Algebraic and homotopical nilpotency

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joint work with Dominique Bourn

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1. Central extensions and affine extensions
2. Algebraic nilpotency and cross-effects
3. Homotopical nilpotency and cocategory

Definition

A group G is *n-nilpotent* if commutators of length $n+1$ vanish:

$$[x_1, [x_2, [x_3, \ldots, [x_n, x_{n+1}] \cdots]]] = e_G \quad \forall x_1, \ldots, x_{n+1} \in G.$$

Definition

A *central group extension* is a surjective group homomorphism $f : G \twoheadrightarrow H$ with kernel $K[f]$ contained in the center of G.

Lemma

A group G is n-nilpotent iff it is an *n-fold central extension* of the trivial group, i.e. $G \overset{f_n}{\twoheadrightarrow} G_{n-1} \overset{f_{n-1}}{\twoheadrightarrow} \cdots G_2 \overset{f_2}{\twoheadrightarrow} G_1 \overset{f_1}{\twoheadrightarrow} \star$ with f_i central.

Problem

In which categories is there a notion of central extension ?

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In which categories is there a notion of central extension ?

Definition (Janelidze-Màrki-Tholen '01)

A category is called *semi-abelian* if it is

- σ-pointed (i.e. with null-object and binary sums);
- exact (Barr '71);
- protomodular (Bourn '91).

Proposition (Bourn '96)

- A pointed category with pullbacks is *protomodular* iff section and kernel of every split epi $f : X \to Y$ strongly generate X;
- A σ-pointed category with pullbacks is *protomodular* iff for every split epi $f : X \to Y$ with section $s_f : Y \rightarrowtail X$ the morphism $< s_f, i_f >: Y + K[f] \to X$ is a strong epimorphism;
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Definition

A *variety* V_T is the cat. of algebras for a *finitary* monad T on *sets*.

Theorem (Lawvere)

Each variety V_T is determined by an *algebraic theory* Θ_T where

$$\begin{cases} \mathrm{Ob}\,\Theta_T = \mathbb{N}; \\ \Theta_T(m,n) = \mathrm{Alg}_T(F_T(\{1,\dots,m\}), F_T(\{1,\dots,n\})). \end{cases}$$

Theorem (Mal'cev)

The variety V_T is a Mal'cev category iff $\exists p \in \Theta_T(1,3)$ such that

$$p^*(x,y,y) = x \text{ and } p^*(x,x,y) = y \text{ for any } T\text{-algebra.}$$

Example (Mal'cev operation for groups)

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Basic hypothesis (on the ambient category $\mathbb{E}$)

$(\mathbb{E}, \star_{\mathbb{E}})$ is a σ-pointed exact Mal'cev category

e.g. any semi-abelian categoryor any pointed Mal'cev variety.

Lemma

Exact Mal'cev categories have reflexive coequalizers;
σ-pointed exact Mal'cev categories have all finite colimits.

Examples (of semi-abelian categories)

Groups, Lie algebras, cocommutative Hopf algebras over a field of characteristic zero, Heyting algebras, loops, ...

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A subobject N of X is *central* iff the inclusion of N into X commutes with the identity of X (in the sense of Huq).
An equivalence relation R on X is *central* iff R commutes with the indiscrete equivalence relation on X (in the sense of Smith).

Proposition (Gran-Van der Linden '08)

In a pointed protomodular category with pullbacks, a regular epimorphism $f : X \twoheadrightarrow Y$ has a central kernel $K[f]$ if and only if the kernel pair $R[f] \rightrightarrows X$ is a central equivalence relation.

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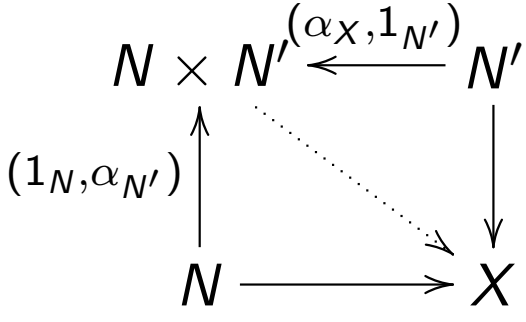

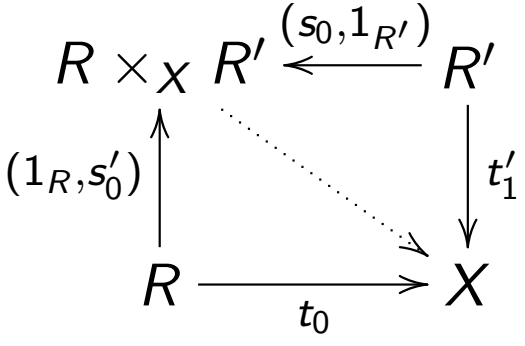

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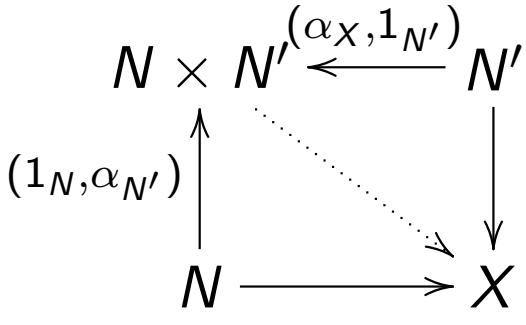

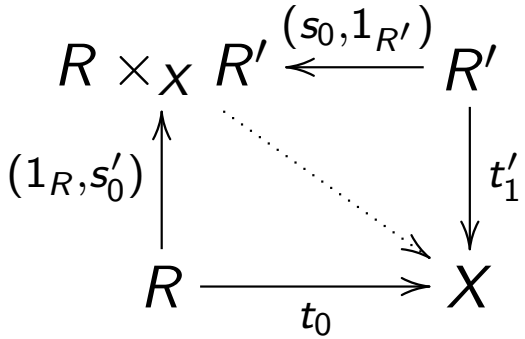

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Definition (Nilpotency)

- Central extension = regular epi with central kernel pair;
- An *n-nilpotent object* is an n-fold central extension of $\star_{\mathbb{E}}$;
- $\mathrm{Nil}^n(\mathbb{E})$ is the subcategory spanned by the n-nilpotent objects;
- A category is *n-nilpotent* iff all its objects are n-nilpotent.

Remark

The abstract notion of n-nilpotent object yields for groups (Lie algebras) the classical notion of n-nilpotent group (Lie algebra).

Proposition

$$\mathbb{E} \text{ 1-nilpotent} \iff \mathbb{E} \text{ abelian.}$$

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A reflective subcategory $\mathbb{D}$ of $\mathbb{E}$ is a *Birkhoff subcategory* iff $\mathbb{D}$ is closed under taking subobjects and quotients in $\mathbb{E}$.

Lemma

This is the case iff for each regular epi $f : X \twoheadrightarrow Y$ the reflection $I : \mathbb{E} \to \mathbb{D}$ induces a *cocartesian* naturality square of reg. epi's

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & I(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle I(f)} \\ Y & \xrightarrow[\eta_Y]{} & I(Y) \end{array}$$

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The relative Birkhoff reflections $I^{n,n+1} : \mathrm{Nil}^{n+1}(\mathbb{E}) \to \mathrm{Nil}^n(\mathbb{E})$ defined by $\mathrm{Nil}^n(\mathrm{Nil}^{n+1}(\mathbb{E})) = \mathrm{Nil}^n(\mathbb{E})$ are *central* reflections.

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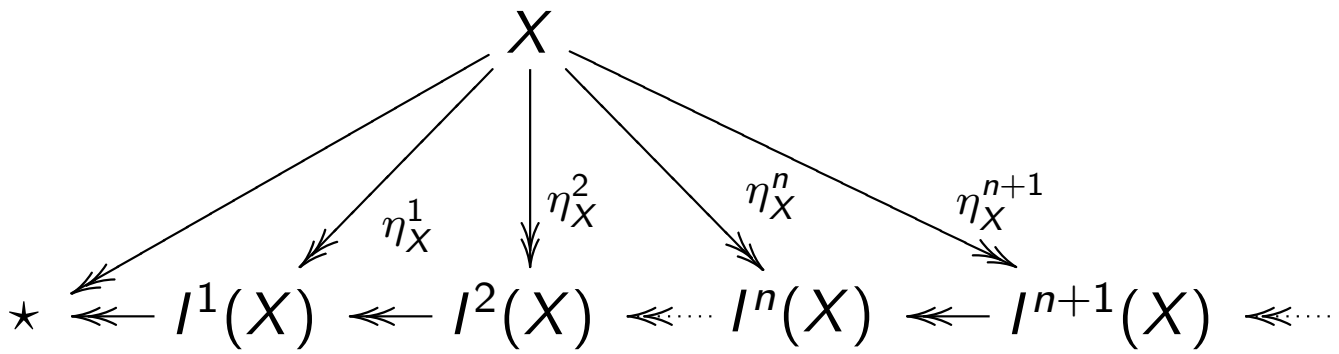

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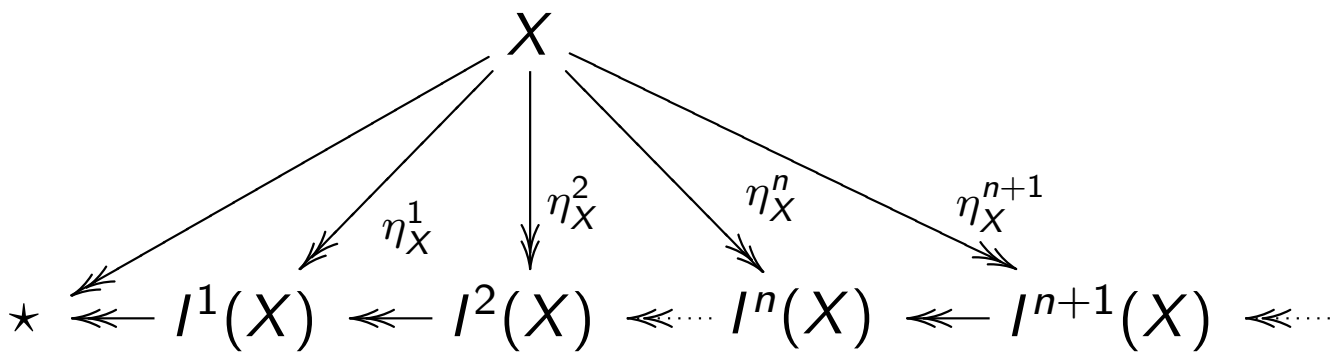
Any morphism inverted by a central reflection is *affine*.

Lemma

The first Birkhoff reflection $I^1 : \mathbb{E} \to \mathrm{Nil}^1(\mathbb{E})$ is *abelianization* and $\mathrm{Nil}^1(\mathbb{E})$ is the full subcategory of *abelian group objects* of $\mathbb{E}$.

Lemma

The relative Birkhoff reflections $I^{n,n+1} : \mathrm{Nil}^{n+1}(\mathbb{E}) \to \mathrm{Nil}^n(\mathbb{E})$ defined by $\mathrm{Nil}^n(\mathrm{Nil}^{n+1}(\mathbb{E})) = \mathrm{Nil}^n(\mathbb{E})$ are *central* reflections.



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Definition (Fibration of "points", Bourn '96)

Let $\mathrm{Pt}_X(\mathbb{E})$ be the category of split epis of $\mathbb{E}$ with codomain X.
Each $f : X \to Y$ induces an adjunction $f_! : \mathrm{Pt}_X(\mathbb{E}) \leftrightarrows \mathrm{Pt}_Y(\mathbb{E}) : f^*$.
A morphism f is *affine* iff $(f_!, f^*)$ is an adjoint equivalence.

Example (split epimorphisms in groups)

A split epimorphism $f : G \twoheadrightarrow X$ in groups

- exhibits G as a *semi-direct product* $X \ltimes_\phi K[f]$;
- determines (and is determined by) a group homomorphism $\phi : X \to \mathrm{Aut}(K[f])$, i.e. an *internal X-representation*.

Corollary (Gray '12)

For $\mathbb{E} =$(groups) or $\mathbb{E} =$(Lie algebras), pointed base-change $f^* : \mathrm{Pt}_{\mathbb{E}}(Y) \to \mathrm{Pt}_{\mathbb{E}}(X)$ has a left adjoint $f_!$ *and* a right adjoint f_*.

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Proposition

Pointed base-change along regular epi's is fully faithful.

Proposition

A regular epi $f : X \twoheadrightarrow Y$ is an *affine extension* iff for all objects Z either of the following two squares is cartesian

$$\begin{array}{ccc} X + Z & \xrightarrow{f+Z} & Y + Z \\ {\scriptstyle \pi_X^Z}\downarrow & & \downarrow{\scriptstyle \pi_Y^Z} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} X + Z & \xrightarrow{f+Z} & Y + Z \\ {\scriptstyle \theta_{X,Z}}\downarrow & & \downarrow{\scriptstyle \theta_{Y,Z}} \\ X \times Z & \xrightarrow[f\times Z]{} & Y \times Z \end{array}$$

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$f : X \twoheadrightarrow Y$ affine $\iff$ $f \diamond Z : X \diamond Z \cong Y \diamond Z$ invertible $\forall Z$

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Proof.

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$\theta_{X,X}$ $(n-1)$-fold central ext. $\Longrightarrow$ η^1_X $(n-1)$-fold central ext. $\square$

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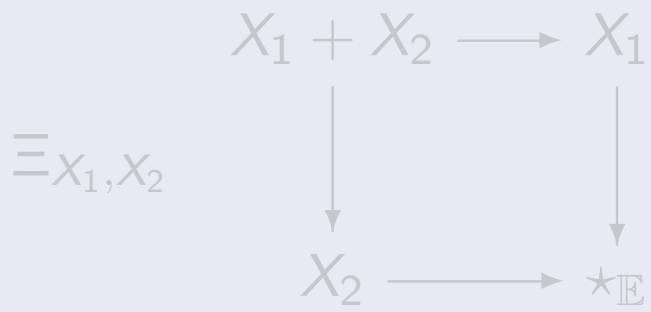
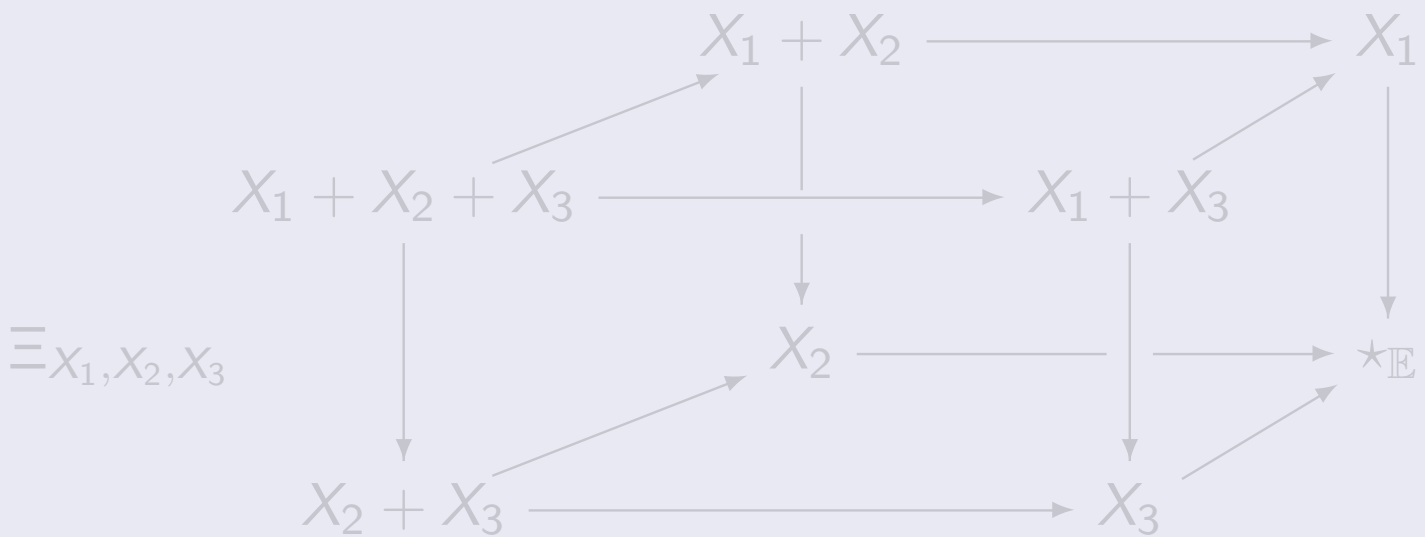
Definition (Goodwillie-cubes $\Xi_{X_1,\ldots,X_n}$ for $n = 2, 3$)





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$$\Xi_{X_1,X_2}:\quad \begin{array}{ccc} X_1 + X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & \star_{\mathbb{E}} \end{array}$$

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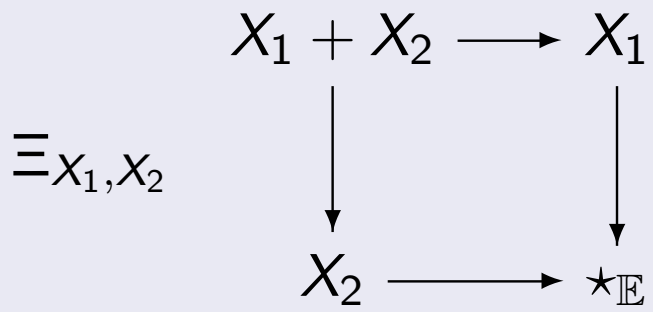

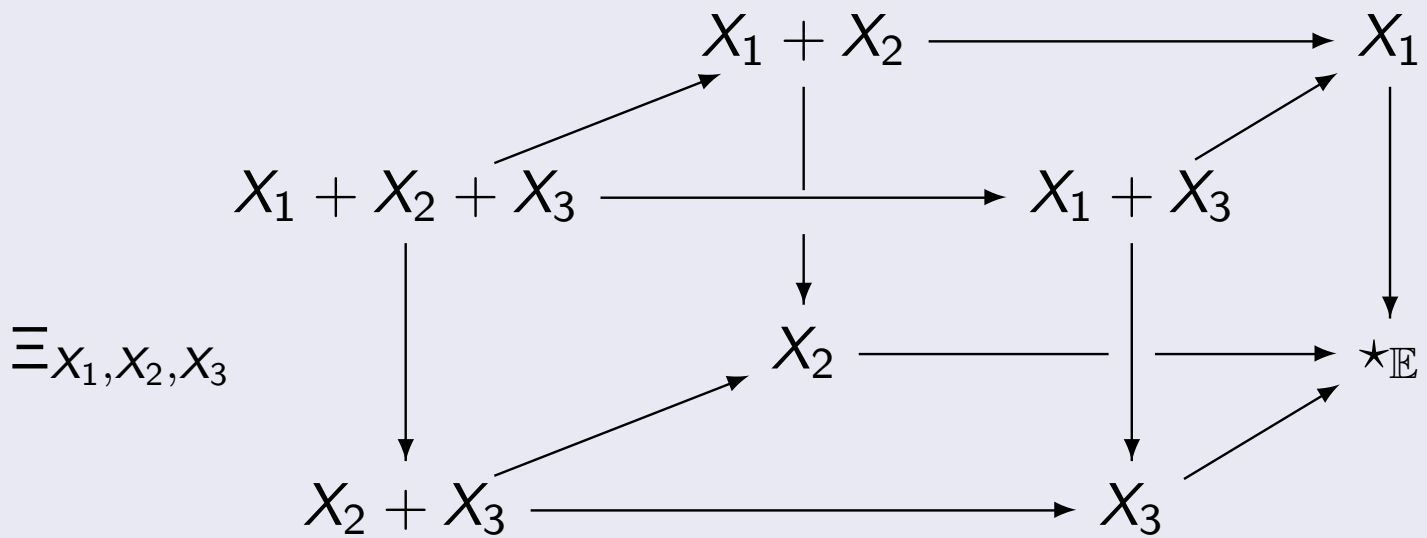

Definition (cubical cross-effects)

- $P_{X_1,\dots,X_n} = \varprojlim_{[0,1]^n \setminus \{(0,\dots,0)\}} \check{\Xi}$ (limit of the punctured cube);
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- $cr_n(X_1,\dots,X_n) = K[\theta_{X_1,\dots,X_n}] =$ "total kernel" of $\Xi_{X_1,\dots,X_n}$;
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- $\theta_{X_1,X_2} : X_1 + X_2 \to X_1 \times X_2$, i.e. $cr_2(X_1,X_2) = X_1 \diamond X_2$

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- $\mathbb{E}$ has linear identity functor iff $\mathbb{E}$ is a linear category !

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- comparison map $\theta_{X_1,\dots,X_n} : X_1 + \cdots + X_n \to P_{X_1,\dots,X_n}$;
- $cr_n(X_1,\dots,X_n) = K[\theta_{X_1,\dots,X_n}] =$ "total kernel" of $\Xi_{X_1,\dots,X_n}$;
- The *identity functor* of $\mathbb{E}$ is said to be *of degree* $\leq n$ if $\Xi_{X_1,\dots,X_{n+1}}$ is cartesian, i.e. $\theta_{X_1,\dots,X_{n+1}}$ invertible.

Example (linear identity functors)

- $\theta_{X_1,X_2} : X_1 + X_2 \to X_1 \times X_2$, i.e. $cr_2(X_1,X_2) = X_1 \diamond X_2$

  2^{nd} cross-effect=co-smash product

- $\mathbb{E}$ has linear identity functor iff $\mathbb{E}$ is a linear category !

Proposition (for semi-abelian categories)

$cr_n(X_1,\dots,X_n) = K[\pi_1] \cap \cdots \cap K[\pi_n]$ (Hartl-Van der Linden '13)

Proposition

If $\mathbb{E}$ has an identity functor of degree $\leq n$ then $\mathbb{E}$ is n-nilpotent.

Theorem

$\mathbb{E}$ has a quadratic identity functor iff $\mathbb{E}$ is 2-nilpotent and moreover one of the following two conditions is satisfied for all X, Y, Z:

- $(X \times Y) + Z \cong (X + Z) \times_Z (Y + Z)$ (alg. codistributivity)
- $(X \times Z) +_Z (Y \times Z) \cong (X + Y) \times Z$ (alg. distributivity)

Corollary

If $\mathbb{E}$ is algebraically distributive then $\mathrm{Nil}^2(\mathbb{E})$ has a quadratic identity functor. In particular, iterated Huq=Higgins commutator: $[X,[X,X]] = [X,X,X]$ (cf. Cigoli-Gray-Van der Linden '14).

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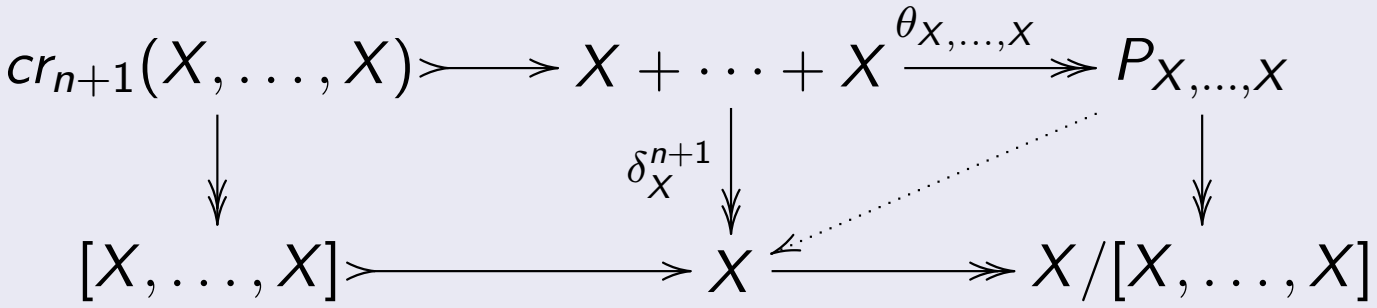
Definition (n-additivity)



X is *n-additive* iff δ_X^{n+1} factors through $\theta_{X,\dots,X}$. For semi-abelian $\mathbb{E}$, this amounts to vanishing Higgins commutator of length $n+1$.

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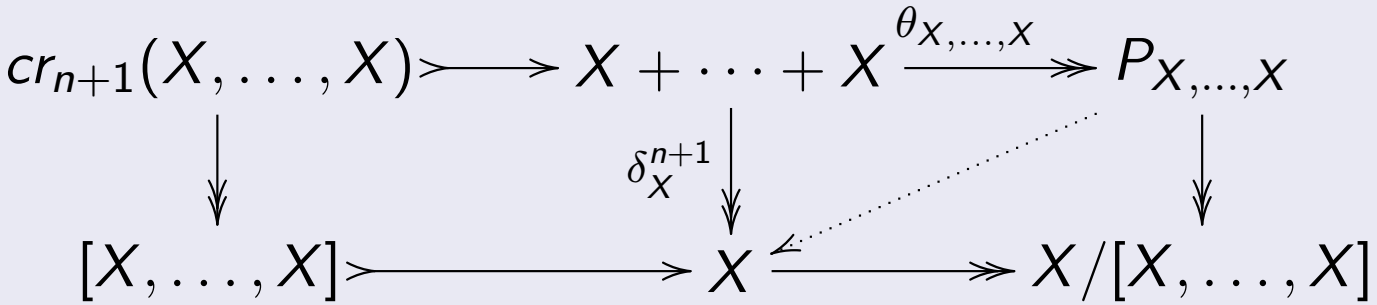

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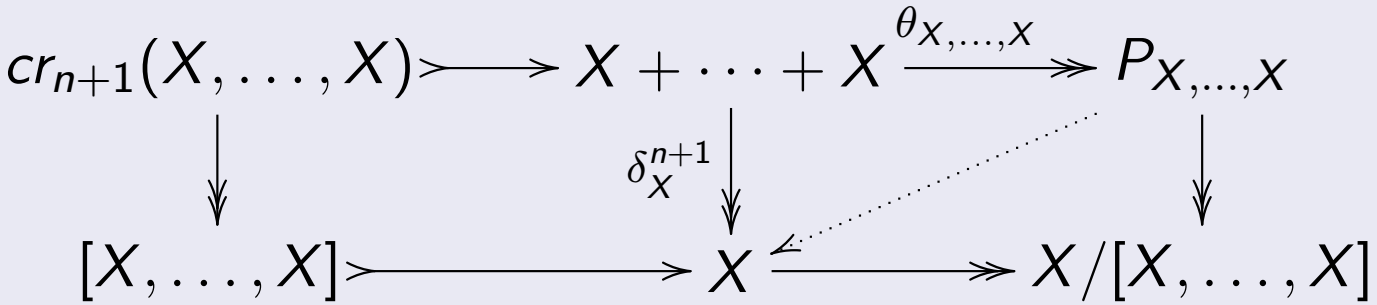

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$$(c) \Longrightarrow (b) \Longrightarrow (a)$$

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$F : \mathbb{E} \to \mathbb{E}'$ is θ-linear iff $F(\theta_{X,Y})$ invertible $\forall X, Y$.

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Abelianization $\mathbb{E} \to \mathrm{Nil}^1(\mathbb{E})$ is θ-linear.

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If $\mathbb{E}$ is alg extensive and has multi-θ-linear n-th cross-effect then $\mathbb{E}$ has an identity functor of degree $\leq n$.

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Any alg extensive n-nilpotent $\mathbb{E}$ has multi-θ-linear n-th cross-effect.

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If $\mathbb{E}$ is alg extensive then $\mathrm{Nil}^n(\mathbb{E})$ has an identity of degree $\leq n$.
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This is the case for the category of groups, resp. Lie algebras.

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Which constructions of $\mathbb{E}$ carry over to $\mathrm{Ho}(\mathbb{E})$?

Example (pullback vs homotopy pullback)

Even if α, β, γ are we's, δ is NOT a we in general. Yet, if moreover f, g, f', g' are fibrations between fibrant objects, then δ is a we !

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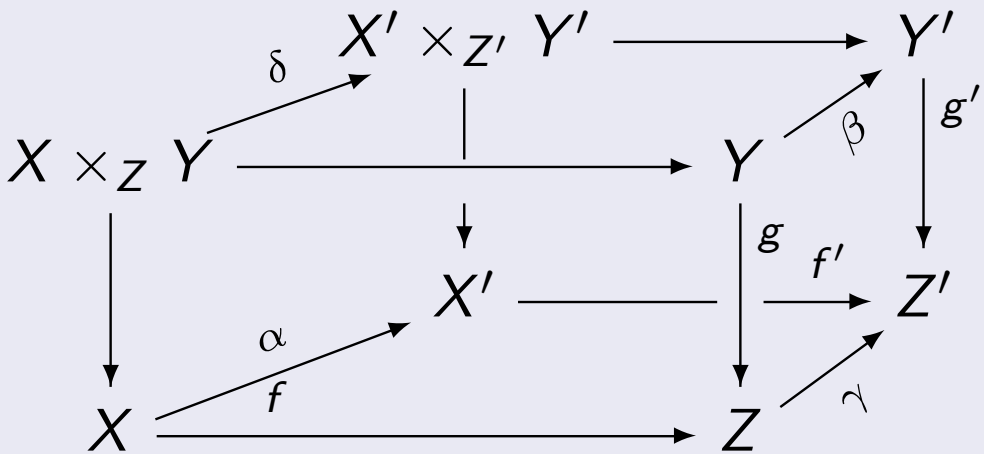

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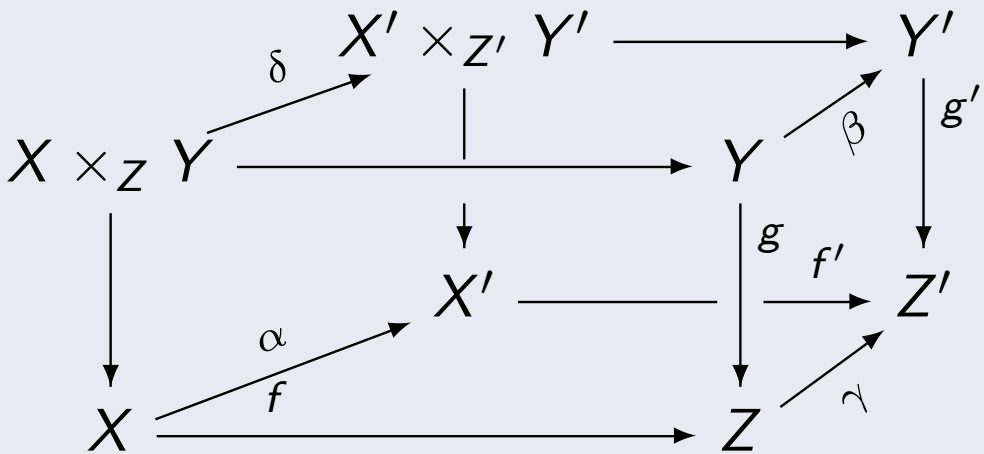

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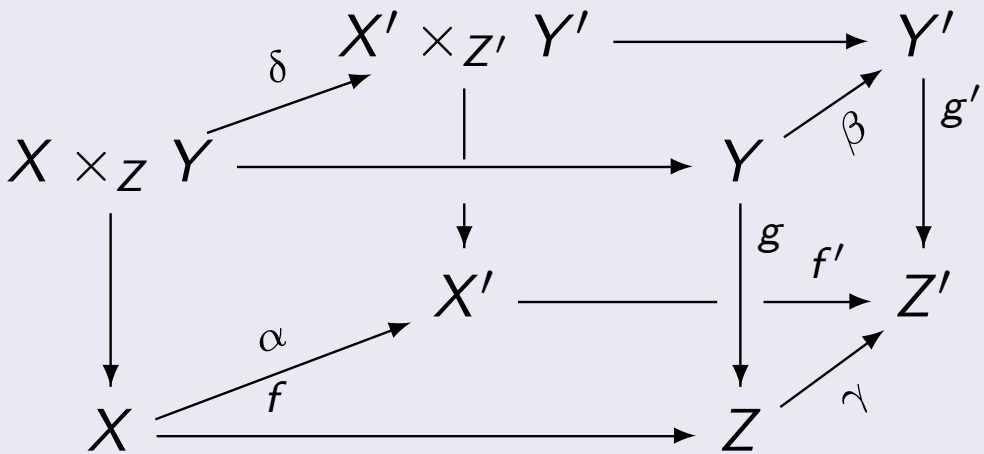

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- The adjunction $|-| : s\mathrm{Sets} \leftrightarrows \mathrm{Top} : \mathrm{Sing}$ is a Quillen equivalence: the simplicial fibrations are the *Kan fibrations*;
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A variety V_T of T-algebras is a Mal'cev variety if and only if $U_T : sV_T \to s\mathrm{Sets}$ takes values in fibrant simplicial sets.

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For each pointed Mal'cev variety V_T $\exists$ *model structure* on sV_T sth

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Proposition

For cofibrant objects $X_1, \ldots, X_n$ in sV_T the n-th "algebraic" cross-effect $cr_n(X_1, \ldots, X_n)$ is *homotopy-invariant*.

Definition (Homotopical nilpotency degrees)

Let X be a cofibrant object in sV_T.

- $\mathrm{nil}_1^T(X) = n$ iff n is the least integer for which $\eta_X^n : X \to I^n(X)$ is a trivial fibration;
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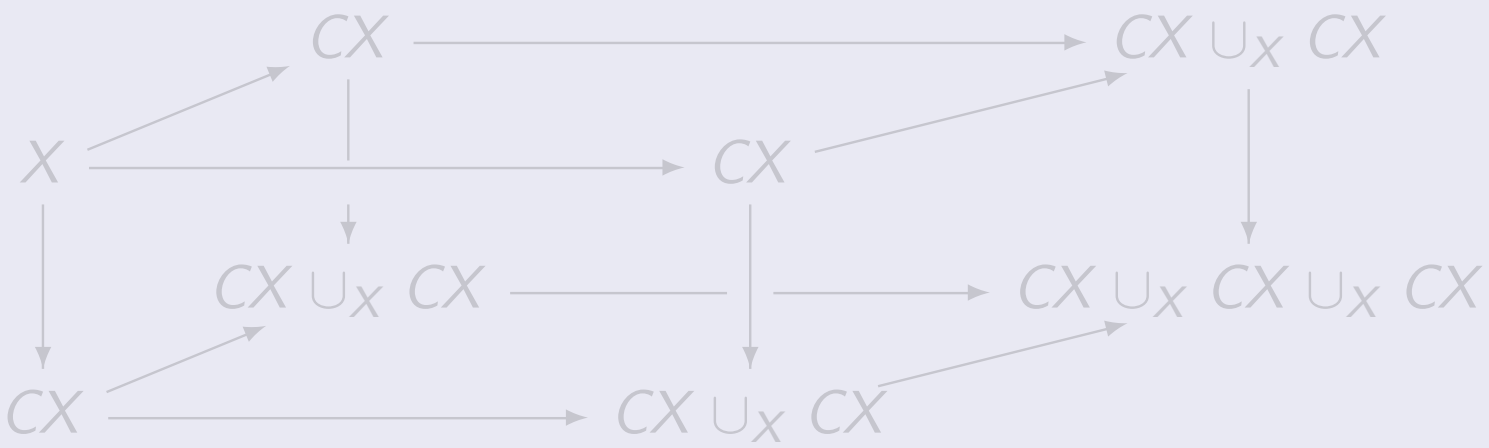
A homotopy functor is n-excisive if it takes cofibration $(n+1)$-cubes to h-cartesian $(n+1)$-cubes.

Enough to check the following cofibration cubes ($n = 3$):



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- F 1-excisive if and only if $F \simeq \Omega F \Sigma$;
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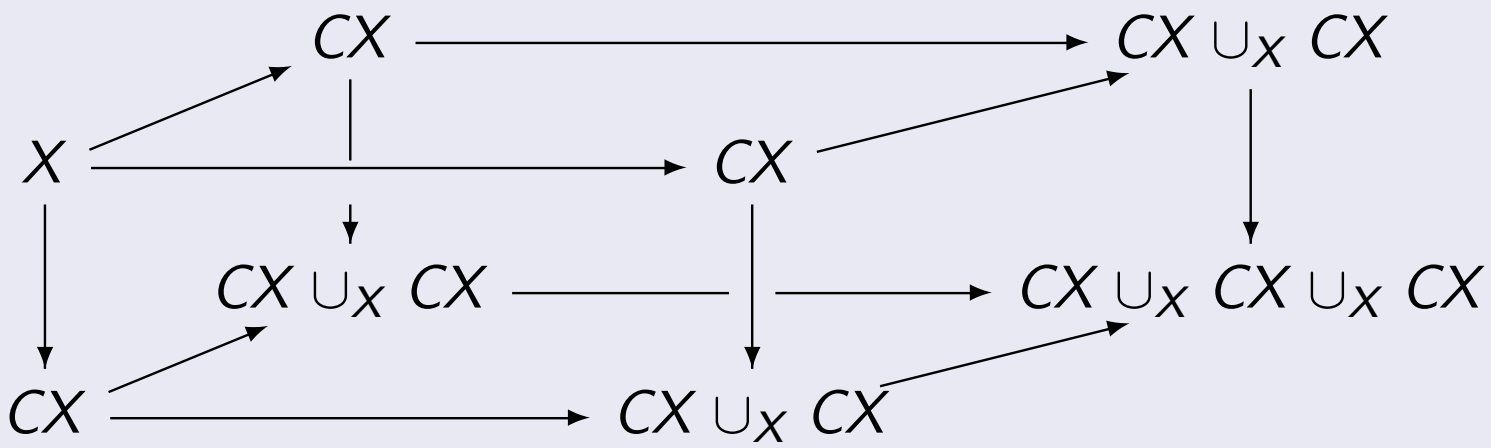

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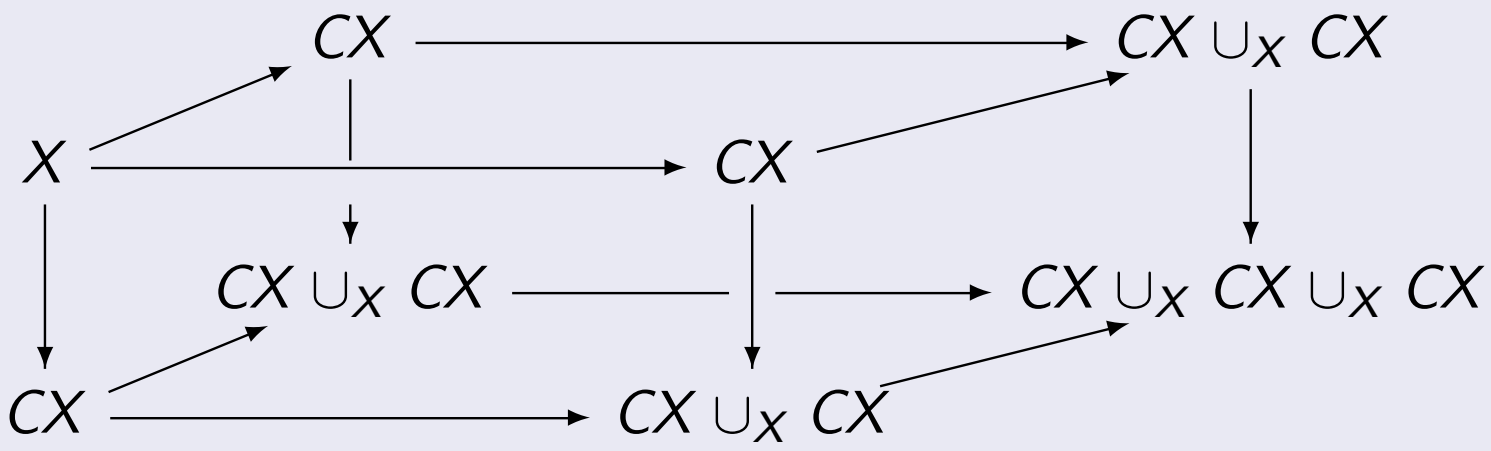

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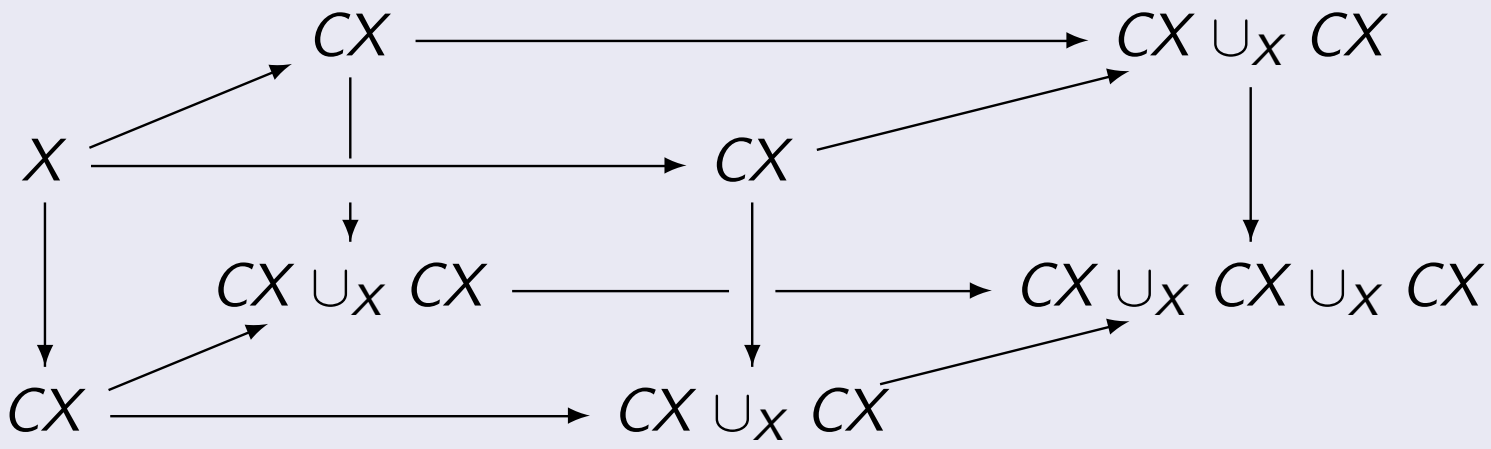

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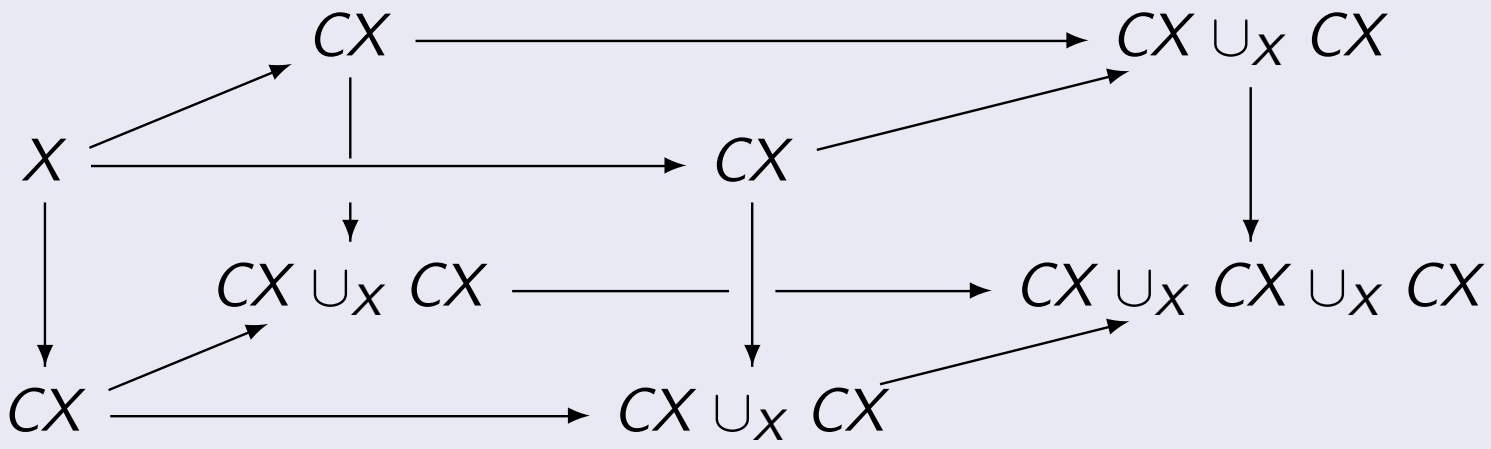

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Theorem (special case $V_T = (\mathrm{groups})$, Kan '56, Quillen '66)

The free-forgetful adjunction $U : sGr \leftrightarrows s\mathrm{Sets}_* : F$ breaks into two adjunctions

$$sGr \underset{G}{\overset{W}{\rightleftarrows}} s\mathrm{Sets}_{red} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} s\mathrm{Sets}_*$$

the first of which is a Quillen equivalence.

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Simplicial groups model loop spaces/loop maps, resp. based *connected* spaces and based maps. The free simpl. group gen. by a pointed simpl. set X is a model of $\Omega\Sigma|X|$, cf. James.

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For a reduced simplical set X one has

- $\mathrm{nil}_1^{Gr}(GX) = \mathrm{nil}_{Berstein-Ganea}(\Omega|X|)$;
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Remark (Lusternik-Schnirelmann '34, Whitehead '56)

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Corollary (Eldred '13, Costoya-Scherer-Viruel '15)

For any based connected space X one has

$$\mathrm{nil}_{BG}(\Omega X) \leq \mathrm{cocat}_{Hov}(X) \leq \mathrm{nil}_{BD}(\Omega X)$$

Remark (Lusternik-Schnirelmann '34, Whitehead '56)

$\mathrm{cat}_{LS}(X) \leq n \iff \exists$ open cover $X \subset U_1 \cup \cdots \cup U_{n+1}$ sth. $U_i \hookrightarrow X$ null-homotopic $\forall\, i \iff$ the diagonal $X \to X^{n+1}$ can be deformed into the fat wedge $Q_{X,\ldots,X} = \{x \in X^{n+1} \mid \exists\, i : x_i = \star\}$.

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