Topological spaces, categorically

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CT 2007

The talk is based on joint work with M.M. Clementino and W. Tholen.

Motivation

"The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that *fundamental* structures are themselves categories."

F. William Lawvere.
Metric spaces, generalized logic, and closed categories.
Rend. Sem. Mat. Fis. Milano, 43:135–166 (1974), 1973.
Also in: *Repr. Theory Appl. Categ.* 1:1–37, 2002.

Examples

Metric spaces, $(\mathrm{P}_+ = [0,\infty]^{\mathrm{op}}, +, 0)$

X with $d : X \times X \longrightarrow \mathrm{P}_+$ such that

$$0 \geq d(x,x), \qquad d(x,y) + d(y,z) \geq d(x,z).$$

Categories, $(\mathrm{Set}, \times, 1)$

X with $\mathrm{hom} : X \times X \longrightarrow \mathrm{Set}$ such that

$$1 \longrightarrow \mathrm{hom}(x,x), \qquad \mathrm{hom}(x,y) \times \mathrm{hom}(y,z) \longrightarrow \mathrm{hom}(x,z)$$

and . . . (commutative diagrams in Set).

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and ... (commutative diagrams in Set).

Ordered sets, $\quad (2 = \{\mathrm{false}, \mathrm{true}\}, \&, \mathrm{true})$

X with $\leq : X \times X \longrightarrow 2$ such that

$$\mathrm{true} \models (x \leq x), \qquad (x \leq y \;\&\; y \leq z) \models x \leq z.$$

The ordered category V-Rel

Quantale

$\mathsf{V} = (\mathsf{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

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$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

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- Involution: $r^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ where $r^\circ(y, x) = r(x, y)$ for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

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- Involution: $r^\circ : Y {\longrightarrow\!\!\!\!\!\!\!+}\; X$ where $r^\circ(y, x) = r(x, y)$ for $r : X {\longrightarrow\!\!\!\!\!\!\!+}\; Y$.
- For each Set-map f: $f \dashv f^\circ$.

V-Cat

V-categories

A V-category is a pair $(X, a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X)$ such that

$$k \leq a(x,x) \qquad a(x,y) \otimes a(y,z) \leq a(x,z)$$

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V-functors

A V-functor $f : (X,a) \longrightarrow (Y,b)$ is a Set-map such that

$$a(x,x') \leq b(f(x), f(x')) \qquad \text{respectively} \qquad f \cdot a \leq b \cdot f.$$

M. Barr 1970

Topological spaces $\qquad 2 = (2, \&, \mathrm{true}), \quad \mathbb{U} = (U, e, m)$

X with $\longrightarrow : UX \relbar\!\!\mapsto X$ such that

$$\mathrm{true} \models (\dot{x} \longrightarrow x), \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \,\&\, \mathfrak{x} \longrightarrow x) \models m_X(\mathfrak{X}) \longrightarrow x.$$

Here $\longrightarrow : UX \relbar\!\!\mapsto X$ is naturally extended to $\longrightarrow : UUX \relbar\!\!\mapsto UX$.

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Topological spaces $\quad \mathsf{2} = (2, \&, \mathrm{true}), \quad \mathbb{U} = (U, e, m)$

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In fact, $U : \mathrm{Set} \longrightarrow \mathrm{Set}$ can be extended to a functor
$U : \mathrm{Rel} \longrightarrow \mathrm{Rel}$ such that e and m become oplax.

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3. $\varphi : X \multimap Y$ is a V-module iff $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ is a V-functor.
4. In particular $a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$ is a V-functor. Its mate $y = \ulcorner a \urcorner : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$ is fully faithful. More general, we have

$$[y(x), \varphi] = \varphi(x).$$

5. . . .

Topological theory

Definition

A topological theory $\mathcal{T}$ is a triple $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of

a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and

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such that

$(\mathrm{M_e})$ $\mathrm{id}_\mathsf{V} \leq \xi \cdot e_\mathsf{V}$, $\qquad (\mathrm{M_m})$ $\xi \cdot T\xi \leq \xi \cdot m_\mathsf{V}$,

$(\mathrm{Q}_\otimes)$
$$\begin{tikzcd} T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V}\times\mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V}, \end{tikzcd}$$

$(\mathrm{Q_k})$
$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V}, \end{tikzcd}$$

$(\mathrm{Q}_\bigvee)$ $(\xi_X)_X : P_\mathsf{V} \longrightarrow P_\mathsf{V} T$ is a natural transformation.

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- $\mathcal{L}_{\mathsf{V}}^{\otimes} = (\mathbb{L}, \mathsf{V}, \xi_{\otimes})$ is a strict topological theory where

$$\xi_{\otimes} : L\mathsf{V} \longrightarrow \mathsf{V}, \quad (v_1, \dots, v_n) \longmapsto v_1 \otimes \dots \otimes v_n.$$

Extending T : Set $\longrightarrow$ Set to V-Rel

We define T_ξ : V-Rel $\longrightarrow$ V-Rel as follows:

Extending T : Set $\longrightarrow$ Set to V-Rel

We define T_ξ : V-Rel $\longrightarrow$ V-Rel as follows:

Given $r : X \times Y \longrightarrow \mathsf{V}$, we put

$$\begin{aligned} T_\xi r : TX \times TY &\longrightarrow \mathsf{V} \\ (\mathfrak{x}, \mathfrak{y}) &\longmapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \,\middle|\, \mathfrak{w} \in T(X \times Y), \mathfrak{w} \longmapsto \mathfrak{x}, \mathfrak{y} \right\}, \end{aligned}$$

that is,

$$\begin{tikzcd} T(X \times Y) \arrow[rr, "\tau_{X,Y}"] \arrow[dr, "\xi \cdot Tr"'] & {} \arrow[d, phantom, "\leq"] & TX \times TY \arrow[dl, dotted, "T_\xi r"] \\ & \mathsf{V} & \end{tikzcd}$$

Properties of T_ξ

Theorem

The following statements hold.

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3. $T_\xi s \cdot T_\xi r \leq T_\xi(s \cdot r)$ *provided that* T *satisfies (BC), and* $T_\xi s \cdot T_\xi r \geq T_\xi(s \cdot r)$ *provided that* $(Q^{=}_{\otimes})$ *holds.*

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4. *The natural transformations* e *and* m *become op-lax, that is, for every* V*-relation* $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ *we have the inequalities:*

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"', "\shortmid" marking] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "T_\xi r", "\shortmid"' marking] \\ Y \arrow[r, "e_Y"'] & TY \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "T_\xi T_\xi r"', "\shortmid" marking] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "T_\xi r", "\shortmid"' marking] \\ TTY \arrow[r, "m_Y"'] & TY \end{tikzcd}$$

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Some properties

We have

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- $a \circ (b \circ c) \geq a \circ b \circ c \leq (a \circ b) \circ c$.
- If $\mathcal{T}$ is a strict theory, then Kleisli convolution is associative.

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \relbar\joinrel\mapsto Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

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We consider now

$$(\_)_\# : \text{V-Rel} \longrightarrow \mathcal{T}\text{-Rel}, \quad r : X \nrightarrow Y \longmapsto r_\# = e_Y \cdot T_\xi r : X \nrightarrow Y$$

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- $(1_Y)_\# \circ a = e_Y^\circ \circ a$ and $a \circ (1_X)_\# = a \circ e_X^\circ$.

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- $(1_Y)_\# \circ a = e_Y^\circ \circ a$ and $a \circ (1_X)_\# = a \circ e_X^\circ$.
- $r_\#$ is unitary.
- T satisfies (BC) $\quad\Rightarrow\quad s_\# \circ r_\# \le (s \cdot r)_\#$.
- $(\mathrm{Q}^=_\otimes) \quad\Rightarrow\quad s_\# \circ r_\# \ge (s \cdot r)_\#$.

$\mathcal{T}$-Cat

$\mathcal{T}$-category

A $\mathcal{T}$–category is a pair $(X, a : TX \to\mapsto X)$ such that

$$k \leq a(e_X(x), x), \quad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x) \quad \text{respectively}$$

$$\mathrm{id}_X \leq a \cdot e_X, \quad a \cdot T_\xi a \leq a \cdot m_X \quad \text{respectively}$$

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$\mathcal{T}$-functor

A map $f : (X, a) \longrightarrow (Y, b)$ is a $\mathcal{T}$-functor if

$$a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x)) \qquad \text{respectively} \qquad f \cdot a \le b \cdot Tf.$$

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From now on we consider a strict theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$.

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We have $(\_)_\# \dashv \mathsf{S}$ where

$$\begin{aligned} \mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} &\longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \\ (X, a) &\longmapsto (X, a \cdot e_X) \end{aligned} \qquad \begin{aligned} (\_)_\# : \mathsf{V}\text{-}\mathsf{Cat} &\longrightarrow \mathcal{T}\text{-}\mathsf{Cat}. \\ X = (X, r) &\longmapsto X_\# = (X, r_\#) \end{aligned}$$

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$$(X, a) \longmapsto (X, a \cdot e_X) \qquad X = (X, r) \longmapsto X_\# = (X, r_\#)$$

T_ξ induces an endofunctor

$$T_\xi : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \qquad (X, r) \longmapsto (TX, T_\xi r)$$

Some functors

We have an embedding $\mathsf{Set}^{\mathbb{T}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and put $|X| = (TX, m_X)$.

We have $(\_)_{\#} \dashv \mathsf{S}$ where

$$\begin{aligned} \mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} &\longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \\ (X, a) &\longmapsto (X, a \cdot e_X) \end{aligned} \qquad \begin{aligned} (\_)_{\#} : \mathsf{V}\text{-}\mathsf{Cat} &\longrightarrow \mathcal{T}\text{-}\mathsf{Cat}. \\ X = (X, r) &\longmapsto X_{\#} = (X, r_{\#}) \end{aligned}$$

T_ξ induces an endofunctor

$$T_\xi : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \qquad (X, r) \longmapsto (TX, T_\xi r)$$

and we have

$$\begin{tikzcd} & \mathcal{T}\text{-}\mathsf{Cat} \arrow[dr, "\mathsf{M}"] & \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[ur, "(\_)_{\#}"] \arrow[rr, "T_\xi"'] & & \mathsf{V}\text{-}\mathsf{Cat} \end{tikzcd}$$

where $\mathsf{M} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$, $(X, a) \longmapsto (TX, T_\xi a \cdot m_X^\circ)$.

The $\mathcal{T}$-category V

We define

$$\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V},\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is a $\mathcal{T}$-category.

The $\mathfrak{T}$-category V

We define

$$\hom_\xi : TV \times V \longrightarrow V,\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $\mathsf{V} = (V, \hom_\xi)$ is a $\mathfrak{T}$-category.

Some maps

1. $\bigwedge : V^I \longrightarrow V$ is a $\mathfrak{T}$-functor.

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2. $\hom(v, \_) : \mathsf{V} \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor for each $v \in \mathsf{V}$ which satisfies $\xi \cdot Tv \geq v \cdot !$.
3. $v \otimes \_ : \mathsf{V} \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor for each $v \in \mathsf{V}$ which satisfies $\xi \cdot Tv \leq v \cdot !$.

Compatible monoidal structures on V

We assume that a monoidal structure $(\mathrm{V}, \oplus, l)$ on V is given such that

1. $(u_1 \oplus v_1) \otimes (u_2 \oplus v_2) \leq (u_1 \otimes u_2) \oplus (v_1 \otimes v_2)$,
2. $l \otimes l \leq l$ and $k \leq k \oplus k$,

Compatible monoidal structures on V

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2. $I \otimes I \leq I$ and $k \leq k \oplus k$,
3. $$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\oplus)"] \arrow[d, "\langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\oplus"'] & \mathsf{V}, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T1 \arrow[r, "TI"] \arrow[d, "!"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "I"'] & \mathsf{V}. \end{tikzcd}$$

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Examples

- $\oplus = \otimes$ (since $\mathcal{T}$ is strict).
- $\oplus = \wedge$.

Monoidal structures on V-Rel

Extending $\oplus$ to V-Rel

- For sets X and Y we put $X \oplus Y = X \times Y$.
- For V-relations $r : X \relbar\joinrel\mapsto X'$ and $s : Y \relbar\joinrel\mapsto Y'$ we define $r \oplus s : X \times Y \relbar\joinrel\mapsto X' \times Y'$ by

$$r \oplus s((x, y), (x', y')) = r(x, x') \oplus s(y, y').$$

Then $\oplus : \text{V-Rel} \times \text{V-Rel} \longrightarrow \text{V-Rel}$ is a lax functor, is associative and with $I : 1 \relbar\joinrel\mapsto 1$ as neutral element.

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Of course, we obtain a monoidal structure on V-Cat where $(X,a) \oplus (Y,b) = (X \times Y, a \oplus b)$ with neutral element $E = (1, I)$.

I. Moerdijk, 1999

Hopf monad

A Hopf monad on a monoidal category E is a monad $\mathbb{T} = (T, e, m)$ on E equipped with a natural transformation

$$\tau : T(\_ \otimes \_) \longrightarrow T(\_) \otimes T(\_)$$

and a map $\theta : T(N) \longrightarrow N$ such that ...

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Theorem

There is a bijective correspondence between such structures τ, θ on $\mathbb{T}$ and liftings of the monoidal structure on E *to* $\mathrm{E}^{\mathbb{T}}$*.*

Here:

$$(X, \alpha) \otimes (Y, \beta) = (X \otimes Y, (\alpha \otimes \beta) \cdot \tau_{X,Y}).$$

Lax Hopf monad

With $\tau_{X,Y} : T(X \times Y) \longrightarrow TX \times TY$ and $! : T1 \longrightarrow 1$, in our situation we have

$$\begin{tikzcd}
T(X \oplus Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "T_\xi(r\oplus s)"'] \arrow[dr, phantom, "\leq"] & TX \oplus TY \arrow[d, "T_\xi r\oplus T_\xi s"] \\
T(X' \oplus Y') \arrow[r, "\tau_{X',Y'}"'] & TX' \oplus TY'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T1 \arrow[r, "!"] \arrow[d, "T_\xi l"'] \arrow[dr, phantom, "\leq"] & 1 \arrow[d, "l"] \\
T1 \arrow[r, "!"'] & 1
\end{tikzcd}$$

making (T_ξ, e, m) a lax Hopf monad on V-Rel.

Extending $\oplus$ to $\mathcal{T}$-Rel...

Let $r : X \relbar\joinrel\mapsto X'$ and $s : Y \relbar\joinrel\mapsto Y'$ be $\mathcal{T}$-relations. We put $X \boxplus Y = X \times Y$ and define $r \boxplus s : X \times Y \relbar\joinrel\mapsto X' \times Y'$ as

$$r \boxplus s = (r \oplus s) \cdot \tau_{X,Y}.$$

and $l_! : 1 \relbar\joinrel\mapsto 1$ as the composite $T1 \xrightarrow{!} 1 \xrightarrow{l} 1$.

Extending $\oplus$ to $\mathcal{T}$-Rel…

Let $r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y'$ be $\mathcal{T}$-relations. We put $X \boxplus Y = X \times Y$ and define $r \boxplus s : X \times Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup X' \times Y'$ as

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and $l_! : 1 \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ as the composite $T1 \xrightarrow{!} 1 \xrightarrow{l} 1$.Then

- $e_X^\circ \boxplus e_Y^\circ \geq e_{X\times Y}^\circ$,
- $(r' \boxplus s') \circ (r \boxplus s) \leq (r' \circ r) \boxplus (s' \circ s)$.

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- $e_X^\circ \boxplus e_Y^\circ \geq e_{X\times Y}^\circ$,
- $(r' \boxplus s') \circ (r \boxplus s) \leq (r' \circ r) \boxplus (s' \circ s)$.

For $(\_)_\# : \mathsf{V}\text{-Rel} \longrightarrow \mathcal{T}\text{-Rel}$ we have

- $(r \oplus r')_\# \leq r_\# \boxplus r'_\#$.
- $l_\# \leq l_!$.

. . . and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, l)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where* $(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, l_!)$.

. . . and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, I)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where* $(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, I_!)$.

- For $(\_)_\# : \mathsf{V}\text{-Cat} \longrightarrow \mathcal{T}\text{-Cat}$ we have $\mathcal{T}$-functors

$$(X \oplus Y)_\# \longrightarrow X_\# \oplus Y_\# \qquad \text{and} \qquad E_\# \longrightarrow E.$$

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$$(X \oplus Y)_\# \longrightarrow X_\# \oplus Y_\# \qquad \text{and} \qquad E_\# \longrightarrow E.$$
- For $\mathsf{S} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{V}\text{-Cat}$ we have $\mathcal{T}$-isomorphisms
$$\mathsf{S}(X \oplus Y) \longrightarrow \mathsf{S}(X) \oplus \mathsf{S}(Y) \qquad \text{and} \qquad \mathsf{S}(E) \longrightarrow E.$$

...and to $\mathcal{T}$-Cat

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- For $\mathsf{M} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{V}\text{-Cat}$ we have $\mathcal{T}$-functors
$$\tau_{X,Y} : \mathsf{M}(X \oplus Y) \longrightarrow \mathsf{M}(X) \oplus \mathsf{M}(Y) \quad \text{and} \quad ! : \mathsf{M}(E) \longrightarrow E.$$

Closedness of $\mathcal{T}$-Gph

Assume now that $u \oplus \_ : \mathsf{V} \to \mathsf{V}$ has right adjoint $u \multimap \_ : \mathsf{V} \to \mathsf{V}$.

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Let $X = (X, a)$, $Y = (Y, b)$ be $\mathcal{T}$-graphs. Then

$$X \multimap Y = \{f : X \longrightarrow Y \mid f : X \oplus G \longrightarrow Y \text{ is a } \mathcal{T}\text{-functor}\}$$

(where $G = (1, e_X^\circ)$)

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(where $G = (1, e_X^\circ)$) with structure

$$a \multimap b(\mathfrak{p}, h) = \bigwedge_{\substack{\mathfrak{q} \in T(X \times (X \multimap Y)), x \in X \\ \mathfrak{q} \longmapsto \mathfrak{p}}} (a(T\pi_X(\mathfrak{q}), x) \multimap b(T\mathrm{ev}(\mathfrak{q}), h(x))).$$

is a $\mathcal{T}$-graph as well. In fact, $X \oplus \_ \dashv X \multimap \_$.

Closed $\mathcal{T}$-categories

Lemma

$$\begin{tikzcd}
T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T(\oplus)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] & T\mathsf{V} \arrow[d, "\xi"] \\
\mathsf{V}\times\mathsf{V} \arrow[r, "\oplus"'] & \mathsf{V}
\end{tikzcd}
\quad\Longrightarrow\quad
\begin{tikzcd}
T(X\times Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "T_\xi(r\oplus s)"', "\shortmid" marking] & TX\times TY \arrow[d, "T_\xi r\oplus T_\xi s", "\shortmid" marking] \\
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Closed $\mathcal{T}$-categories

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$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\oplus)} & T\mathsf{V} \\ {\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\oplus]{} & \mathsf{V} \end{array} \quad\Longrightarrow\quad \begin{array}{ccc} T(X\times Y) & \xrightarrow{\tau_{X,Y}} & TX\times TY \\ {\scriptstyle T_\xi(r\oplus s)}\downarrow & & \downarrow{\scriptstyle T_\xi r\oplus T_\xi s} \\ T(X'\times Y') & \xrightarrow[\tau_{X',Y'}]{} & TX'\times TY'. \end{array}$$

Theorem

$(\mathsf{V},\oplus,I)$ *closed, strictly compatible with* $\mathcal{T}$; $X=(X,a)\in\mathcal{T}\text{-}\mathsf{Cat}$.

1. $a \multimap b$ *is transitive for each* $\mathcal{T}$*-category* $Y=(Y,b)$ *if*

$$(*)\quad \bigvee_{\mathfrak{x}\in TX}(T_\xi a(\mathfrak{X},\mathfrak{x})\oplus u)\otimes(a(\mathfrak{x},x_0)\oplus v)\geq a(m_X(\mathfrak{X}),x_0)\oplus(u\otimes v).$$

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2. *If* $a\multimap \hom_\xi$ *is transitive, then* $(*)$ *for all* $\mathfrak{X}\in T^2X$, $x_0\in X$ *and* $u,v\in\mathsf{V}$ *with* $\xi\cdot Tu=u\cdot!$ *and* $\xi\cdot Tv\leq v\cdot!$.

Closed $\mathcal{T}$-categories

Corollary

Consider $\oplus = \otimes$. *Let* $X = (X, a)$ *be a* $\mathcal{T}$*-category. Then*

1. *If* $a \cdot T_{\xi} a = a \cdot m_X$, *then* $\hom(a, b)$ *is transitive for each* $\mathcal{T}$*-category* $Y = (Y, b)$.
2. $a \cdot T_{\xi} a = a \cdot m_X$ *provided that* $\hom(a, \hom_{\xi})$ *is transitive.*

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Corollary

Consider $\oplus = \otimes$. Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

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3. *Each Eilenberg-Moore algebra (X, α) is closed in $\mathcal{T}$-Cat.*

Closed $\mathcal{T}$-categories

Corollary

Consider $\oplus = \otimes$. Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

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2. *$a \cdot T_{\xi} a = a \cdot m_X$ provided that $\mathrm{hom}(a, \mathrm{hom}_{\xi})$ is transitive.*
3. *Each Eilenberg-Moore algebra (X, α) is closed in $\mathcal{T}$-Cat.*
4. *If $Te_X \cdot e_X = m_X^{\circ} \cdot e_X$, then $X_{\#} = (X, r_{\#})$ is closed for each V-category $X = (X, r)$.*

Compactness

- Degree of compactness: $\mathrm{comp}(X) = \bigwedge_{\mathfrak{x} \in TX} \bigvee_{x \in X} a(\mathfrak{x}, x)$.

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Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. TFAE.

(i). *X is $\oplus$-compact.*

(ii). $\bigvee : (X \multimap \mathsf{V}) \longrightarrow \mathsf{V}$ *is a $\mathcal{T}$-functor (where $X \oplus \_ \dashv X \multimap \_$).*

(iii). $\gamma : |X|_I \longrightarrow \mathsf{V},\ \mathfrak{x} \longmapsto \bigvee_{x \in X} a(\mathfrak{x}, x)$ *is a $\mathcal{T}$-functor.*

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Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. TFAE.

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(iii). *$\gamma : |X|_I \longrightarrow \mathsf{V}$, $\mathfrak{x} \longmapsto \bigvee_{x\in X} a(\mathfrak{x}, x)$ is a $\mathcal{T}$-functor.*

Corollary

A $\mathcal{T}$-category $X = (X, a)$ is $\oplus$-compact iff $\pi_Y : Y \oplus X \longrightarrow Y$ is closed for each $\mathcal{T}$-category $Y = (Y, b)$.

$\mathcal{T}$-modules

A $\mathcal{T}$-module $\varphi : (X, a) \multimap (Y, b)$ is a $\mathcal{T}$-relation $\varphi : X \nrightarrow Y$ such that

$$b \circ \varphi \leq \varphi \qquad \text{and} \qquad \varphi \circ a \leq \varphi.$$

$\mathcal{T}$-modules

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Each $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ defines $\mathcal{T}$-modules $f_* \dashv f^*$:

$$f_* : (X, a) \multimap (Y, b); \;\; f_*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$$

$$f^* : (Y, b) \multimap (X, a); \;\; f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$$

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A $\mathcal{T}$-module $\varphi : (X, a) \multimapinv (Y, b)$ is a $\mathcal{T}$-relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ such that

$$b \circ \varphi \leq \varphi \qquad \text{and} \qquad \varphi \circ a \leq \varphi.$$

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$$f^* : (Y, b) \multimapinv (X, a); \;\; f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$$

$f : (X, a) \longrightarrow (Y, b)$ is fully faithful iff $a = (\mathrm{id}_X)_* = f^* \circ f_*$.

Liftings and extensions

In V-Rel

For $\psi : X \mathrel{\longrightarrow\!\!\!\!\!\mapsto} Z$, the composition maps

$$\_ \cdot \psi : \text{V-Rel}(Z, Y) \longrightarrow \text{V-Rel}(X, Y) \qquad \text{and}$$
$$\psi \cdot \_ : \text{V-Rel}(Y, X) \longrightarrow \text{V-Rel}(Y, Z)$$

have respective right adjoints

$$\_ \bullet\!\!- \psi : \text{V-Rel}(X, Y) \longrightarrow \text{V-Rel}(Z, Y) \qquad \text{and}$$
$$\psi -\!\!\bullet \_ : \text{V-Rel}(Y, Z) \longrightarrow \text{V-Rel}(Y, X).$$

$$\begin{tikzcd}
X \arrow[r, "\varphi", "|"] \arrow[d, "\psi"', "-"] \arrow[dr, phantom, "\Uparrow"] & Y \\
Z \arrow[ur, dotted, "\varphi \bullet\!- \psi"'] &
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
Z \arrow[dr, phantom, "\Uparrow"] & Y \arrow[l, "\varphi"', "|"] \arrow[dl, dotted, "\psi -\!\bullet \varphi"] \\
X \arrow[u, "\psi", "-"'] &
\end{tikzcd}$$

(extension) (lifting)

Liftings and extensions

In $\mathcal{T}$-Rel

For $\psi : X \longrightarrow\!\!\!\!\!\!\!+ \;\; Z$, the composition maps $\_ \circ \psi$ still has a right adjoint but $\psi \circ \_$ in general not.

Liftings and extensions

In $\mathcal{T}$-Rel

For $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$, the composition maps $\_ \circ \psi$ still has a right adjoint but $\psi \circ \_$ in general not. We pass from

$$\begin{tikzcd} X \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "\psi"', "\shortmid"{marking}] & Y \\ Z & \end{tikzcd} \qquad \text{to} \qquad \begin{tikzcd} TX \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "m_X^\circ"', "\shortmid"{marking}] & Y \\ TTX \arrow[d, "T_\xi\psi"', "\shortmid"{marking}] & \\ TZ & \end{tikzcd}$$

(in $\mathcal{T}$-Rel) (in V-Rel)

and define $\varphi \mathrel{\circ\!\!-} \psi = \varphi \mathrel{\bullet\!\!-} (T_\xi\psi \cdot m_X^\circ)$.

Modules as functors

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Theorem

For $\mathcal{T}$-categories (X, a) and (Y, b), and a $\mathcal{T}$-relation $\psi : X \rightharpoondown Y$, the following assertions are equivalent.

i. $\psi : (X, a) \multimap (Y, b)$ *is a $\mathcal{T}$-module.*
ii. *Both* $\psi : |X| \otimes Y \longrightarrow \mathsf{V}$ *and* $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *are $\mathcal{T}$-functors.*

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\begin{aligned} \alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) &\longrightarrow \mathcal{T}\text{-Map}(Y, X). \\ f &\longmapsto f_* \end{aligned}$$

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- In Met: L-complete=Cauchy-complete.
- In Top: L-complete=weakly sober.

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- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.

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- $\varphi \dashv \psi \iff \mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \,.\, \mathfrak{x} \to x\}$ & $\exists \mathfrak{x} \in \mathcal{A} \,.\, A \in \mathfrak{x}$

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$$\begin{aligned} \varphi \text{ is left adjoint } &\iff \exists \mathfrak{x} \in UX \,.\, (A \in \mathfrak{x} \,\&\, \mathfrak{x} \to A) \\ &\iff A \text{ is irreducible.} \end{aligned}$$

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$$\varphi \text{ is representable by } x \iff A = \overline{\{x\}}.$$

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$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$$

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Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

1. *For all $\mathfrak{x} \in TX$ and $\psi \in \mathsf{V}^{|X|}$, $[\![T\mathcal{y}(\mathfrak{x}), \psi]\!] \leq \psi(\mathfrak{x})$.*

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2. *Let $\psi \in \mathsf{V}^{|X|}$. Then*

$$\forall \mathfrak{x} \in TX \,.\, \psi(\mathfrak{x}) \leq [\![T\mathscr{y}(\mathfrak{x}), \psi]\!] \iff \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor.}$$

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We put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

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$\boxed{\text{From now on we assume } T1 = 1.}$

L-closure

Definition

Let $X = (X, a)$ be a $\mathcal{T}$-category. For $M \subseteq X$ we define

$$\overline{M} = \{x \in X \mid i^* \circ x_* \dashv x^* \circ i_*\}.$$

and call $\overline{M}$ the L-closure of M.

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Theorem

Then the following assertions are equivalent.

i. $x \in \overline{M}$.
ii. *For all $\mathcal{T}$-functors $\varphi, \psi : X \longrightarrow Y$ with L-separated codomain: if $\varphi|_M = \psi|_M$, then $\varphi(x) = \psi(x)$.*
iii. *For all $\mathcal{T}$-functors $\varphi, \psi : X \longrightarrow \mathsf{V}$: if $\varphi|_M = \psi|_M$, then $\varphi(x) = \psi(x)$.*

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- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.

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$\psi \in \hat{X}$ is a right adjoint $\mathfrak{T}$-module if and only if $\psi \in \overline{y[X]}$.

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Proof.

$\ldots \varphi = (\mathrm{id}_X)_* \multimap \psi$ and observe that $\varphi(x) = \hat{a}(e_{\hat{X}}(\psi)\, y(x))$ and

$$\xi \cdot T\varphi(\mathfrak{x}) = T_\xi \hat{a}(Te_{\hat{X}} \cdot e_{\hat{X}}(\psi), Ty(\mathfrak{x})) \ldots \qquad \square$$

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

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Theorem

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i. *X is L-complete.*
ii. *X is injective with respect to fully faithful dense $\mathcal{T}$-functor.*
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- X with $a \cdot T_{\xi} a = a \cdot m_X$, Y L-complete $\Rightarrow$ Y^X L-complete.
- $\mathsf{V}^{|X|}$, $\hat{X}$, $\tilde{X}$ are L-complete.