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Regularity for relational algebras and the case of approach spaces

Topological and approach spaces are known to be relational $\mathbb{T}$-algebras for suitable monads $\mathbb{T}$ on Set, laxly extended to Rel.

Top is known to be isomorphically described as $(\beta, 2)-\mathsf{Cat}$ for the ultrafiltermonad with the Barr extension [1] or as $(\mathbb{F}, 2) - \mathsf{Cat}$ for the filtermonad [7]. App can be isomorphically described as $(\mathbb{B}, 2) - \mathsf{Cat}$ for the prime functional ideal monad $\mathbb{B}$ [6] or as $(\mathbb{I}, 2)-\mathsf{Cat}$ for the functional ideal monad $\mathbb{I}$ [2], [3]. Both $\mathbb{F}$ and $\mathbb{I}$ are power-enriched monads and their extension to Rel is the Kleisli extension. For the monad $\mathbb{B}$, we use the initial extension to Rel.

In this talk we look at objects in $(\mathbb{T}, 2) - \mathsf{Cat}$ as spaces and we explore the topological property of $\mathbb{T}$-regularity. When applied to $(\beta, 2) - \mathsf{Cat}$, β-regularity is known to be equivalent to the usual regularity of the topological space [5].

Consider now a monad $\mathbb{T}$, power-enriched by the monad morphism $\tau : \mathbb{P} \to \mathbb{T}$, with the Kleisli extension. We prove that under mild conditions, even when abandoning improper elements, $\mathbb{T}$-regularity is too strong since it implies the space to be indiscrete. Both $\mathbb{I}$ and $\mathbb{F}$ satisfy these mild conditions. In the case of the functional ideal monad $\mathbb{I}$, we present weaker conditions in order to describe the usual notion of regularity in App.

For the prime functional ideal monad $\mathbb{B}$, a submonad of $\mathbb{I}$, the situation is different. Again abandoning improper elements to avoid uninteresting results, we explain that $\mathbb{B}$-regularity is equivalent to the approach space being topological and regular. We also present weaker conditions based on the monad $\mathbb{B}$ describing the usual regularity in App.

References:

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