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Higher-order categorical substructural logics

Whereas the concept of topos is seemingly difficult to generalise beyond higher-order intuitionisitc logic and its variants, the concept of tripos or higher-order hyperdoctrine, which allows us to present all toposes via the tripos-to-topos construction (but not *vice versa*), is based on a more general, fibrational mechanism, thus looking more promising for developing "categorical universal logic" (the concept of tripos was originally introduced in Hyland-Johnstone-Pitts [2], then extended to allow for general base categories).

In this talk, we introduce the concept of full Lambek tripos, and show that full Lambek triposes give complete semantics for higher-order full Lambek calculus HoFL. Relativising this result to different classes of additional axioms, we can obtain higher-order categorical completeness theorems for a broad variety of logical systems, including classical, intuitionistic, fuzzy, relevant, paraconsistent, and (both commutative and non-commutative) linear logics. The general framework thus developed allows us to give a tripos-theoretical account of Girard's and Gödel's translation for higher-order logic.

Higher-order full Lambek calculus HoFL extends quantified full Lambek calculus FL (see Galatos et al. [1] or Ono [6]) so that HoFL equipped with all the structural rules boils down to higher-order intuitionistic logic, the logic of topos (see Lambek-Scott [4]). HoFL is a so-called "logic over type theory" or "logic-enriched type theory" in Aczel's terms; there is an underlying type theory, upon which logic is built (see Jacobs [3]). The type theory of HoFL is given by simply typed λ-calculus extended with finite product types (i.e., 1 and $\times$; these amount to the structure of Cartesian closed categories), and moreover, with the special, distinguished type

$$\mathrm{Prop}$$

which is a "proposition" type, intended to represent a truth-value object Ω on the categorical side. The logic of HoFL is given by full Lambek calculus FL. The Prop type plays the key rôle of reflecting the logical or propositional structure into the type or term structure: every formula or proposition φ may be seen as a term of type Prop.

The algebras of propositional FL are FL algebras. The algebras of first-order FL are FL hyperdoctrines as argued in Maruyama [5]; note that complete FL algebras only give us completeness in the presence of the *ad hoc* condition of so-called safe valuations, and yet FL hyperdoctrines allow us to prove completeness without any such *ad hoc* condition, and at the same time, to recover the complete FL algebra semantics as a special, set-theoretical instance of the FL hyperdoctrine semantics (in a nutshell, the condition of safe valuations is only necessary to show completeness with respect to the restricted class of FL hyperdoctrines with the category of sets their base categories). FL triposes introduced in this talk are arguably the (fibred) algebras of higher-order FL, allowing us to prove higher-order completeness, again

without any *ad hoc* condition such as safe valuations or Henkin-style restrictions on quantification (set-theoretical semantics is only complete under this condition).

FL hyperdoctrines and triposes are defined as follows. An FL hyperdoctrine is a contravariant functor

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$$

such that the base category $\mathbf{C}$ of P is a category with finite products, and that the adjointness conditions to express quantifiers are satisfied (see Maruyama [5]). Now, an FL tripos, or higher-order FL hyperdoctrine, is an FL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ such that:

- The base category $\mathbf{C}$ is a CCC (Cartesian Closed Category);
- There is an object $\Omega \in \mathbf{C}$ such that

$$P \simeq \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega).$$

We then call Ω the truth-value object of the FL tripos P. Building upon these concepts, we establish the following two results in this talk: (i) higher-order completeness via full Lambek tripos, which can be instantiated for any of classical, intuitionistic, fuzzy, relevant, paraconsistent, and (both commutative and non-commutative) linear logics; (ii) tripos-theoretical accounts of Girard's ! translation and Gödel's $\neg\neg$ translation for higher-order logic, in which the internal language of tripos is at work. As illustrated by this account of logical translation, the general framework thus developed allows us to compare different categorical logics within the one universal setting.

References

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