Dualities for distributive spaces

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2011 Spring Southeastern AMS Sectional Meeting
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$X$ ordered set: $x \leq x$, $(x \leq y \leq z) \Rightarrow (x \leq z)$
For an ordered set $X$

- $x \leq y$

For a topological space $X$

- $\xi \rightarrow x$

$X$ top. space: $\hat{x} \rightarrow x$, $(\mathcal{X} \rightarrow \xi \rightarrow x) \Rightarrow (m_{X}(\mathcal{X}) \rightarrow x)$
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$X$ top. space: $\mathcal{X} \to \xi \to x$, $(\mathcal{X} \to \xi \to x) \Rightarrow (m_X(\mathcal{X}) \to x)$ where $\mathcal{X} = \{A \subseteq X \mid x \in A\}$, $m_X(\mathcal{X}) = \{A \subseteq X \mid \mathcal{X} \in A^\#$

$A^\# = \{\xi \in UX \mid A \in \xi\}$.
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$= \text{morphism of type } X \to 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)
### For an ordered set $X$

$\ x \leq y$

up-closed subset

### For a topological space $X$

$\xi \rightarrow x$

closed subset

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$= \text{morphism of type } X \rightarrow 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

Example: $\uparrow x = \{y \in X \mid x \leq y\}$
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$= \text{morphism of type } X \rightarrow 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

Example: $\uparrow \xi = \{x \in X \mid \xi \rightarrow x\}$
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= morphism of type $X^{\text{op}} \to 2$
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= morphism of type $X^{\text{op}} \rightarrow 2$ resp. $X \rightarrow 2^{\text{op}}$
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$= \text{morphism of type } X^{op} \rightarrow 2 \text{ resp. } X \rightarrow 2^{op}$

Example: $\downarrow y = \{x \in X \mid x \leq y\}$
For an ordered set $X$

- $x \leq y$
  - up-closed subset
  - down-closed subset

For a topological space $X$

- $\xi \to x$
  - closed subset

= morphism of type $X^{\text{op}} \to 2$ resp. $X \to 2^{\text{op}}$

Example: $\downarrow x = \{\xi \in X \mid \xi \to x\} \subseteq UX$
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Hence: $X^\text{op} = (UX, \tau)$
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Hence: $\mathcal{X}^{\text{op}} = (UX, \tau)$, 
$\mathcal{A} \subseteq UX$ closed $\iff$ $\mathcal{A} = \{\tau \mid \tau \supseteq f\}$ for some filter of opens $f \subseteq \mathcal{O}X$. 

**Note:** The text contains repetition and possible redundancy, so the statement about $\mathcal{X}$ being cocomplete is not really needed. The key point is the correspondence between up-closed and closed subsets under the duality.
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Hence: $X^{\text{op}} = (UX, \tau)$, 
$\mathcal{A} \subseteq UX$ closed $\iff$ $\mathcal{A} = \{r \mid r \supseteq f\}$ for some filter of opens $f \subseteq \mathcal{O}X$. 
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$x \leq y$ whenever $\xi \to y$
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$x \leq y$ whenever $\hat{x} \rightarrow y$ ($\iff y \in \{x\}$)
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Adjunction: for $f : X \rightarrow Y$ and $g : Y \rightarrow X$,  
$f \dashv g \iff 1_X \leq g \cdot f \& f \cdot g \leq 1_Y \iff (Uf(\xi) \rightarrow y \iff \xi \rightarrow g(y))$
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Adjunction: for $f : X \rightarrow Y$ and $g : Y \rightarrow X$,  
$f \dashv g \iff 1_X \leq g \cdot f \& f \cdot g \leq 1_Y \iff (Uf(\xi) \rightarrow y \iff \xi \rightarrow g(y))$  
Fact: $f$ left adjoint $\Rightarrow f$ preserves smallest limit points.
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In Ord: $X$ cocomplete $\iff y_X : X \rightarrow 2^{X^{\text{op}}}$, $x \mapsto \downarrow x$ has left adjoint.
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In Top: we wish to have a left adjoint of $y_{\mathcal{X}} : \mathcal{X} \rightarrow 2^{\mathcal{X}^{\text{op}}}$, $x \mapsto \downarrow x$. 
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In Top: we wish to have a left adjoint of $y_X : X \to 2^{X^{\text{op}}}$, $x \mapsto \downarrow x$. 
Note: $\to : X^{\text{op}} \times X \to 2$ is continuous, $X^{\text{op}}$ exponentiable.
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$\mathbb{D} = (D, y, m)$
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$\mathcal{D} = (D, y, m)$, where $DX = 2^{X^{\text{op}}}$,

$y_X : X \to DX, \ x \mapsto \downarrow x$
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$\mathbb{D} = (D, y, m)$, where $DX = 2^{X^{\text{op}}}$,

$y_X : X \rightarrow DX$, $x \mapsto \downarrow x$, and

$m_X : DDX \rightarrow DX$, $\mathcal{A} \mapsto \bigcup \mathcal{A}$
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$X$ algebra $\iff y_X$ right adjoint, $X$ $T_0$ $\iff \alpha \cdot y_X = 1_X$. 
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For $A \subseteq X$ down-closed
For an ordered set $X$

- $x \leq y$
- up-closed subset
- down-closed subset
- upper bound
- supremum
- cocomplete
- cocomplete
- down-set monad $\mathbb{D}$
- non-empty down-closed subset
- directed down-closed subset

For a topological space $X$

- $\xi \to x$
- closed subset
- filter of opens
- limit point
- smallest limit point
- cocomplete (but not really)
- continuous lattice
- filter monad $\mathbb{F}$

For $A \subseteq X$ down-closed

$$\text{Up}(X) \to 2, \quad B \mapsto \left[ \exists x \in X . x \in A \land x \in B \right] = \left[ A \cap B \neq \emptyset \right]$$
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For $A \subseteq X$ down-closed resp. $\xi \in FX$ ($= A \subseteq UX$ closed):

$\text{Up}(X) \rightarrow 2, \quad B \mapsto [\exists x \in X . \, x \in A \; \& \; x \in B] = [A \cap B \neq \emptyset]$

$\text{Cl}(X) \rightarrow 2, \quad B \mapsto [\exists \xi \in UX . \, \xi \in A \; \& \; \xi \in UB] = [A \cap UB \neq \emptyset]$
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Completely distributive:
Em Ord: $X$ (cd) $\iff y_X \vdash \text{Sup}_X \vdash t$,.
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Completely distributive:

**Em Ord**: $X$ (cd) $\iff y_X \vdash \operatorname{Sup}_X \vdash t$, $\operatorname{CDOrd}_{\text{sup}} \simeq \operatorname{kar}(\text{Rel})$. 
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<tr>
<td>$x \leq y$</td>
<td>$\xi \to x$</td>
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<td>up-closed subset</td>
<td>closed subset</td>
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<td>down-closed subset</td>
<td>filter of opens</td>
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<td>upper bound</td>
<td>limit point</td>
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<td>supremum</td>
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<tr>
<td>cocomplete</td>
<td>cocomplete (but not really)</td>
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<tr>
<td>cocomplete</td>
<td>continuous lattice</td>
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<tr>
<td>down-set monad $\mathbb{D}$</td>
<td>filter monad $\mathbb{F}$</td>
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<tr>
<td>non-empty down-closed subset</td>
<td>proper filter of opens</td>
</tr>
<tr>
<td>directed down-closed subset</td>
<td></td>
</tr>
</tbody>
</table>

Completely distributive:

Em Ord: $X$ (cd) $\iff y_X \vdash \text{Sup}_X \vdash t$, $\text{CDOrd}_{\text{sup}} \simeq \text{kar}(\text{Rel})$.

Em Top: $X$ (cd) $\iff y_X \vdash \text{Sup}_X \vdash t$, $\text{CDTop}_{\text{sup}} \simeq ???$. 
We will consider $\mathbb{T} = (T, y, m)$ being
- the filter monad $\mathbb{F}$ on Top.
We will consider $\mathbb{T} = (T, y, m)$ being
- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.

Theorem (Rosebrugh and Wood, 2004).
For a monad $D$ on a category $C$ where idempotents split:
$$\text{kar}(C_D) \cong \text{Spl}(C_D).$$
$(X, \alpha) \in \text{Spl}(C_D)$ whenever $\alpha \cdot t = 1_X$ for some homomorphism $t: X \to DX$ ($\iff X$ is projective wrt. those homomorphisms which split in $C$)
We will consider $\mathbb{T} = (T, y, m)$ being
- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.

Theorem (Rosebrugh and Wood, 2004).
For a monad $\mathbb{D}$ on a category $\mathbb{C}$ where idempotents split:
\[
\text{kar}(\mathbb{C} \mathbb{D}) \cong \text{Spl}(\mathbb{C} \mathbb{D}).
\]
\[\text{(X,}\alpha) \in \text{Spl}(\mathbb{C} \mathbb{D}) \text{ whenever } \alpha \cdot t = 1_X \text{ for some homom. } t : X \to \mathbb{D}X\]
\[
\text{(}\iff X \text{ is projective wrt. those homomorphisms which split in } \mathbb{C}.)}
We will consider $T = (T, y, m)$ being
- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.
- the completely prime filter monad $\mathbb{F}_\Omega$ on Top.

Theorem (Rosebrugh and Wood, 2004).
For a monad $D$ on a category $C$ where idempotents split:

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We will consider $\mathbb{T} = (T, y, m)$ being

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- the proper filter monad $\mathbb{F}_1$ on $\text{Top}$.
- the prime filter monad $\mathbb{F}_\omega$ on $\text{Top}$.
- the completely prime filter monad $\mathbb{F}_\Omega$ on $\text{Top}$.

**Theorem** (Rosebrugh and Wood, 2004). For a monad $\mathbb{D}$ on a category $\mathbb{C}$ where idempotents split: $\text{kar}(\mathbb{C}_\mathbb{D}) \simeq \text{Spl}(\mathbb{C}^{\mathbb{D}})$.

$(X, \alpha) \in \text{Spl}(\mathbb{C}^{\mathbb{D}})$ whenever $\alpha \cdot t = 1_X$ for some homomorphism $t : X \to DX$.

$(\iff X$ is projective wrt. those homomorphisms which split in $\mathbb{C})$

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \downarrow \\
X & \longrightarrow & \\
\end{array}
\]
We will consider $\mathbb{T} = (T, y, m)$ being
- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.
- the completely prime filter monad $\mathbb{F}_\Omega$ on Top.

**Theorem** (Rosebrugh and Wood, 2004). *For a monad $\mathbb{D}$ on a category $\mathcal{C}$ where idempotents split:* $\text{kar}(\mathcal{C}_\mathbb{D}) \simeq \text{Spl}(\mathcal{C}_\mathbb{D})$.

$(X, \alpha) \in \text{Spl}(\mathcal{C}_\mathbb{D})$ whenever $\alpha \cdot t = 1_X$ for some homomorphism $t : X \to DX$

( $\iff$ $X$ is projective wrt. those homomorphisms which split in $\mathcal{C}$)

**Theorem** (Gleason, 1954). *The projective compact Hausdorff spaces are precisely the extremely disconnected ones.*

Recall: $\text{CompHaus} \simeq \text{Set}^\mathcal{U}$, $X$ extremely disconnected whenever $\overline{A}$ open for every open $A \subseteq X$. 
Let $X$ be a topological space.
Let $X$ be a topological space.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff X$ is sober, core-compact, stable.

Let $X$ be a topological space.

1. $U \ll V$ if every prime filter $f$ with $U \in f$ has a limit point in $V$.
2. $X$ is **core-compact** if $U \in \mathcal{O}(x) \Rightarrow \exists \ V \in \mathcal{O}(x)$ with $U \ll V$.
3. $X$ is **stable** if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll (\bigcap_{i=1}^{n} U_i)$.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff$ $X$ is sober, core-compact, stable.

Let $X$ be a topological space.

**Proposition.** $X$ is a $\mathbb{F}_\omega$-algebra $\iff X$ is $T_0$, has “suprema” of prime filters, and $f \mapsto \text{Sup } f$ is continuous.

1. $U \ll V$ if every prime filter $f$ with $U \in f$ has a limit point in $V$.
2. $X$ is **core-compact** if $U \in O(x) \Rightarrow \exists V \in O(x)$ with $U \ll V$.
3. $X$ is **stable** if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll (\bigcap_{i=1}^n U_i)$.

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**Lemma.** For $X$ core-compact:

$X$ is stable $\iff \lim f$ is irreducible for all $f$ prime.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff X$ is sober, core-compact, stable.

---

Let $X$ be a topological space.

**Proposition.** $X$ is a $T_\alpha$-algebra $\iff$ $X$ is $T_0$, has “suprema” of $\alpha$-filters, and $f \mapsto \text{Sup} f$ is continuous.

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in $V$.
2. $X$ is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists V \in \mathcal{O}(x)$ with $U \ll V$.
3. $X$ is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll (\bigcap_{i=1}^n U_i)$.

**Lemma.** For $X$ core-compact:

$X$ is stable $\iff$ lim $\mathfrak{f}$ is irreducible for all $\mathfrak{f}$ prime.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff$ $X$ is sober, core-compact, stable.

---

Let $X$ be a topological space.

**Proposition.** $X$ is a $\mathbb{T}$-algebra $\iff$ $X$ is $T_0$, has “suprema” of $\alpha$-filters, and $f \mapsto \text{Sup } f$ is continuous.

1. $U \ll_{\mathbb{T}} V$ if every $f \in TX$ with $U \in f$ has a limit point in $V$.
2. $X$ is $\mathbb{T}$-core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists V \in \mathcal{O}(x)$ with $V \ll_{\mathbb{T}} U$.
3. $X$ is $\mathbb{T}$-stable if $V_i \ll_{\mathbb{T}} U_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll_{\mathbb{T}} (\bigcap_{i=1}^n U_i)$.

**Lemma.** For $X$ core-compact:
$X$ is stable $\iff$ lim $f$ is irreducible for all $f$ prime.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff$ $X$ is sober, core-compact, stable.

---

Let $X$ be a topological space.

**Proposition.** $X$ is a $\mathbb{T}$-algebra $\iff X$ is $T_0$, has “suprema” of $\alpha$-filters, and $f \mapsto \text{Sup } f$ is continuous.

1. $U \ll_T V$ if every $f \in TX$ with $U \in f$ has a limit point in $V$.
2. $X$ is $\mathbb{T}$-core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists V \in \mathcal{O}(x)$ with $V \ll_T U$.
3. $X$ is $\mathbb{T}$-stable if $V_i \ll_T U_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll_T (\bigcap_{i=1}^{n} U_i)$.

**Lemma.** For $X$ $\mathbb{T}$-core-compact:

$X$ is $\mathbb{T}$-stable $\iff$ lim $f$ is irreducible for all $f \in TX$.

**Theorem.** $X$ is $\mathbb{F}_\omega$-algebra $\iff X$ is sober, core-compact, stable.

---

Let \( X \) be a topological space.

**Proposition.** \( X \) is a \( \mathbb{T} \)-algebra \( \iff \) \( X \) is \( T_0 \), has “suprema” of \( \alpha \)-filters, and \( f \mapsto \text{Sup } f \) is continuous.

1. \( U \ll_{\mathbb{T}} V \) if every \( f \in TX \) with \( U \in f \) has a limit point in \( V \).
2. \( X \) is \( \mathbb{T} \)-core-compact if \( U \in \mathcal{O}(x) \Rightarrow \exists \ V \in \mathcal{O}(x) \) with \( V \ll_{\mathbb{T}} U \).
3. \( X \) is \( \mathbb{T} \)-stable if \( V_i \ll_{\mathbb{T}} U_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll_{\mathbb{T}} (\bigcap_{i=1}^{n} U_i) \).

**Lemma.** For \( X \) \( \mathbb{T} \)-core-compact:
\( X \) is \( \mathbb{T} \)-stable \( \iff \) \( \lim f \) is irreducible for all \( f \in TX \).

**Theorem.** \( X \) is \( \mathbb{T} \)-algebra \( \iff \) \( X \) is sober, \( \mathbb{T} \)-core-compact, \( \mathbb{T} \)-stable.

Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \to X$. 

$X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}_X$. 
Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \rightarrow X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$. 

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Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$.
- $X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

For $A, B \in \mathcal{O}X$ and $(A_i)_{i \in I}$ a family of opens, $\#(I) < \alpha \in \{0, 1, \omega, \Omega\}$:

1. $A \subseteq \mu(A)$.
2. If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$.
3. $\mu(\bigcup_{i \in I} A_i) \subseteq \bigcup_{i \in I} \mu(A_i)$
4. $A \cap \mu(B) \subseteq \mu(A \cap B)$.
5. If $X$ is $\mathbb{T}$-disconnected, then $\mu\mu(A) \subseteq \mu(A)$.
6. If $X$ is $\mathbb{T}$-disconnected, then $\mu(A \cap B) = \mu(A) \cap \mu(B)$.
Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X | x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$.
- $X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

**Lemma.** If $t \vdash \text{Sup}$, then $t(x) = \{A \in \mathcal{O}X | x \in \mu(A)\}$.
Let \( X \) be a \( \mathbb{T} \)-algebra with structure \( \text{Sup} : TX \to X \).

- For \( A \in \mathcal{O}X \): \( \mu(A) = \{ x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f \} \).
- \( X \) is called \( \mathbb{T} \)-disconnected if \( \mu(A) \) is open, for every \( A \in \mathcal{O}X \).

**Lemma.** If \( t \dashv \text{Sup} \), then \( t(x) = \{ A \in \mathcal{O}X \mid x \in \mu(A) \} \).

- If \( t \dashv \text{Sup} \), then \( \mu(A) = t^{-1}(A\#) \).
Let $X$ be a $T$-algebra with structure $\text{Sup} : TX \to X$.

- For $A \in \mathcal{OX}$: $\mu(A) = \{x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$.
- $X$ is called $T$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{OX}$.

**Lemma.** If $t \dashv \text{Sup}$, then $t(x) = \{A \in \mathcal{OX} \mid x \in \mu(A)\}$.

- If $t \dashv \text{Sup}$, then $\mu(A) = t^{-1}(A\#)$.
- If $X$ is $T$-disconnected, then $t(x) := \{A \in \mathcal{OX} \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.
Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{ x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f \}$.
- $X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

**Lemma.** If $t \dashv \text{Sup}$, then $t(x) = \{ A \in \mathcal{O}X \mid x \in \mu(A) \}$.

- If $t \dashv \text{Sup}$, then $\mu(A) = t^{-1}(A\#)$.
- If $X$ is $\mathbb{T}$-disconnected, then $t(x) := \{ A \in \mathcal{O}X \mid x \in \mu(A) \}$ defines a continuous map $t : X \to TX$.

**Lemma.** If $X$ is $\mathbb{T}$-disconnected, then $t$ is a $\mathbb{T}$-homomorphism.
Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \rightarrow X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$.
- $X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

**Lemma.** If $t \dashv \text{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.

- If $t \dashv \text{Sup}$, then $\mu(A) = t^{-1}(A\#)$.
- If $X$ is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \rightarrow TX$.

**Lemma.** If $X$ is $\mathbb{T}$-disconnected, then $t$ is a $\mathbb{T}$-homomorphism.

**Lemma.** If $X$ is $\mathbb{T}$-disconnected, then $\text{Sup}(t(x)) = x$. 
Let $X$ be a $\mathbb{T}$-algebra with structure $\text{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \text{Sup}(f) \text{ for some } f \in TX, A \in f\}$.
- $X$ is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

\textbf{Lemma.} If $t \vdash \text{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.

- If $t \vdash \text{Sup}$, then $\mu(A) = t^{-1}(A#)$.
- If $X$ is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.

\textbf{Lemma.} If $X$ is $\mathbb{T}$-disconnected, then $t$ is a $\mathbb{T}$-homomorphism.

\textbf{Lemma.} If $X$ is $\mathbb{T}$-disconnected, then $\text{Sup}(t(x)) = x$.

\textbf{Theorem.} $X$ is $\mathbb{T}$-distributive $\iff$ $X$ is $\mathbb{T}$-disconnected.
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$

- idempotent:

\[
\begin{array}{ccc}
X & \xrightarrow{e} & X \\
\downarrow & & \downarrow \\
\Downarrow & & \Downarrow \\
\end{array}
\]

$e \cdot e = e$
About \( \text{kar}(C_T) \cong \text{Spl}(C_T) \)

- idempotent:

\[
\begin{align*}
    & \quad \quad \quad E \\
    r & \quad \rightarrow & s \\
    \quad \quad \quad X & \quad \rightarrow & X \\
    \quad \quad \quad e & \quad \rightarrow & X \\
\end{align*}
\]

\[ r \cdot s = 1_E \quad \quad e \cdot e = e \]
About $\text{kar}(C^T) \simeq \text{Spl}(C^T)$

- idempotent:

\[
\begin{array}{c}
\xymatrix{ & E 
  \ar[dl]_r 
  \ar[dr]^s \\
X & X 
  \ar[r]^{1_E} & X \\
X & X 
  \ar[ur]_e 
}
\end{array}
\]

$r \cdot s = 1_E$

$e \cdot e = e$
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$

- idempotent:

\[
\begin{array}{ccc}
  & E & \\
  r & s & \\
 X & \xrightarrow{e} & X \xrightarrow{\frac{1}{e}} X \\
  e & & e \cdot e = e
\end{array}
\]

\[r \cdot s = 1_E\]

- idempotents split in $C$ whenever all are of this form.
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$

- idempotent: \[ \begin{array}{ccc}
  & E & \\
 r \downarrow & & s \downarrow \\
 X & \xrightarrow{e} & X \\
 1 \quad e \downarrow & & e \downarrow \\
 e \quad & X & \\
\end{array} \]

\[ r \cdot s = 1_E \]
\[ e \cdot e = e \]

- idempotents split in $C$ whenever all are of this form.
- idempotents split completion can be calculated as follows:
About $\text{kar}(\mathbb{C}_T) \simeq \text{Spl}(\mathbb{C}_T)$

- idempotent: \[ E \xrightarrow{r} s \] \[ r \cdot s = 1_E \]

\[ X \xrightarrow{e} X \xrightarrow{1} X \]
\[ e \cdot e = e \]

- idempotents split in $\mathbb{C}$ whenever all are of this form.

- idempotents split completion can be calculated as follows:
  - embed $\mathbb{C}$ fully into a category $X$ where idempotents split,
About \( \text{kar}(C_T) \simeq \text{Spl}(C^T) \)

- idempotent:
  
  \[
  \begin{array}{ccc}
  & E & \\
  r & \downarrow & s \\
  X & \rightarrow & X \overset{1}{\rightarrow} X \\
  e & \downarrow & e \\
  & X & \\
  \end{array}
  \]

  \[ r \cdot s = 1_E \]

- idempotents split in \( C \) whenever all are of this form.

- idempotents split completion can be calculated as follows:
  - embed \( C \) fully into a category \( X \) where idempotents split,
  - then the idempotents split completion of \( C \) is "the closure of \( C \) in \( X \)".

\[ X \in \overline{C} : \iff X \text{ splits an idempotent in } C \iff X \text{ is a split subobject of some object in } C \]
About $\text{kar}(C_T) \simeq \text{Spl}(C_T^T)$

- idempotent:
  \[ \begin{array}{ccc}
  & & E \\
  & r & \\
  X & \xrightarrow{e} & X \\
  & s & \\
  & \downarrow 1 & \downarrow e \\
  & & X
  \end{array} \quad r \cdot s = 1_E \\
  e \cdot e = e
  
- idempotents split in $C$ whenever all are of this form.

- idempotents split completion can be calculated as follows:
  - embed $C$ fully into a category $X$ where idempotents split,
  - then the idempotents split completion of $C$ is “the closure of $C$ in $X$”.

\[ X \in \overline{C} : \iff X \text{ splits an idempotent in } C \iff X \text{ is a split subobject of some object in } C \]

Example: $C_T \xrightarrow{\cdot \cdot \cdot} C^T_T$

\[ \begin{array}{ccc}
  & & C_T \\
  & \downarrow & \\
  C_T & \xrightarrow{\cdot \cdot \cdot} & C^T_T
  \end{array} \]
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$

- idempotent:

\[ E \xrightarrow{r} X \xrightarrow{e} X \xrightarrow{1} X \xrightarrow{e} X \]
\[ e \cdot e = e \quad r \cdot s = 1_E \]

- idempotents split in $C$ whenever all are of this form.

- idempotents split completion can be calculated as follows:
  
  1. embed $C$ fully into a category $X$ where idempotents split,
  2. then the idempotents split completion of $C$ is “the closure of $C$ in $X$”.

\[ X \in \overline{C} : \iff X \text{ splits an idempotent in } C \]
\[ \iff X \text{ is a split subobject of some object in } C \]

Example:  

\[
\begin{array}{ccc}
C_T & \to & C^T \\
\downarrow & & \downarrow \\
\overline{C_T} & \to & \overline{C}^T
\end{array}
\]

$\overline{C_T} \simeq \text{Spl}(C^T)$
About $\text{kar}(C_{\mathbb{T}}) \simeq \text{Spl}(C^\mathbb{T})$, $\mathbb{T} = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$. 
About $\text{kar}(C_T) \simeq \text{Spl}(C_T^\top)$, $T = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$.

$T$ is also induced by $\text{SLat}^{\text{op}}_{\wedge, \alpha} \Rightarrow \xrightarrow{\eta} \xrightarrow{\mathcal{O}} \xleftarrow{\mathcal{O}} \xleftarrow{\varepsilon} \text{Top}$,
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$, $\mathbb{T} = F_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$.

$\mathbb{T}$ is also induced by $\text{SLat}^{\text{op}}_{\land, \alpha} \nabla \xrightarrow{\eta} \xleftarrow{\epsilon} \text{Top}$, which gives fully faithful $\mathcal{O} : \text{Top}_T \to \text{SLat}^{\text{op}}_{\land, \alpha}$.
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$, $T = F_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$.

$T$ is also induced by $\text{SLat}^{\text{op}}_{\land, \alpha} \xrightarrow{\eta} \xleftarrow{\varepsilon} \text{Top}$, $\text{Top} \xrightarrow{\mathcal{O}} \xleftarrow{\mathcal{F}_\alpha} \text{SLat}^{\text{op}}_{\land, \alpha}$.

which gives fully faithful $\mathcal{O} : \text{Top}_T \rightarrow \text{SLat}^{\text{op}}_{\land, \alpha}$.

**Lemma.** $L \in \overline{\text{Top}_T} \iff L$ is a frame, $\text{hom}(L, 2)$ separates points.
About \( \text{kar}(C_T) \cong \text{Spl}(C^T) \), \( T = F_\alpha, \alpha \in \{0, 1, \omega, \Omega\} \), \( C = \text{Top} \).

\( T \) is also induced by \( \text{SLat}^{\text{op}}_{\land, \alpha} \xrightarrow{\eta} \xleftarrow{\phi} \xrightarrow{\epsilon} \text{Top} \),

which gives fully faithful \( \phi : \text{Top}_T \rightarrow \text{SLat}^{\text{op}}_{\land, \alpha} \).

**Lemma.** \( L \in \overline{\text{Top}}_T \iff L \) is a frame, \( \text{hom}(L, 2) \) separates points.

**Theorem.** \( \text{Frm}^{\text{op}}_{\land, \alpha} \cong \text{Spl}(\text{Top}^{F_\alpha}) \). Furthermore, a topological space \( X \) is \( F_\alpha \)-distributive if and only if \( X \) is the \( \alpha \)-filter space of a frame.
About \( \text{kar}(C_T) \simeq \text{Spl}(C^T) \), \( T = F_\alpha, \ \alpha \in \{0, 1, \omega, \Omega\}, \ C = \text{Top} \).

\( T \) is also induced by \( \text{SLat}^{\text{op}}_{\land, \alpha} \xrightarrow{\eta} F_\alpha \xrightarrow{\epsilon} \text{Top} \),

which gives fully faithful \( \mathcal{O} : \text{Top}_T \to \text{SLat}^{\text{op}}_{\land, \alpha} \).

**Lemma.** \( L \in \overline{\text{Top}_T} \iff L \) is a frame, \( \text{hom}(L, 2) \) separates points.

**Theorem.** \( \text{Frm}^{\text{op}}_{\land, \alpha} \simeq \text{Spl}(\text{Top}^{F_\alpha}) \). Furthermore, a topological space \( X \) is \( F_\alpha \)-distributive if and only if \( X \) is the \( \alpha \)-filter space of a frame.

\( \alpha = \Omega \): Duality between spatial frames and sober spaces.
About \( \text{kar}(C_T) \simeq \text{Spl}(C^T) \), \( T = F_\alpha \), \( \alpha \in \{0, 1, \omega, \Omega\} \), \( C = \text{Top} \).

\( T \) is also induced by \( \text{SLat}^{\text{op}}_{\wedge, \alpha} \xrightarrow{\eta} \xleftarrow{O} \xrightarrow{F_\alpha} \xleftarrow{\varepsilon} \text{Top} \), which gives fully faithful \( O : \text{Top}_T \to \text{SLat}^{\text{op}}_{\wedge, \alpha} \).

**Lemma.** \( L \in \overline{\text{Top}}_T \iff L \) is a frame, \( \text{hom}(L, 2) \) separates points.

**Theorem.** \( \text{Frm}^{\text{op}}_{\wedge, \alpha} \simeq \text{Spl}(\text{Top}^{F_\alpha}) \). Furthermore, a topological space \( X \) is \( F_\alpha \)-distributive if and only if \( X \) is the \( \alpha \)-filter space of a frame.

\( \alpha = \Omega \): Duality between spatial frames and sober spaces.

\( \alpha = \omega \): Restriction of Priestley duality to frames and f-spaces.


About \( \text{kar}(C_T) \simeq \text{Spl}(C^T) \), \( T = \mathbb{F}_\alpha \), \( \alpha \in \{0, 1, \omega, \Omega\} \), \( C = \text{Top} \).

\( T \) is also induced by \( \text{SLat}^{\text{op}}_{\land, \alpha} \xrightarrow{\eta} \xrightarrow{F_\alpha} \xleftarrow{\mathcal{O}} \text{Top} \),

which gives fully faithful \( \mathcal{O} : \text{Top}_T \rightarrow \text{SLat}^{\text{op}}_{\land, \alpha} \).

**Lemma.** \( L \in \overline{\text{Top}}_T \iff L \) is a frame, \( \text{hom}(L, 2) \) separates points.

**Theorem.** \( \text{Frm}^{\text{op}}_{\land, \alpha} \simeq \text{Spl}(\text{Top}^{\mathbb{F}_\alpha}) \). Furthermore, a topological space \( X \) is \( \mathbb{F}_\alpha \)-distributive if and only if \( X \) is the \( \alpha \)-filter space of a frame.

\( \alpha = \Omega \) : Duality between spatial frames and sober spaces.

\( \alpha = \omega \) : Restriction of Priestley duality to frames and f-spaces.

\( \alpha = 0 \) : \( \text{Frm}_{\land}^{\text{op}} \simeq \text{CDTop}_{\text{sup}} \)
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$, $T = F_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$.

$T$ is also induced by $\text{SLat}_{\land, \alpha}^{\text{op}} \xrightarrow{\eta} F_\alpha \xleftarrow{\varepsilon} \text{Top}$, which gives fully faithful $\mathcal{O} : \text{Top}_T \to \text{SLat}_{\land, \alpha}^{\text{op}}$.

**Lemma.** $L \in \overline{\text{Top}_T} \iff L$ is a frame, $\text{hom}(L, 2)$ separates points.

**Theorem.** $\text{Frm}_{\land, \alpha}^{\text{op}} \simeq \text{Spl}(\text{Top}_{F_\alpha})$. Furthermore, a topological space $X$ is $F_\alpha$-distributive if and only if $X$ is the $\alpha$-filter space of a frame.

$\alpha = \Omega$ : Duality between spatial frames and sober spaces.

$\alpha = \omega$ : Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$ : $\text{Frm}_{\land}^{\text{op}} \simeq \text{CDTop}_{\text{sup}}$, hence

$$\text{Leftadjoints}(\text{Frm}_{\land}) \simeq \text{Rightadjoints}(\text{CDTop}_{\text{sup}})$$
About \text{kar}(C_T) \simeq \text{Spl}(C^{\mathbb{T}})\ , \ \mathbb{T} = \mathcal{F}_\alpha, \ \alpha \in \{0,1,\omega,\Omega\}, \ C = \text{Top}.

\mathbb{T} is also induced by \ SLat^{\text{op}}_{\land,\alpha} \xrightarrow{\eta} \xleftarrow{\mathcal{O}} \xrightarrow{\mathcal{F}_\alpha} \xleftarrow{\varepsilon} \text{Top},

which gives fully faithful \ \mathcal{O} : \text{Top}_\mathbb{T} \rightarrow \text{SLat}_{\land,\alpha}^{\text{op}}.

\textbf{Lemma.} \ L \in \overline{\text{Top}}_\mathbb{T} \iff \ L \ is \ a \ frame, \ \text{hom}(L,2) \ separates \ points.

\textbf{Theorem.} \ \text{Frm}_{\land,\alpha}^{\text{op}} \simeq \text{Spl}(\text{Top}^{\mathcal{F}_\alpha})\ . \ Furthermore, \ a \ topological \ space \ X \ is \ \mathcal{F}_\alpha\text{-distributive \ if \ and \ only \ if \ X \ is \ the} \ \alpha\text{-filter \ space \ of \ a \ frame.}

\alpha = \Omega \ : \ \text{Duality \ between \ spatial \ frames \ and \ sober \ spaces.}

\alpha = \omega \ : \ \text{Restriction \ of \ Priestley \ duality \ to \ frames \ and \ f-spaces.}

\alpha = 0 \ : \ \text{Frm}_{\land}^{\text{op}} \simeq \text{CDTop}^{\text{sup}}, \ hence

\text{Frm} \simeq \text{Leftadjoints}(\text{Frm}_{\land}) \simeq \text{Rightadjoints}(\text{CDTop}_{\text{sup}}^{\text{sup}})
About $\text{kar}(C_T) \simeq \text{Spl}(C^T)$, $T = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $C = \text{Top}$.

$T$ is also induced by $\text{SLat}^{\text{op}}_{\land, \alpha} \ni \eta \xrightarrow{\mathcal{O}} F_\alpha \xleftarrow{\varepsilon} \text{Top}$, which gives fully faithful $\mathcal{O} : \text{Top}_T \to \text{SLat}^{\text{op}}_{\land, \alpha}$.

**Lemma.** $L \in \overline{\text{Top}}_T \iff L$ is a frame, $\text{hom}(L, 2)$ separates points.

**Theorem.** $\text{Frm}^{\text{op}}_{\land, \alpha} \simeq \text{Spl}(\text{Top}^{\mathbb{F}_\alpha})$. Furthermore, a topological space $X$ is $\mathbb{F}_\alpha$-distributive if and only if $X$ is the $\alpha$-filter space of a frame.

$\alpha = \Omega$ : Duality between spatial frames and sober spaces.

$\alpha = \omega$ : Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$ : $\text{Frm}^{\text{op}}_{\land} \simeq \text{CDTop}_{\text{sup}}$, hence

$$\text{Frm} \simeq \text{Leftadjoints(} \text{Frm}_{\land}) \simeq \text{Rightadjoints(} \text{CDTop}_{\text{sup}}) \simeq \text{CDTop}.$$