$ of the slices category $\mathbb{C}/Y$ are naturally Mal’tsev ones.

In particular its Mal’tsev core $\Sigma\mathbb{C}^\#$ is a naturally Mal’tsev category.

- The Mal’tsev core of $\text{CoM}$ is the category $\text{Ab}$ of abelian groups.

- The Mal’tsev core of $\text{AQd}$ is the category $\text{LAQd}$ of latin autonomous quandles.
Proposition

When $\mathbb{C}$ is a point congruous $\Sigma$-naturally Mal’tsev category, then the full subcategory $\text{Spl}_Y$ of the slices category $\mathbb{C}/Y$ are naturally Mal’tsev ones.

In particular its Mal’tsev core $\Sigma\mathbb{C}_\#$ is a naturally Mal’tsev category.

- the Mal’tsev core of $\text{CoM}$ is the category $\text{Ab}$ of abelian groups.

- the Mal’tsev core of $\text{AQd}$ is the category $\text{LAQd}$ of latin autonomous quandles.
Finally let us emphasize that there appears some subtle phenomena:

The category $CoM$ is a $\Sigma'$-Mal'tsev category with respect to the class $\Sigma'$ of weakly Schreier split epimorphisms and $\Sigma$-naturally Mal'tsev for the subclass $\Sigma$ of Schreier split epimorphisms.

Similarly the category $AQd$ is a $\Sigma'$-Mal'tsev category with respect to the class $\Sigma'$ of puncturing split epimorphisms and $\Sigma$-naturally Mal’tsev for the subclass $\Sigma$ of acupuncturing split epimorphisms.

When $\mathbb{C}$ is a $\Sigma$-Mal’tsev category, any fibre $\text{Grd}_Y\mathbb{C}$ is a Mal’tsev category (since it is protomodular in any category $\mathbb{C}$), and $\Sigma$-naturally Mal’tsev.
Finally let us emphasize that there appears some subtle phenomena:

- The category $CoM$ is a $\Sigma'$-Mal'tsev category with respect to the class $\Sigma'$ of weakly Schreier split epimorphisms and $\Sigma$-naturally Mal’tsev for the subclass $\Sigma$ of Schreier split epimorphisms.

- Similarly the category $AQd$ is a $\Sigma'$-Mal’tsev category with respect to the class $\Sigma'$ of puncturing split epimorphisms and $\Sigma$-naturally Mal’tsev for the subclass $\Sigma$ of acupuncturing split epimorphisms.

- When $C$ is a $\Sigma$-Mal’tsev category, any fibre $Grd_\gamma C$ is a Mal’tsev category (since it is protomodular in any category $C$), and $\Sigma$-naturally Mal’tsev.
Finally let us emphasize that there appears some subtle phenomena:

The category $CoM$ is a $\Sigma'$-Mal'tsev category with respect to the class $\Sigma'$ of weakly Schreier split epimorphisms and $\Sigma$-naturally Mal'tsev for the subclass $\Sigma$ of Schreier split epimorphisms.

Similarly the category $AQd$ is a $\Sigma'$-Mal'tsev category with respect to the class $\Sigma'$ of puncturing split epimorphisms and $\Sigma$-naturally Mal'tsev for the subclass $\Sigma$ of acupuncturing split epimorphisms.

When $\mathcal{C}$ is a $\Sigma$-Mal'tsev category, any fibre $Grd_\gamma\mathcal{C}$ is a Mal'tsev category (since it is protomodular in any category $\mathcal{C}$), and $\Sigma$-naturally Mal'tsev.
Finally let us emphasize that there appears some subtle phenomena:

The category $CoM$ is a $\Sigma'$-Mal’tsev category with respect to the class $\Sigma'$ of weakly Schreier split epimorphisms and $\Sigma$-naturally Mal’tsev for the subclass $\Sigma$ of Schreier split epimorphisms.

Similarly the category $AQd$ is a $\Sigma'$-Mal’tsev category with respect to the class $\Sigma'$ of puncturing split epimorphisms and $\Sigma$-naturally Mal’tsev for the subclass $\Sigma$ of acupuncturing split epimorphisms.

When $\mathcal{C}$ is a $\Sigma$-Mal’tsev category, any fibre $Grd_\gamma \mathcal{C}$ is a Mal’tsev category (since it is protomodular in any category $\mathcal{C}$), and $\Sigma$-naturally Mal’tsev.