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Generalising the functor of points approach

The passage from commutative rings to schemes has three main steps: first, one identifies a distinguished class of ring homomorphisms corresponding to open immersions of schemes; second, one defines the notion of an open covering in terms of these distinguished homomorphisms; and finally, one embeds the opposite of the category of commutative rings in an ambient category in which one can glue (the formal duals of) commutative rings along (the formal duals of) distinguished homomorphisms. Traditionally, the ambient category is taken to be the category of locally ringed spaces, but following [1], one could equally well work in the category of sheaves for the large Zariski site – this is the so-called ‘functor of points approach’.

The three procedures described above can be generalised to other contexts. The first step essentially amounts to reconstructing the class of open embeddings from the class of closed embeddings. Once we have a suitable class of open embeddings, the class of open coverings is a subcanonical Grothendieck pretopology. We then define a notion of ‘charted space’ in the category of sheaves. This gives a uniform way of defining locally Hausdorff spaces, schemes, locally finitely presented C^∞ -schemes etc. as special sheaves on their respective categories of local models, taking as input just the class of closed embeddings. We can also get many variations on manifolds by skipping the first step and working directly with a given class of open embeddings.

References:

- [1] Demazure, Michel and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Paris: Masson & Cie, Éditeur (1970) xxvi+700.