Dualities for distributive spaces

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For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$

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X ordered set: $x \leq x, \quad (x \leq y \leq z) \Rightarrow (x \leq z)$

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X top. space: $\dot{x} \to x$, $\quad (\mathfrak{X} \to \mathfrak{x} \to x) \Rightarrow (m_X(\mathfrak{X}) \to x)$

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X top. space: $\dot{x} \to x$, $\quad (\mathfrak{X} \to \mathfrak{x} \to x) \Rightarrow (m_X(\mathfrak{X}) \to x)$ where

$\dot{x} = \{A \subseteq X \mid x \in A\}$, $m_X(\mathfrak{X}) = \{A \subseteq X \mid \mathfrak{X} \in A^{\#}\}$

$(A^{\#} = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\})$.

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$=$ morphism of type $X \to 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

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Example: $\uparrow x = \{y \in X \mid x \leq y\}$

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$=$ morphism of type $X^{\mathrm{op}} \to 2$ resp. $X \to 2^{\mathrm{op}}$

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Example: $\downarrow y = \{x \in X \mid x \leq y\}$

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Example: $\downarrow x = \{\mathfrak{x} \in X \mid \mathfrak{x} \to x\} \subseteq UX$

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Hence: $X^{\mathrm{op}} = (UX, \tau)$

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$\mathcal{A} \subseteq UX$ closed $\iff$ $\mathcal{A} = \{\mathfrak{x} \mid \mathfrak{x} \supseteq \mathfrak{f}\}$ for some filter of opens $\mathfrak{f} \subseteq \mathcal{O}X$.

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upper bound	

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$x \leq y$ whenever $\dot{x} \to y \quad (\iff \; y \in \overline{\{x\}})$

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Adjunction: for $f : X \to Y$ and $g : Y \to X$,

$$f \dashv g \iff 1_X \leq g \cdot f \ \& \ f \cdot g \leq 1_Y \iff (Uf(\mathfrak{x}) \to y \Leftrightarrow \mathfrak{x} \to g(y))$$

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Fact: f left adjoint $\Rightarrow$ f preserves smallest limit points.

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In Ord: X cocomplete $\iff$ $y_X : X \to 2^{X^{\mathrm{op}}}$, $x \mapsto {\downarrow}x$ has left adjoint.

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In Top: we wish to have a left adjoint of $y_X : X \to 2^{X^{\mathrm{op}}}$, $x \mapsto \downarrow x$.

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$\mathbb{D} = (D, y, m)$

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$y_X : X \to 2^{X^{\mathrm{op}}}$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$

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$\mathbb{F}$ is of Kock–Zöberlein type:

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X algebra $\iff$ y_X right adjoint, X T_0 $\iff$ $\alpha \cdot y_X = 1_X$.

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For $A \subseteq X$ down-closed

$$\mathrm{Up}(X) \to 2,\ B \mapsto [\![\exists x \in X\,.\, x \in A \ \&\ x \in B]\!] = [\![A \cap B \neq \varnothing]\!]$$

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Completely distributive:

Em Ord: X (cd) $\iff$ $y_X \vdash \mathrm{Sup}_X \vdash t$,

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Em Top: X (cd) $\iff$ $y_X \vdash \mathrm{Sup}_X \vdash t$, $\quad \mathrm{CDTop}_{\mathrm{sup}} \simeq ???$.

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Theorem (Rosebrugh and Wood, 2004). *For a monad $\mathbb{D}$ on a category* C *where idempotents split:* $\mathsf{kar}(\mathsf{C}_{\mathbb{D}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{D}})$.

$(X, \alpha) \in \mathsf{Spl}(\mathsf{C}^{\mathbb{D}})$ whenever $\alpha \cdot t = 1_X$ for some homom. $t : X \to DX$

($\iff$ X is projective wrt. those homomorphisms which split in C)

$$\begin{tikzcd} A \arrow[r, two heads] & B \\ X \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

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Theorem (Gleason, 1954). *The projective compact Hausdorff spaces are precisely the extremely disconnected ones.*

Recall: $\mathsf{CompHaus} \simeq \mathsf{Set}^{\mathbb{U}}$,

X extremely disconnected whenever $\overline{A}$ open for every open $A \subseteq X$.

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Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

Simmons, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $U \ll V$.
3. X is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll (\bigcap_{i=1}^{n} U_i)$.

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Proposition. *X is a $\mathbb{F}_\omega$-algebra $\iff$ X is T_0, has "suprema" of prime filters, and $\mathfrak{f} \mapsto \operatorname{Sup} \mathfrak{f}$ is continuous.*

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Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

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1. $A \subseteq \mu(A)$.
2. If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$.
3. $\mu(\bigcup_{i \in I} A_i) \subseteq \bigcup_{i \in I} \mu(A_i)$
4. $A \cap \mu(B) \subseteq \mu(A \cap B)$.
5. If X is $\mathbb{T}$-disconnected, then $\mu\mu(A) \subseteq \mu(A)$.
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Priestley, H.A. (1970), Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2**, 186–190.

Pultr, A. and Sichler, J. (1988), Frames in Priestley's duality, *Cahiers Topologie Géom. Différentielle Catég.* **29** (3), 193–202.

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$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\hom(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is $\mathbb{F}_\alpha$-distributive if and only if X is the α-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$: $\mathsf{Frm}^{\mathrm{op}}_{\wedge} \simeq \mathsf{CDTop}_{\mathsf{sup}}$, hence

$$\mathsf{Frm} \simeq \mathsf{Leftadjoints}(\mathsf{Frm}_\wedge) \simeq \mathsf{Rightadjoints}(\mathsf{CDTop}_{\mathrm{sup}})$$

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