

Richard J. Wood

Dalhousie University

Waves and total distributivity

Following joint work with Francisco Marmolejo and Bob Rosebrugh on “Completely and totally distributive categories”, I reported at CT2013 on the *wave functor* $W : \mathcal{K} \rightarrow \widehat{\mathcal{K}} = \mathbf{set}^{\mathcal{K}^{\text{op}}}$ of a totally cocomplete category \mathcal{K} . Indeed, if the defining property of \mathcal{K} is given by an adjunction $\bigvee \dashv Y : \mathcal{K} \rightarrow \widehat{\mathcal{K}}$, where Y is the Yoneda functor, then W is given for objects A and K in \mathcal{K} by

$$W(A)(K) = \mathbf{set}^{\widehat{\mathcal{K}}}(\mathcal{K}(A, -), \bigvee, [K, -])$$

where $[K, -]$ denotes evaluation of an object of $\widehat{\mathcal{K}}$ at K . \mathbf{set} denotes the category of *small sets* and it requires some work to show that W is well-defined. W arises along with natural transformations $\beta : W \bigvee \rightarrow 1_{\widehat{\mathcal{K}}}$ and $\gamma : \bigvee W \rightarrow 1_{\mathcal{K}}$ that satisfy $\bigvee \beta = \gamma \bigvee$ and $\beta W = W \gamma$. A total \mathcal{K} is said to be *totally distributive* if \bigvee has a left adjoint. It was shown that \mathcal{K} is totally distributive iff γ is invertible iff $W \dashv \bigvee$.

For any total \mathcal{K} there is a well-defined, associative composition of waves. If we write $\widetilde{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}$ for the small profunctor determined by W then composition becomes an arrow $\circ : \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}}$, although $\widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}}$ is not in general small. Moreover, there is also an augmentation $(-)^{\sim} : \widetilde{\mathcal{K}}(-, -) \rightarrow \mathcal{K}(-, -)$, corresponding to a natural transformation $\delta : W \rightarrow Y$ constructed via β . We will show that if \mathcal{K} is totally distributive then $\circ : \widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} \rightarrow \widetilde{\mathcal{K}}$ is invertible, meaning that composition of waves is *interpolative*, and $\widetilde{\mathcal{K}}$ thus supports an idempotent comonad structure. In fact, $\widetilde{\mathcal{K}} \circ_{\mathcal{K}} \widetilde{\mathcal{K}} = \widetilde{\mathcal{K}} \circ_{\widetilde{\mathcal{K}}} \widetilde{\mathcal{K}}$ so that $\widetilde{\mathcal{K}}$ becomes a *taxon* structure, in the sense of Koslowski, on the objects of \mathcal{K} . In the paper with Marmolejo and Rosebrugh we showed that, for any small taxon \mathcal{T} , the category of taxon functors $\mathbf{Tax}(\mathcal{T}^{\text{op}}, \mathbf{set})$ is totally distributive. To this we now add, for any totally distributive \mathcal{K} , there is an equivalence of categories $\mathcal{K} \rightarrow \mathbf{Tax}(\widetilde{\mathcal{K}}^{\text{op}}, \mathbf{set})$.