Injective spaces via adjunction

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I

Prolog

The theory presented in this report has its roots in [Lawvere, 1973] where Lawvere makes the point that "...the kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' ..., as concentrated in the thesis that *fundamental* structures are themselves categories.". While it is trivial to regard an individual group or an individual ordered set as a category, Lawvere made the fundamental observation that, within a natural generalisation of the definition of category, one also obtains metric spaces. For his leading example, he considers the points of a metric space X as the objects of a category X and lets the distance

$$a(x,y) \in [0,\infty]$$

play the role of the hom-set of x and y. In fact, the basic laws

$$0 \geqslant a(x,x) \qquad \text{and} \qquad a(x,y)+a(y,z) \geqslant a(x,z)$$

remind us immediately of the operations "choosing the identity" and "composition"

$$1 \to \hom(x,x) \qquad \text{and} \qquad \hom(x,y) \times \hom(y,z) \to \hom(x,z)$$

of a category. Moreover, Lawvere shows that categorical notions and results translate into significant ones about metric spaces, "so that enriched category theory can suggest new directions of research in metric space theory and conversely" [Lawvere, 1973]. Although Lawvere's suggestions were not embraced by analysists, they did resonate with theoretical computer scientists and led to a substantial body of work as it can be seen, for instance, from [Wagner, 1994; Flagg *et al.*, 1996; Rutten, 1998; Bonsangue *et al.*, 1998; Waszkiewicz, 2009].

In this report we follow the nomenclature of [Lawvere, 1973] and require a metric only to satisfy the two axioms above; then a *classical* metric space becomes a *separated* ($d(x,y)=0=d(y,x)$ implies $x=y$), *symmetric* ($d(x,y)=d(y,x)$) and *finitary* ($d(x,y)<\infty$) metric space. Furthermore, with this categorical interpretation, a "functor" between metric spaces (X,a) and (Y,b) is a map $f:X\to Y$ satisfying $a(x,y)\geqslant b(f(x),f(y))$ for all $x,y\in X$, that is, $f:(X,a)\to(Y,b)$ is non-expansive. The category of metric spaces and non-expansive maps is denoted by

$$\mathsf{Met}.$$

In a similar way, our work with topological spaces presented as convergence structures shaped the idea that *topological spaces are categories*, and therefore can be studied using notions and techniques from Category Theory. Here we consider the points of a space X as the objects of our category, and interpret the convergence $\mathfrak{x} \to x$ of an ultrafilter $\mathfrak{x}$ on X to a point $x \in X$ as a morphism in X. With this interpretation, the convergence relation

$$\to : UX \times X \to 2$$

becomes the "hom-functor" of X; however, we have to make the concession that a morphism in X does not just have an object but rather an ultrafilter of objects as domain. Nevertheless, our intuition is supported by the observation (due to Barr in 1970) that a relation $\mathfrak{x} \to x$ between ultrafilters and points of a set X is the convergence relation of a topology on X if and only if

$$e_X(x) \to x \qquad \text{and} \qquad (\mathfrak{X} \to \mathfrak{x} \;\&\; \mathfrak{x} \to x) \;\Rightarrow\; m_X(\mathfrak{X}) \to x, \qquad (*)$$

for all $x \in X$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$; where $m_X(\mathfrak{X})$ is the filtered sum of the filters in $\mathfrak{X}$ and $e_X(x) = \dot{x}$ the principal ultrafilter generated by $x \in X$. Note that the second condition of $(*)$ talks about the convergence of an ultrafilter of ultrafilters $\mathfrak{X}$ to an ultrafilter $\mathfrak{x}$, which comes from applying the ultrafilter functor U to the *relation* $a : UX \relbar\!\!\mapsto X$. In general, for a relation $r : X \relbar\!\!\mapsto Y$ from X to Y and ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$, one puts

$$\mathfrak{x}\,(Ur)\,\mathfrak{y} \quad \text{whenever} \quad \forall A \in \mathfrak{x}, B \in \mathfrak{y}\, \exists x \in A, y \in B \,.\, x\,r\,y,$$

and obtains this way an extension of the Set-functor U to a 2-functor $U : \mathsf{Rel} \to \mathsf{Rel}$. In our interpretation, the first condition of $(*)$ postulates the existence of an "identity arrow" on x, whereby the second one requires the existence of a "composite" of "composable pairs of arrows". Furthermore, a function $f : X \to Y$ between topological spaces is continuous whenever $\mathfrak{x} \to x$ in X implies $f(\mathfrak{x}) \to f(x)$ in Y, that is, f associates to each object in X an object in Y and to each arrow in X an arrow in Y between the corresponding (ultrafilter of) objects in Y. In the sequel

$$\mathsf{Top}$$

denotes the category of topological spaces and continuous maps.

> *Yes, I have drawn the conclusion; now, however, does it draw me.*
> Friedrich Nietzsche, from "Thus spoke Zarathustra".

These categorical perspectives suggest two features which usually play no prominent role in Topology: firstly, both categories Top and Met are actually 2-categories; secondly, the notion of distributor enters the scene. In fact, one amazing insight of [Lawvere, 1973] is a characterisation of the notion of Cauchy completeness for metric spaces using adjoint distributors, giving further evidence to MacLane's motto "adjoints occur almost everywhere" MacLane [1971]. Eventually, this characterisation amounts to saying that a metric space is Cauchy complete if and only if it admits suprema of right adjoint distributors; here *supremum* has to be understood in the sense of *weighted colimit* of enriched category theory [Eilenberg and Kelly, 1966; Kelly, 1982]. Other types of distributors specify other properties of metric spaces: forward Cauchy nets can be represented by flat distributors (see [Vickers, 2005], called flat modules there) and their limit points as suprema of these distributors, and the formal ball model of a metric space relates to its cocompletion with respect to yet another type of distributors (see [Rutten, 1998; Kostanek and Waszkiewicz, 2011]). Similarly, a topological T0-space is sober if and only if it has suprema of right adjoint distributors (suitably defined, see [Clementino and Hofmann, 2009a]), and a continuous lattice (introduced in [Scott, 1972]) can be described as a topological T0-space admitting colimits of all distributors. Due to the analogy with categories, our study of spaces is primary based on concepts and results like distributor, colimit, adjunction, dual space and the Yoneda lemma; however, we stress that this path leads us naturally to more traditional topics such as compact, locally compact and stably compact spaces, to the filter space and the Vietories construction, and to sober spaces and continuous lattices. We hope to be able to show in this report that this perspective is very fruitful for the study of topological and other kind of spaces.

Last but not least, the title of this report is reminiscent of the chapter *Ordered sets via adjunction* by Wood of the book [Pedicchio and Tholen, 2004]. This chapter is largely based on Wood's joint work with Rosebrugh which constitutes a continuing source of inspiration for our investigation of spaces.

II

Models of topological theories

In this chapter we describe our general framework, namely that of a topological theory $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$ consisting of a monad $\mathbb{T} = (T, m, e)$ on Set, a quantale $\mathcal{V}$ and a $\mathbb{T}$-algebra structure $\xi : T\mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$. The associated notion of $\mathcal{T}$-category embodies several types of spaces such as topological, metric or approach spaces, and together with $\mathcal{T}$-functors and $\mathcal{T}$-distributors defines the categories $\mathcal{T}\text{-}\mathsf{Cat}$ and $\mathcal{T}\text{-}\mathsf{Dist}$ respectively. We recall succinctly the main constructions and some fundamental results.

The study of $\mathcal{T}$-categories, under the designation $(\mathbb{T}, \mathcal{V})$-categories, $(\mathbb{T}, \mathcal{V})$-algebras or lax algebras, started with the papers [Clementino and Tholen, 2003; Clementino and Hofmann, 2003] which synthesise the notion of relational algebra (see [Barr, 1970; Manes, 1974]) with Lawvere's presentation of metric spaces as enriched categories.

II.1. Topological theories

Our starting point is the concept of topological theory as introduced in [Hofmann, 2007]. More precise, what we define here is called *strict* topological theory there; however, in this report all theories are assumed to be strict and therefore we use the term topological theory in the stronger sense.

Definition II.1.1. A ***topological theory*** $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$ consists of

(1). a monad $\mathbb{T} = (T, m, e)$ on Set (with multiplication m and unit e),
(2). a commutative and unital quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ and
(3). a function $\xi : T\mathcal{V} \to \mathcal{V}$

such that

(a). T preserves weak pullbacks and each naturality square of m is a weak pullback,
(b). the pair $(\mathcal{V}, \xi)$ is an Eilenberg–Moore algebra for $\mathbb{T}$ and the monoid structure on $\mathcal{V}$ in $(\mathsf{Set}, \times, 1)$ lifts to a monoid structure on $(\mathcal{V}, \xi)$ in $(\mathsf{Set}^{\mathbb{T}}, \times, 1)$, that is, the diagrams

$$\begin{tikzcd} T(\mathcal{V} \times \mathcal{V}) \arrow[r, "T(-\otimes-)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] & T\mathcal{V} \arrow[d, "\xi"] & T1 \arrow[l, "T(k)"'] \arrow[d, "!"] \\ \mathcal{V} \times \mathcal{V} \arrow[r, "-\otimes-"'] & \mathcal{V} & 1 \arrow[l, "k"] \end{tikzcd}$$

commute and,

(c). writing $P_{\mathcal{V}} : \mathsf{Set} \to \mathsf{Ord}$ for the functor that sends a function $f : X \to Y$ to the left adjoint of the "inverse image" function $f^{-1} : \mathcal{V}^Y \to \mathcal{V}^X$, $\varphi \mapsto \varphi \cdot f$ (where $\mathcal{V}^X$ is the set of functions from X to $\mathcal{V}$, with pointwise order), the functions $\xi_X : V^X \to V^{TX}$, $f \mapsto \xi \cdot Tf$ (for X in Set) are the components of a natural transformation $(\xi_X)_X : P_{\mathcal{V}} \to P_{\mathcal{V}}T$.

Throughout we will *always assume that $\mathcal{V}$ is non-trivial*, that is, $\perp \neq k$. Consequently, since $\mathcal{V}$ is assumed to be a $\mathbb{T}$-algebra, the monad $\mathbb{T}$ must be non-trivial. We recall here that there are two trivial monads $\mathbb{T} = (T, m, e)$ on Set: one with $TX = 1$ for every set X, and one with $TX = 1$ for every non-empty set and $T\varnothing = \varnothing$; a monad $\mathbb{T} = (T, m, e)$ on Set different from these two is called non-trivial.

The algebra structure $\xi : T\mathcal{V} \to \mathcal{V}$ is also compatible with the ***internal hom*** in $\mathcal{V}$ defined by

$$x \otimes y \leq z \iff x \leq \hom(y, z),$$

as shown in [Hofmann, 2007, Lemma 3.2]:

Lemma II.1.2. *For every topological theory $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$,*

$$\begin{tikzcd} T(\mathcal{V}\times\mathcal{V}) \arrow[r, "T(\hom)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & T\mathcal{V} \arrow[d, "\xi"] \\ \mathcal{V}\times\mathcal{V} \arrow[r, "\hom"'] & \mathcal{V}. \end{tikzcd}$$

Our leading examples are the following:

Examples II.1.3. (1). For any quantale $\mathcal{V}$ we can consider the theory whose monad-part is the identity monad on Set and where $\xi : \mathcal{V} \to \mathcal{V}$ is the identity function. We write this trivial topological theory as $\mathcal{I}_{\mathcal{V}}$.

(2). Let $\mathcal{V}$ be the 2-element chain $\mathsf{2}$, and consider the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set. This together with the “identity” function $\xi : U\mathsf{2} \to \mathsf{2}$ is a topological theory which we denote by $\mathcal{U}_{\mathsf{2}}$.

(3). More generally, for a non-trivial monad $\mathbb{T} = (T, m, e)$ on Set where T preserves weak pullbacks and each naturality square of m is a weak pullback and every completely distributive complete lattice $\mathcal{V}$ (considering $\otimes = \wedge$ and $k = \top$), $(\mathbb{T}, \mathcal{V}, \xi)$ is a topological theory where

$$\xi : T\mathcal{V} \to \mathcal{V},\ \ \mathfrak{x} \mapsto \bigvee\{v \in \mathcal{V} \mid \mathfrak{x} \in T(\uparrow v)\}.$$

(4). In particular, for the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set and the complete lattice $[0, \infty]$ ordered by the “greater or equal” relation $\geqslant$ (so that the infimum of two numbers is their maximum and the supremum of $S \subseteq [0, \infty]$ is given by $\inf S$), we write $\mathsf{P}_{\wedge} = ([0, \infty], \max, 0)$ for the corresponding quantale and $\mathcal{U}_{\mathsf{P}_{\wedge}} = (\mathbb{U}, \mathsf{P}_{\wedge}, \xi)$ for the corresponding theory where

$$\xi : U([0, \infty]) \to [0, \infty],\ \ \mathfrak{x} \mapsto \inf\{v \in [0, \infty] \mid [0, v] \in \mathfrak{x}\}.$$

Also note that

$$\hom(u, v) = \begin{cases} 0 & \text{if } u \geqslant v, \\ v & \text{otherwise} \end{cases}$$

in $\mathsf{P}_{\wedge}$.

(5). Let $\mathcal{V}$ be the quantale $\mathsf{P}_{+} = ([0, \infty], +, 0)$ of extended non-negative real numbers ordered by the “greater or equal” relation (see [Lawvere, 1973]), and consider again the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set. Together with the function $\xi : U([0, \infty]) \to [0, \infty]$ as above this makes up a topological theory, denoted as $\mathcal{U}_{\mathsf{P}_+}$. For later use we record here that the internal hom of the quantale P_+ is given by truncated minus:

$$\hom(u,v) = v \ominus u := \max\{v-u,0\}.$$

(6). For any quantale $\mathcal{V}$, the word monad $\mathbb{L} = (L,m,e)$ on Set together with the function

$$\xi : L(\mathcal{V}) \to \mathcal{V},\ (v_1,\dots,v_n) \mapsto v_1 \otimes \dots \otimes v_n,\ (\ ) \mapsto k$$

determine a topological theory.

II.2. Syntactic constructions

The components of a topological theory $\mathcal{T} = (\mathbb{T},\mathcal{V},\xi)$ allow for a sequence of constructions and definitions which we describe now.

Firstly, we can form the quantaloid $\mathcal{V}\text{-}\mathsf{Rel}$ with sets as objects, and a morphism $r : X \rightarrow\!\!\!\!\!\mapsto Y$ from X to Y is a ***$\mathcal{V}$-relation*** $r : X \times Y \to \mathcal{V}$ (called $\mathcal{V}$-matrix in [Betti *et al.*, 1983]). The composition of $\mathcal{V}$-relations $r : X \rightarrow\!\!\!\!\!\mapsto Y$ and $s : Y \rightarrow\!\!\!\!\!\mapsto Z$ is defined as matrix multiplication

$$s \cdot r(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z),$$

and the identity arrow $1_X : X \rightarrow\!\!\!\!\!\mapsto X$ is the $\mathcal{V}$-relation which sends all diagonal elements (x,x) to k and all other elements to the bottom element $\perp$ of $\mathcal{V}$. The set $\mathcal{V}\text{-}\mathsf{Rel}(X,Y)$ of all $\mathcal{V}$-relations from X to Y becomes a complete ordered set by putting

$$r \le r' \quad \text{whenever} \quad \forall x \in X\ \forall y \in Y\ .\ r(x,y) \le r'(x,y),$$

for $\mathcal{V}$-relations $r, r' : X \rightarrow\!\!\!\!\!\mapsto Y$; and suprema of $\mathcal{V}$-relations are calculated pointwise. For every $\mathcal{V}$-relation $t : X \rightarrow\!\!\!\!\!\mapsto Z$, the composing-with-$t$ functions

$$- \cdot t : \mathcal{V}\text{-}\mathsf{Rel}(Z,Y) \to \mathcal{V}\text{-}\mathsf{Rel}(X,Y) \qquad \text{and} \qquad t \cdot - : \mathcal{V}\text{-}\mathsf{Rel}(Y,X) \to \mathcal{V}\text{-}\mathsf{Rel}(Y,Z)$$

preserve suprema and therefore have respective right adjoints

$$(-) \bullet\!\!- t : \mathcal{V}\text{-}\mathsf{Rel}(X,Y) \to \mathcal{V}\text{-}\mathsf{Rel}(Z,Y) \qquad \text{and} \qquad t -\!\!\bullet (-) : \mathcal{V}\text{-}\mathsf{Rel}(Y,Z) \to \mathcal{V}\text{-}\mathsf{Rel}(Y,X).$$

Here, for $\mathcal{V}$-relations $r : X \rightarrow\!\!\!\!\!\mapsto Y$ and $s : Y \rightarrow\!\!\!\!\!\mapsto Z$,

$$(r \bullet\!\!- t)(z,y) = \bigwedge_{x\in X} \hom(t(x,z), r(x,y)) \qquad (t -\!\!\bullet s)(y,x) = \bigwedge_{z\in Z} \hom(t(x,z), s(y,z)).$$

We call $r \bullet\!\!- t$ the ***extension of*** r ***along*** t, and $t -\!\!\bullet s$ the ***lifting of*** s ***along*** t. The category $\mathcal{V}\text{-}\mathsf{Rel}$ has an involution $(r : X \rightarrow\!\!\!\!\!\mapsto Y) \mapsto (r^\circ : Y \rightarrow\!\!\!\!\!\mapsto X)$ where $r^\circ(y,x) = r(x,y)$, satisfying

$$1_X^\circ = 1_X, \qquad (s\cdot r)^\circ = r^\circ \cdot s^\circ, \qquad r^{\circ\circ} = r,$$

as well as $r^\circ \le s^\circ$ whenever $r \le s$. Finally, there is a faithful functor

$$\mathsf{Set} \to \mathcal{V}\text{-}\mathsf{Rel},\ (f : X \to Y) \mapsto (f : X \rightarrow\!\!\!\!\!\mapsto Y)$$

sending a map $f : X \to Y$ to its graph $f : X \rightarrow\!\!\!\!\!\mapsto Y$ defined by

$$f(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

In the sequel we will not distinguish between the function f and the $\mathcal{V}$-relation f and simply write $f : X \to Y$. We also note that $f \dashv f^\circ$ in the quantaloid $\mathcal{V}\text{-}\mathsf{Rel}$.

Secondly, the Set-functor T extends to a 2-functor $T_\xi : \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$. Here, T_ξ assigns to each $\mathcal{V}$-relation $r : X \times Y \to \mathcal{V}$ the unique $\mathcal{V}$-relation $T_\xi r : TX \times TY \to \mathcal{V}$ satisfying, for every map $s : TX \times TY \to \mathcal{V}$,

$$\xi \cdot Tr \leq s \cdot \langle T\pi_1, T\pi_2 \rangle \iff T_\xi r \leq s.$$

$$\begin{tikzcd}
TX \times TY \arrow[drr, dotted, "T_\xi r"] & & \\
& \leq & \\
T(X \times Y) \arrow[uu, "\langle T\pi_1, T\pi_2 \rangle"] \arrow[rr, "\xi \cdot Tr"'] & & \mathcal{V}
\end{tikzcd}$$

In other words, regarding TX, TY and $TX \times TY$ as discrete ordered sets, $T_\xi r$ is the left Kan extension in Ord of $\xi \cdot Tr$ along $\langle T\pi_1, T\pi_2 \rangle$. Hence, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}.$$

THEOREM II.2.1. *The 2-functor T_ξ preserves the involution in the sense that $T_\xi(r^\circ) = T_\xi(r)^\circ$ (and we write $T_\xi r^\circ$) for each $\mathcal{V}$-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, m becomes a natural transformation $m : T_\xi T_\xi \to T_\xi$ and e an op-lax natural transformation $e : 1 \to T_\xi$, that is, $e_Y \cdot r \leq T_\xi r \cdot e_X$ for all $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathcal{V}\text{-}\mathsf{Rel}$.*

EXAMPLES II.2.2. For $\mathcal{T} = \mathcal{U}_2$, the extension above coincides with the one given in Chapter I; and for $\mathcal{T} = \mathcal{U}_{\mathsf{P}_+}$ and $\mathcal{T} = \mathcal{U}_{\mathsf{P}_\wedge}$ one obtains

$$U_\xi r(\mathfrak{x}, \mathfrak{y}) = \sup_{A \in \mathfrak{x}, B \in \mathfrak{y}} \inf_{x \in A, y \in B} r(x, y)$$

for all $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$ (see also [Clementino and Tholen, 2003]).

Finally, of particular interest to us will be $\mathcal{V}$-relations of the form $\alpha : TX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. We refer to these relations as $\mathcal{T}$-***relations*** and think of α as an arrow $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$. Given two $\mathcal{T}$-relations $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ and $\beta : Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$, their ***Kleisli convolution*** $\beta \circ \alpha : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$ is defined as

$$\beta \circ \alpha = \beta \cdot T_\xi \alpha \cdot m_X^\circ.$$

This operation is associative and has the $\mathcal{T}$-relation $e_X^\circ : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ as a lax identity:

$$a \circ e_X^\circ = a \qquad \text{and} \qquad e_Y^\circ \circ a \geq a,$$

for any $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$. A $\mathcal{T}$-relation $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ is called ***unitary*** whenever $e_Y^\circ \circ a = a$. We denote the 2-category of sets and unitary $\mathcal{T}$-relations with Kleisli convolution as compositional structure by

$$\mathcal{T}\text{-}\mathsf{URel}.$$

EXAMPLE II.2.3. Unitary $\mathcal{U}_2$-relations are characterised in Hofmann [2005] as precisely those $\mathcal{U}_2$-relations $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ where, for each $y \in Y$, $\{\mathfrak{x} \in UX \mid \mathfrak{x}\, a\, y\} \subseteq UX$ is closed in UX with respect to the Zariski closure. Recall that $\mathfrak{a} \in UX$ belongs to the ***Zariski closure*** of $\mathcal{A} \subseteq UX$ whenever $\bigcap \mathcal{A} \subseteq \mathfrak{a}$.

We note that an arbitrary infimum of unitary $\mathcal{T}$-relations $X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ is again unitary, hence $\mathcal{T}\text{-}\mathsf{URel}(X, Y)$ is a complete lattice. Moreover, for every unitary $\mathcal{T}$-relation $\alpha : X \relbar\joinrel\circ\joinrel\rightharpoonup Y$, the composition function $- \circ \alpha$ has a right adjoint

$$(-) \circ \alpha \dashv (-) \multimapinv \alpha$$

where, for a given unitary $\mathcal{T}$-relation $\gamma : X \relbar\joinrel\circ\joinrel\rightharpoonup Z$, the extension $\gamma \circ\!\!- \alpha : Y \relbar\joinrel\circ\joinrel\rightharpoonup Z$ is constructed in $\mathcal{V}$-Rel as the extension $\gamma \bullet\!\!- (T_\xi\alpha \cdot m_X^\circ)$.

$$\begin{tikzcd}
TX \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "m_X^\circ"', "\shortmid"{marking}] & Z \\
TTX \arrow[d, "T_\xi\alpha"', "\shortmid"{marking}] & \\
TY \arrow[uur, dotted, "\shortmid"{marking}] &
\end{tikzcd}$$

Unfortunately, *in general liftings need not exist* in $\mathcal{T}$-URel.

II.3. Categories, functors and distributors

We turn now our attention to models of topological theories.

Definition II.3.1. A $\mathcal{T}$-***category*** is a pair (X, a) consisting of a set X and a $\mathcal{T}$-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ on X such that $e_X^\circ \le a$ and $a \circ a \le a$.

We note that the $\mathcal{T}$-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ of a $\mathcal{T}$-category is necessarily unitary. Furthermore, employing the adjunctions $e_X \dashv e_X^\circ$ and $m_X \dashv m_X^\circ$ in $\mathcal{V}$-Rel, the two conditions $e_X^\circ \le a$ and $a \circ a \le a$ can be written as the *lax Eilenberg–Moore axioms* $1_X \le a \cdot e_X$ and $a \cdot T_\xi a \le a \cdot m_X$. Expressed elementwise, these two conditions read as

$$k \le a(e_X(x), x) \qquad \text{and} \qquad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$ and $x \in X$.

Definition II.3.2. A function $f : X \to Y$ between $\mathcal{T}$-categories (X, a) and (Y, b) is a $\mathcal{T}$-***functor*** whenever $f \cdot a \le b \cdot Tf$.

Since $f \dashv f^\circ$ in $\mathcal{V}$-Rel, this condition is equivalent to $a \le f^\circ \cdot b \cdot Tf$, and in pointwise notation the latter inequality becomes

$$a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x)),$$

for all $\mathfrak{x} \in TX$, $x \in X$. The category of $\mathcal{T}$-categories and $\mathcal{T}$-functors is denoted by

$$\mathcal{T}\text{-}\mathsf{Cat}.$$

For the identity theory $\mathcal{I}_\mathcal{V}$, a $\mathcal{I}_\mathcal{V}$-category is just a $\mathcal{V}$-category and $\mathcal{I}_\mathcal{V}$-functor means $\mathcal{V}$-functor in the sense of [Eilenberg and Kelly, 1966]. Therefore we write $\mathcal{V}$-category instead of $\mathcal{I}_\mathcal{V}$-category, $\mathcal{V}$-functor instead of $\mathcal{I}_\mathcal{V}$-functor, and

$$\mathcal{V}\text{-}\mathsf{Cat}$$

instead of $\mathcal{I}_\mathcal{V}$-Cat.

Examples II.3.3. (1). Clearly, $\mathsf{2}$-Cat is concretely isomorphic to the category Ord of ordered sets and monotone maps, $\mathsf{P}_{\!+}$-Cat to the category Met of metric spaces and non-expansive maps (in the sense of [Lawvere, 1973]) and $\mathsf{P}_{\!\wedge}$-Cat to the category UMet of ultrametric spaces and non-expansive maps. Similarly to our nomenclature for metric spaces, an order relation is only assumed to be reflexive and transitive, and an ultrametrics a on X needs only to satisfy

$$0 \geqslant a(x, x) \qquad \text{and} \qquad \max\{a(x, y), a(y, z)\} \geqslant a(x, z).$$

(2). The main result of [Barr, 1970] states that $\mathcal{U}_2$-Cat is concretely isomorphic to the category Top of topological spaces and continuous maps.

(3). Another inspiring source for our study we found in approach theory initiated by Lowen in 1989; for an extensive presentation we refer to [Lowen, 1997]. By definition, an ***approach space*** is a set X together with a function $\delta : PX \times X \to [0,\infty]$ (here PX denotes the powerset of X) subject to

(a) $\delta(\{x\}, x) = 0$,

(b) $\delta(\varnothing, x) = \infty$,

(c) $\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\}$,

(d) $\delta(A^{(\varepsilon)}, x) + \varepsilon \geqslant \delta(A, x)$;

for all $A, B \subseteq X$, $x \in X$ and $\varepsilon \in [0,\infty]$. Here $A^{(\varepsilon)} = \{x \in X \mid \varepsilon \geqslant \delta(A, x)\}$. For $\delta : PX \times X \to [0,\infty]$ and $\delta' : PY \times Y \to [0,\infty]$, a map $f : X \to Y$ is called ***non-expansive*** if $\delta(A, x) \geqslant \delta'(f[A], f(x))$, for all $A \subseteq X$ and $x \in X$. Approach spaces and non-expansive maps are the objects and morphisms of the category

$$\mathsf{App};$$

and it is shown in [Clementino and Hofmann, 2003] that App is concretely isomorphic to $\mathcal{U}_{\mathsf{P}_+}$-$\mathsf{Cat}$.

There is a canonical forgetful functor $\mathsf{App} \to \mathsf{Top}$ sending an approach space (X, a) to the topological space with the same underlying set X and with the convergence relation defined by

$$\mathfrak{x} \to x \text{ whenever } 0 \geqslant a(\mathfrak{x}, x).$$

This functor has a left adjoint $\mathsf{Top} \to \mathsf{App}$ which one obtains by composing the convergence relation $UX \times X \to \mathsf{2}$ of a topological space with $\mathsf{2} \hookrightarrow [0,\infty]$ where $1 \mapsto 0$ and $0 \mapsto \infty$.

(4). The category $\mathcal{U}_{\mathsf{P}_\wedge}$-$\mathsf{Cat}$ can be identified with the full subcategory UApp of App defined by all those approach spaces (X, a) which satisfy

$$\max(U_\xi a(\mathfrak{X}, \mathfrak{x}), a(\mathfrak{x}, x)) \geqslant a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$ and $x \in X$.

In the sequel we always refer to these presentations when talking about Ord, Met, UMet, Top, App *or* UApp.

Besides functors, there is another important type of morphisms between categories, called distributors. The notion of distributor was introduced by Bénabou in the 1960's and provides "a generalisation of relations between sets to 'relations between (small) categories' " (see [Bénabou, 2000]). Below we provide a corresponding notion for $\mathcal{T}$-categories which is due to [Clementino and Hofmann, 2009a].

DEFINITION II.3.4. A $\mathcal{T}$-relation $\varphi : X \rightharpoonup Y$ between $\mathcal{T}$-categories $X = (X, a)$ and $Y = (Y, b)$ is a ***$\mathcal{T}$-distributor***, written as $\varphi : X \multimapinv Y$, whenever $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$.

Hence, a $\mathcal{T}$-distributor $\varphi : X \multimapinv Y$ comes with a *right action* of the $\mathcal{T}$-relation a and a *left action* of b. This perspective motivates the designation *bimodule* or *module* used by some authors. Note that we always have $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, so that the $\mathcal{T}$-distributor conditions above are in fact equalities. $\mathcal{T}$-categories and $\mathcal{T}$-distributors form a 2-category, denoted by

$$\mathcal{T}\text{-}\mathsf{Dist},$$

with Kleisli convolution as composition and with the 2-categorical structure inherited from $\mathcal{V}$-Rel. The identity in $\mathcal{T}$-Dist on a $\mathcal{T}$-category $X = (X, a)$ is given by $a : X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$. As before, we write

$$\mathcal{V}\text{-}\mathsf{Dist}$$

whenever $\mathcal{T} = \mathcal{I}_{\mathcal{V}}$ is an identity theory, and use $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$ instead of $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ in this case.

Let (X, a) and (Y, b) be $\mathcal{T}$-categories. Each map $f : X \to Y$ induces $\mathcal{T}$-relations

$$f^{\circledast} = b \cdot Tf : Y \mathrel{-\!\!\!\mapstochar\!\!\rightharpoonup} X \qquad \text{and} \qquad f^{\circledast} = f^{\circ} \cdot b : Y \mathrel{-\!\!\!\mapstochar\!\!\rightharpoonup} X;$$

moreover, one has $b \circ f_{\circledast} \leq f_{\circledast}$ and $f^{\circledast} \circ b \leq f^{\circledast}$. These $\mathcal{T}$-relations are actually $\mathcal{T}$-distributors precisely when f is a $\mathcal{T}$-functor:

Lemma II.3.5. *The following are equivalent, for $\mathcal{T}$-categories (X, a) and (Y, b) and a map $f : X \to Y$.*

(i). *f is a $\mathcal{T}$-functor $f : (X, a) \to (Y, b)$.*
(ii). *$f_{\circledast}$ is a $\mathcal{T}$-distributor, that is, $f_{\circledast} \circ a \leq f_{\circledast}$.*
(iii). *$f^{\circledast}$ is a $\mathcal{T}$-distributor, that is, $a \circ f^{\circledast} \leq f_{\circledast}$.*

In fact, these assignments define functors

$$(-)_{\circledast} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Dist} \qquad \text{and} \qquad (-)^{\circledast} : \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \to \mathcal{T}\text{-}\mathsf{Dist}$$

which leave objects unchanged.

For a $\mathcal{V}$-functor $f : X \to Y$ we will, however, use the traditional notation $f_* : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$ and $f^* : Y \mathrel{-\!\!\circ\!\!\rightarrow} X$. This distinction is convenient since at some occasions we will consider simultaneously a $\mathcal{T}$-distributor and a $\mathcal{V}$-distributor induced by the same $\mathcal{T}$-functor $f : (X, a) \to (Y, b)$.

Example II.3.6. We describe now Lawvere's elegant characterisation of Cauchy complete metric spaces mentioned already in Chapter I. To this end, let X be a metric space. Each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X induces P_{+}-distributors

$$\varphi : G \mathrel{-\!\!\circ\!\!\rightarrow} X,\ \varphi(x) = \lim_{n\to\infty} d(x_n, x) \qquad \text{and} \qquad \psi : X \mathrel{-\!\!\circ\!\!\rightarrow} G,\ \psi(x) = \lim_{n\to\infty} d(x, x_n),$$

moreover, $\varphi \dashv \psi$ in P_{+}-Dist. Conversely, every adjunction $\varphi \dashv \psi$ of P_{+}-distributors $\varphi : G \mathrel{-\!\!\circ\!\!\rightarrow} X$ and $\psi : X \mathrel{-\!\!\circ\!\!\rightarrow} G$ is of this form, and Cauchy sequences induce the same adjunction if and only if they are equivalent. Furthermore, $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if the corresponding adjunction is of the form $x_* \dashv x^*$. Hence, X is Cauchy complete if and only if for every left adjoint P_{+}-distributors $\varphi : G \mathrel{-\!\!\circ\!\!\rightarrow} X$ there is some $x \in X$ with $\varphi = x_*$.

II.4. Fundamental properties

The aim of this section is to collect some important properties of $\mathcal{T}$-categories and $\mathcal{T}$-functors.

The forgetful functor $O_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Set}$ sending (X, a) to its underlying set X *is topological*, hence it has a left and a right adjoint. Here the free $\mathcal{T}$-category on a set X is given by (X, e_X°). In particular, the $\mathcal{T}$-category $G = (1, e_1^{\circ})$ is a generator in $\mathcal{T}$-Cat and $O_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Set}$ is naturally isomorphic to the covariant hom-functor $\mathcal{T}\text{-}\mathsf{Cat}(G, -)$. We conclude that $\mathcal{T}$-Cat is complete and cocomplete and the forgetful functor $O_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Set}$ preserves limits and colimits. For example, the product of $\mathcal{T}$-categories (X, a) and (Y, b) can be constructed by first taking the Cartesian product $X \times Y$ of the sets X and Y, and then equipping $X \times Y$ with the structure c defined by

$$c(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(fw), y),$$

for all $\mathfrak{w} \in T(X \times Y)$, $x \in X$ and $y \in Y$.

More important to us is, however, a different structure c on $X \times Y$ derived from the tensor product of $\mathcal{V}$, namely

$$c(\mathfrak{w},(x,y)) = a(T\pi_1(\mathfrak{w}),x) \otimes b(T\pi_2(\mathfrak{w}),y),$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$. We write

$$(X,a) \otimes (Y,b) = (X \times Y, c),$$

and this construction is in an obvious way part of a functor $\otimes : \mathcal{T}\text{-}\mathsf{Cat} \times \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Cat}$. The $\mathcal{T}$-category $E = (1,k)$ is a $\otimes$-neutral object, where 1 is a singleton set and $k : T1 \times 1 \to \mathcal{V}$ the constant relation with value $k \in \mathcal{V}$. In general, this construction does not result in a closed structure on $\mathcal{T}\text{-}\mathsf{Cat}$; however, it does so when defined in the larger category $\mathcal{T}\text{-}\mathsf{Gph}$ of $\mathcal{T}$-graphs and $\mathcal{T}$-graph morphisms (see [Clementino *et al.*, 2003]). Here a ***$\mathcal{T}$-graph*** is a pair (X,a) consisting of a set X and a $\mathcal{T}$-relation $a : X \relbar\joinrel\rightharpoonup X$ which is only required to satisfy the reflexivity axiom $e_X \le a$; ***$\mathcal{T}$-graph morphisms*** are defined as $\mathcal{T}$-functors. There is an obvious full embedding

$$\mathcal{T}\text{-}\mathsf{Cat} \hookrightarrow \mathcal{T}\text{-}\mathsf{Gph}$$

which preserves all limits (in fact, has a left adjoint) and all coproducts.

EXAMPLES II.4.1. By definition, a $\mathcal{U}_2$-graph is a pseudo-topological space as introduced in [Choquet, 1948] and a $\mathcal{U}_2$-graph morphism is a continuous map, that is, $\mathcal{U}_2\text{-}\mathsf{Gph}$ is concretely isomorphic to the category PsTop of pseudo-topological spaces and continuous maps. Similarly, $\mathcal{U}_{\mathsf{P}_+}\text{-}\mathsf{Gph} = \mathcal{U}_{\mathsf{P}_\wedge}\text{-}\mathsf{Gph}$ is concretely isomorphic to the category PsApp the category of pseudo-approach spaces and non-expansive maps (see [Lowen and Lowen, 1988]).

For $\mathcal{T}$-graphs $X = (X,a)$ and $Y = (Y,b)$, their tensor product $X \otimes Y$ is defined as above, but now $X \otimes - : \mathcal{T}\text{-}\mathsf{Gph} \to \mathcal{T}\text{-}\mathsf{Gph}$ has a right adjoint $(-)^X : \mathcal{T}\text{-}\mathsf{Gph} \to \mathcal{T}\text{-}\mathsf{Gph}$ (see [Hofmann, 2007]) where the structure d on

$$Y^X = \{f : X \to Y \mid f \text{ is a } \mathcal{T}\text{-functor of type } G \otimes X \to Y\}$$

is given by

$$d(\mathfrak{p},h) = \bigwedge_{\substack{\mathfrak{q} \in T(Y^X \times X), x \in X \\ \mathfrak{q} \mapsto \mathfrak{p}}} \hom(a(T\pi_2(\mathfrak{q}),x), b(T\mathrm{ev}(\mathfrak{q}),h(x))).$$

Here ev denotes the evaluation map $\mathrm{ev} : Y^X \times X \to Y, (h,x) \mapsto h(x)$. The following result can be found in [Hofmann, 2007].

PROPOSITION II.4.2. *Let $X = (X,a)$ be a $\mathcal{T}$-category with $a \cdot T_\xi a = a \cdot m_X$. Then, for each $\mathcal{T}$-category Y, the structure d on Y^X is transitive. Hence, $X \otimes - : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Cat}$ has a right adjoint $(-)^X : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Cat}$. Moreover, for all $h, h' \in Y^X$,*

$$d_0(h',h) = \bigwedge_{x \in X} b_0(h'(x),h(x))).$$

DEFINITION II.4.3. A $\mathcal{T}$-category $X = (X,a)$ is called ***core-compact*** whenever $a \cdot T_\xi a = a \cdot m_X$.

The designation "core-compact" is motivated by the case of topological spaces. Classically, a topological space X with topology $\mathcal{O}$ is called core-compact whenever, for all $A \in \mathcal{O}$ and all $x \in A$, there exists some $B \in \mathcal{O}$ with $x \in B$ and B is ***relatively compact*** in A; the latter meaning that very open cover of A includes a finite sub-cover of B, or, equivalently, every ultrafilter on B has a convergence point in A. It is shown in [Möbus, 1983; Pisani, 1999] that X is core-compact if and only if its convergence structure $a : UX \relbar\joinrel\rightharpoonup X$ satisfies $a \cdot U_\xi a = a \cdot m_X$. Every locally compact space is core-compact, and for weakly sober spaces also the converse is true. A very nice exposition of the theory of core-compact spaces and function spaces in Topology can be found in [Isbell, 1986]

The category $\mathcal{T}$-Cat *becomes a 2-category* by transporting the order-structure on hom-sets from $\mathcal{T}$-Dist to $\mathcal{T}$-Cat via the functor $(-)^{\circledast} : \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \to \mathcal{T}\text{-}\mathsf{Dist}$: for $\mathcal{T}$-functors $f, g : (X, a) \to (Y, b)$ we define (see [Hofmann and Tholen, 2010, Lemma 4.7])

$$\begin{aligned} f \le g \text{ in } \mathcal{T}\text{-}\mathsf{Cat} \;:\iff\; & f^{\circledast} \le g^{\circledast} \text{ in } \mathcal{T}\text{-}\mathsf{Dist} && \iff g_{\circledast} \le f_{\circledast} \text{ in } \mathcal{T}\text{-}\mathsf{Dist} \\ \iff\; & \forall x \in X \,.\, k \le b_0(f(x), g(x)) && \\ \iff\; & f^* \le g^* \text{ in } \mathcal{V}\text{-}\mathsf{Dist} && \iff g_* \le f_* \text{ in } \mathcal{V}\text{-}\mathsf{Dist}. \end{aligned}$$

We call $f, g : X \to Y$ ***equivalent***, and write $f \simeq g$, if $f \le g$ and $g \le f$. Hence, $f \simeq g$ if and only if $f^{\circledast} = g^{\circledast}$ if and only if $f_{\circledast} = g_{\circledast}$, and also if and only if $f^* = g^*$ if and only if $f_* = g_*$. A $\mathcal{T}$-functor $f : (X, a) \to (Y, b)$ is called ***fully faithful*** whenever $f^{\circledast} \circ f_{\circledast} = a$. Note that f is fully faithful if and only if, for all $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) = b(Tf(\mathfrak{x}), f(x))$.

Since $\mathcal{T}$-Cat is locally ordered, the forgetful functor $O_{\mathcal{T}} \simeq \mathcal{T}\text{-}\mathsf{Cat}(G, -)$ lifts to a 2-functor $\widetilde{O}_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Ord}$. Furthermore, there is a concrete 2-functor $(-)_0 : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ which sends a $\mathcal{T}$-category $X = (X, a)$ to the $\mathcal{V}$-category $X_0 = (X, a_0)$ where $a_0 = a \cdot e_X$; and $(-)_0 : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ has a concrete left adjoint which sends a $\mathcal{V}$-category (X, c) to $(X, e_X^{\circ} \cdot T_{\xi} c)$. Combining $(-)_0 : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ with $\widetilde{O}_{\mathcal{V}} : \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$ yields $\widetilde{O}_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Ord}$.

Remark II.4.4. The underlying order of a topological space X is characterised by

$$x \le y \iff \dot{x} \to y,$$

which is equivalent to saying that every neighbourhood of y contains x. Hence we obtain the *dual* of the specialisation order which is often employed in Domain Theory.

A $\mathcal{T}$-category X is called ***separated*** (see [Hofmann and Tholen, 2010]) whenever $f \simeq g$ implies $f = g$, for all $\mathcal{T}$-functors $f, g : Y \to X$ with codomain X. One easily verifies that it is enough to consider the case $Y = G$, so that X is separated if and only if the ordered set $\mathcal{T}\text{-}\mathsf{Cat}(G, X)$ is anti-symmetric. The full subcategory of $\mathcal{T}$-Cat consisting of all separated $\mathcal{T}$-categories is denoted by

$$\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}.$$

Separateness captures precisely the notion of anti-symmetry in ordered sets and the T0-axiom in topological spaces; and a metric space $X = (X, d)$ is separated if and only if, for all $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies $x = y$.

The 2-categorical structure on $\mathcal{T}$-Cat *allows us to consider adjoint* $\mathcal{T}$*-functors*: a $\mathcal{T}$-functor $f : X \to Y$ is ***left adjoint*** if there exists a $\mathcal{T}$-functor $g : Y \to X$ such that $1_X \le g \cdot f$ and $1_Y \ge f \cdot g$. Considering the corresponding $\mathcal{T}$-distributors, f is left adjoint to g in $\mathcal{T}$-Cat if and only if $g_{\circledast} \dashv f_{\circledast}$ in $\mathcal{T}$-Dist, that is, if and only if $f_{\circledast} = g^{\circledast}$. More generally, thanks to Lemma II.3.5, if $f : X \to Y$ and $g : Y \to X$ are maps with $f_{\circledast} = g^{\circledast}$, then f and g are $\mathcal{T}$-functors and $f \dashv g$ in $\mathcal{T}$-Cat.

III

Representable categories

This chapter is devoted to the important notions of representable $\mathcal{T}$-category and dualisable $\mathcal{T}$-graph. Our interest in representable $\mathcal{T}$-categories derives from the fact that these are precisely those $\mathcal{T}$-categories for which the associated dual $\mathcal{T}$-graph is a $\mathcal{T}$-category. For topological T0-spaces, the concept of representability specialises to the classical notion of stably compact space which is closely related to Nachbin's ordered compact Hausdorff spaces introduced in [Nachbin, 1950]. The designation "representable" is borrowed from [Hermida, 2000, 2001] where the notion of representable multicategory via a "monadic 2-adjuntion between the 2-category of strict monoidal categories and that of multicategories" is introduced and analised. In comparison to the topological case, strict monoidal categories are to multicategories what ordered compact Hausdorff spaces are to topological spaces.

III.1. Extending the monad

Based on the lax extension of the Set-monad $\mathbb{T} = (T, m, e)$ to $\mathcal{V}$-Rel described in Section II.2, the Set-monad $\mathbb{T}$ admits a natural extension to a monad on $\mathcal{V}$-Cat, in the sequel also denoted as $\mathbb{T} = (T, m, e)$ (see [Tholen, 2009]). Here the functor $T : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ sends a $\mathcal{V}$-category (X, a_0) to $(TX, T_\xi a_0)$, and with this definition $e_X : X \to TX$ and $m_X : TTX \to TX$ become $\mathcal{V}$-functors for each $\mathcal{V}$-category X. We also note that $T : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ is actually a 2-functor: if $f^* \leq g^*$, then $(Tf)^* = T_\xi(f^*) \leq T_\xi(g^*) = (Tg)^*$. The Eilenberg–Moore algebras for this monad can be described as triples (X, a_0, α) where (X, a_0) is a $\mathcal{V}$-category and (X, α) is an algebra for the Set-monad $\mathbb{T}$ such that $\alpha : T(X, a_0) \to (X, a_0)$ is a $\mathcal{V}$-functor. For $\mathbb{T}$-algebras (X, a_0, α) and (Y, b_0, β), a map $f : X \to Y$ is a homomorphism $f : (X, a_0, \alpha) \to (Y, b_0, \beta)$ precisely if f preserves both structures, that is, whenever $f : (X, a_0) \to (Y, b_0)$ is a $\mathcal{V}$-functor and $f : (X, \alpha) \to (Y, \beta)$ is a $\mathbb{T}$-homomorphism. Since the extension T_ξ of T commutes with the involution $(-)^\circ$, with (X, a_0, α) also (X, a_0°, α) is a $\mathbb{T}$-algebra. It follows from Lemma II.1.2 that the internal hom in $\mathcal{V}$ combined with the $\mathbb{T}$-algebra structure ξ induces the Eilenberg–Moore algebra $\mathcal{V} = (\mathcal{V}, \hom, \xi)$ for the monad $\mathbb{T}$ on $\mathcal{V}$-Cat.

There is a canonical functor

$$K : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}} \to \mathcal{T}\text{-}\mathsf{Cat}.$$

which associates to each $X = (X, a_0, \alpha)$ in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ the $\mathcal{T}$-category $KX = (X, a)$ where $a = a_0 \cdot \alpha$. Note that $(a_0 \cdot \alpha)_0 = a_0$, hence our notation remains consistent. The category $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ is actually a 2-category with the order relation on hom-sets inherited from $\mathcal{V}$-Cat, and one easily verifies that K is a 2-functor. Applying K to $\mathcal{V} = (\mathcal{V}, \hom, \xi)$ produces the $\mathcal{T}$-category $\mathcal{V} = (\mathcal{V}, \hom_\xi)$ where

$$\hom_\xi : T\mathcal{V} \times \mathcal{V} \to \mathcal{V},\ (\mathfrak{v}, v) \mapsto \hom(\xi(\mathfrak{v}), v).$$

We note that $\mathcal{V} = (\mathcal{V}, \hom_\xi)$ is separated since $\mathcal{T}\text{-}\mathsf{Cat}(G, \mathcal{V}) \simeq \mathcal{V}$ in Ord. In the topological case, $\mathsf{2}$ is the ***Sierpiński space*** where $\{1\}$ is closed and $\{0\}$ is open; and in the approach case we obtain the ***Sierpiński approach space*** P_+ with approach convergence structure defined by $\lambda(\mathfrak{x}, x) = x \ominus \xi(\mathfrak{x})$ (see [Lowen, 1997, Example 1.8.33(2)]).

THEOREM III.1.1. *The functor* $K : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}} \to \mathcal{T}\text{-}\mathsf{Cat}$ *has a left adjoint* $M : \mathcal{T}\text{-}\mathsf{Cat} \to (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{T}}$ *sending a* $\mathcal{T}$*-category* (X, a) *to* $(TX, T_\xi a \cdot m_X^\circ, m_X)$ *and a* $\mathcal{T}$*-functor* f *to* Tf. *Moreover,* M *is a 2-functor.*

Via the adjunction $M \dashv K$ one obtains a lifting of the Set-monad $\mathbb{T} = (T, m, e)$ to a monad on $\mathcal{T}\text{-}\mathsf{Cat}$, also denoted as $\mathbb{T} = (T, m, e)$. As it turns out, this monad has a very pleasant property.

PROPOSITION III.1.2. *The monad* $\mathbb{T} = (T, m, e)$ *on* $\mathcal{T}\text{-}\mathsf{Cat}$ *is of Kock-Zöberlein type.*

III.2. Representable categories

Since the monad $\mathbb{T} = (T, m, e)$ on $\mathcal{T}\text{-}\mathsf{Cat}$ is of Kock-Zöberlein type, an algebra structure $\alpha : TX \to X$ on a $\mathcal{T}$-category X is left adjoint to the unit $e_X : X \to TX$. However, unless X is separated, a left adjoint $\alpha : TX \to X$ to e_X is in general only a pseudo-algebra structure on X, that is,

$$\alpha \cdot e_X \simeq 1_X \qquad \text{and} \qquad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

DEFINITION III.2.1. We call a $\mathcal{T}$-category X ***representable*** whenever $e_X : X \to TX$ has a left adjoint in $\mathcal{T}\text{-}\mathsf{Cat}$. A $\mathcal{T}$-functor $f : X \to Y$ between representable $\mathcal{T}$-categories X and Y, with left adjoint $\alpha : TX \to X$ and $\beta : TY \to Y$ respectively, is called a ***pseudo-homomorphism*** whenever

$$\beta \cdot Tf \simeq f \cdot \alpha.$$

Of course, if Y is separated, then one has equality above. Furthermore, since $\mathbb{T}$ is of Kock-Zöberlein type, every left adjoint $\mathcal{T}$-functor between representable $\mathcal{T}$-categories is a pseudo-homomorphism. We also note that X is representable if and only if there exists a $\mathcal{T}$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, then necessarily $\alpha \dashv e_X$ and $\alpha \cdot T\alpha \simeq \alpha \cdot m_X$. We denote the category of representable $\mathcal{T}$-categories and pseudo-homomorphism by

$$\mathcal{T}\text{-}\mathsf{ReprCat},$$

and its full subcategory defined by the separated representable $\mathcal{T}$-categories by $\mathcal{T}\text{-}\mathsf{ReprCat}_{\mathrm{sep}}$. For a separated representable $\mathcal{T}$-category X, the left adjoint $\alpha : TX \to X$ is actually an algebra structure for $\mathbb{T}$ and therefore $(X, a) \mapsto (X, \alpha)$ defines a functor

$$\mathcal{T}\text{-}\mathsf{ReprCat}_{\mathrm{sep}} \to \mathsf{Set}^{\mathbb{T}}$$

which commutes with the underlying Set-functors.

Below we give a characterisation of representable $\mathcal{T}$-categories.

PROPOSITION III.2.2. *The following assertions are equivalent, for a* $\mathcal{T}$*-category* (X, a).

(i). (X, a) *is representable.*

(ii). (X, a) *is core-compact and there is a map* $\alpha : TX \to X$ *such that* $a = a_0 \cdot \alpha$.

EXAMPLES III.2.3. For $\mathcal{T} = \mathcal{U}_2$, a topological space X is representable if and only if X is core-compact and every ultrafilter on X has a smallest convergence point in X with respect to the underlying order. It is well-known that the T0-spaces satisfying these properties are precisely the

stably compact spaces (= locally compact sober spaces where finite intersections of compact down-sets are compact); and $f : X \to Y$ is a pseudo-homomorphism if and only if $f^{-1}(K)$ is compact, for every compact down-set $K \subseteq Y$. For more information we refer to [Gierz *et al.*, 1980; Simmons, 1982; Jung, 2004; Lawson, 2011]. Similarly, as observed in [Gutierres and Hofmann, 2013], an approach space (X, a) is representable if and only if (X, a) is weakly sober, core-compact and has the property that $a(\mathfrak{x}, -)$ is an approach prime element, for every $\mathfrak{x} \in UX$. We refer to [Banaschewski *et al.*, 2006] and [Van Olmen, 2005] for the theory of sober approach space.

III.3. Dualisable graphs

Our next aim is to introduce a concept of dual $\mathcal{T}$-category which generalises the notion of dual $\mathcal{V}$-category. It turns out to be convenient to consider more generally certain $\mathcal{T}$-graphs.

Definition III.3.1. A $\mathcal{T}$-graph (X, a) is called ***dualisable*** whenever the reflexive $\mathcal{V}$-relation $a_0 = a \cdot e_X$ is also transitive and $a = a_0 \cdot \alpha$, for some map $\alpha : TX \to X$.

For a dualisable $\mathcal{T}$-graph $X = (X, a)$, we write X_0 to denote its underlying $\mathcal{V}$-category (X, a_0). We consider TX as a discrete $\mathcal{V}$-category so that $\alpha : TX \to X_0$ is a $\mathcal{V}$-functor. With this notation, $a_0 \cdot \alpha = \alpha_*$ and, if also $a = a_0 \cdot \beta = \beta_*$ for some map $\beta : TX \to X$, then necessarily $\alpha^* = \beta^*$ and therefore

$$a_0^\circ \cdot \alpha = (\alpha^*)^\circ = (\beta^*)^\circ = a_0^\circ \cdot \beta.$$

Lemma III.3.2. *Let $X = (X, a)$ be a dualisable $\mathcal{T}$-graph. Then $(X, a_0^\circ \cdot \alpha)$ is a dualisable $\mathcal{T}$-graph as well, and the underlying $\mathcal{V}$-category of $(X, a_0^\circ \cdot \alpha)$ is $(X_0)^{\mathrm{op}}$.*

Definition III.3.3. Let $X = (X, a)$ be a dualisable $\mathcal{T}$-graph. Then the ***dual*** $\mathcal{T}$-graph of X is $X^{\mathrm{op}} = (X, a_0^\circ \cdot \alpha)$.

By the discussion before Lemma III.3.2, this definition is independent of the choice of α. Of course, the dual of a $\mathcal{V}$-category in the sense above is just the usual dual. Also note that, even if X is a $\mathcal{T}$-category, X^{op} need not be a $\mathcal{T}$-category.

Example III.3.4. For a topological space X, the pseudo-topological space 2^X is dualisable. We can identify 2^X with the set of all closed subsets of X, then the underlying order of 2^X is given by subset inclusion. We write $\mathcal{O}$ for the collection of all open subsets of X, and $\mathcal{O}(x)$ for the set of all open neighbourhoods of $x \in X$. For any subset $V \subseteq X$, we put

$$V^\Diamond = \{A \in 2^X \mid A \cap V \neq \varnothing\}.$$

With this notation, for an ultrafilter $\mathfrak{p}$ on 2^X, the smallest convergence point $\mu(\mathfrak{p})$ of $\mathfrak{p}$ can be calculated as

$$\mu(\mathfrak{p}) = \{x \in X \mid \forall V \in \mathcal{O}(x) \,.\, V^\Diamond \in \mathfrak{p}\}.$$

By definition, the convergence in $(2^X)^{\mathrm{op}}$ is described by

$$\mathfrak{p} \to A \iff A \subseteq \mu(\mathfrak{p}) \iff \forall V \in \mathcal{O} \,.\, (A \in V^\Diamond \Rightarrow V^\Diamond \in \mathfrak{p});$$

hence it is induced by $\{V^\Diamond \mid V \in \mathcal{O}\}$ and therefore $(2^X)^{\mathrm{op}}$ is actually a topological space. This topology on the set of closed subsets of a topological space is known as the ***lower Vietoris topology*** (see [Clementino and Tholen, 1997], for instance). We find it remarkable that, albeit 2^X belongs to Top if and only if X is core-compact (see [Schwarz, 1984]), its dual $(2^X)^{\mathrm{op}}$ is always a topological space.

Similarly, for an approach space X, the pseudo-approach space $\mathsf{P}_{\!+}^X$ is dualisable where $(-)^X$ is the right adjoint to $-\otimes X$ and the approach structure λ on $\mathsf{P}_{\!+}$ is given by $\lambda(\mathfrak{v},v) = v \ominus \xi(\mathfrak{v})$, for $\mathfrak{v} \in U\mathsf{P}_{\!+}$ and $v \in \mathsf{P}_{\!+}$. As for topological spaces, $\mathsf{P}_{\!+}^X$ is an approach space if and only if X is core-compact, while $(\mathsf{P}_{\!+}^X)^{\mathrm{op}}$ is for every X an approach space. Note that we cannot use here the same argument as for topological spaces since it relies on the description of a topology via open subsets; the crucial observation is that the convergence structure $c : U(\mathsf{P}_{\!+}^X) \longrightarrow\!\!\!\!\!\!\mapsto \mathsf{P}_{\!+}^X$ can be described as a *lifting* in $\mathsf{P}_{\!+}$-Rel. In fact, it is shown in [Hofmann, 2014] that a corresponding result holds for every topological theory $\mathcal{T} = (\mathbb{T}, \mathcal{V}, \xi)$ where $T1 = 1$.

PROPOSITION III.3.5. *Let $X = (X, a)$ be a $\mathcal{T}$-category where $a = a_0 \cdot \alpha$, for some map $\alpha : TX \to X$. Then the following assertions are equivalent.*

(i). *The $\mathcal{T}$-graph X^{op} is actually a $\mathcal{T}$-category.*
(ii). *X is core-compact.*
(iii). *X is representable.*

In conclusion, taking duals gives a functor $(-)^{\mathrm{op}} : \mathcal{T}\text{-}\mathsf{ReprCat} \to \mathcal{T}\text{-}\mathsf{ReprCat}$ which makes the diagram

$$\begin{tikzcd}
\mathcal{T}\text{-}\mathsf{ReprCat} \arrow[r, "(-)^{\mathrm{op}}"] \arrow[d, "(-)_0"'] & \mathcal{T}\text{-}\mathsf{ReprCat} \arrow[d, "(-)_0"] \\
\mathcal{V}\text{-}\mathsf{Cat} \arrow[r, "(-)^{\mathrm{op}}"'] & \mathcal{V}\text{-}\mathsf{Cat}
\end{tikzcd}$$

commutative.

III.4. Distributors revisited

By Propositions II.4.2 and III.2.2, every representable $\mathcal{T}$-category is $\otimes$-exponentiable. For a $\mathcal{T}$-category X, its ***presheaf $\mathcal{T}$-category*** $P_{\mathcal{T}}X$ is defined as $P_{\mathcal{T}}X := \mathcal{V}^{(TX)^{\mathrm{op}}}$ with structure relation denoted $[\![-,-]\!]$. Proposition II.4.2 implies that the underlying $\mathcal{V}$-category $(P_{\mathcal{T}}X)_0$ is a full subcategory of the presheaf $\mathcal{V}$-category of $(MX)_0$, where, for $\psi, \psi' \in P_{\mathcal{T}}X$,

$$[\psi,\psi'] := [\![e_{P_{\mathcal{T}}X}(\psi),\psi']\!] = \bigwedge_{\mathfrak{x}\in TX} \hom(\psi(\mathfrak{x}),\psi'(\mathfrak{x})).$$

A slight adaptation of [Clementino and Hofmann, 2009a, Theorem 2.5] gives the following characterisation of $\mathcal{T}$-distributors.

THEOREM III.4.1. *Let $\varphi : X \longrightarrow\!\!\!\!\!\!\mapsto Y$ be a $\mathcal{T}$-relation. The following assertions are equivalent.*

(i). *φ is a $\mathcal{T}$-distributor $\varphi : X \multimapdotbothvert Y$.*
(ii). *$\varphi : (TX)^{\mathrm{op}} \otimes Y \to \mathcal{V}$ is a $\mathcal{T}$-functor.*
(iii). *$\ulcorner\varphi\urcorner : Y \to P_{\mathcal{T}}X$ is a $\mathcal{T}$-functor.*

The following result (see [Hofmann, 2011]) turns out to be crucial for calculating with $\mathcal{T}$-distributors.

THEOREM III.4.2. *Let $\psi : X \multimapdotbothvert Z$ and $\varphi : X \multimapdotbothvert Y$ be $\mathcal{T}$-distributors. Then, for all $\mathfrak{z} \in TZ$ and $y \in Y$,*

$$[\![T\ulcorner\psi\urcorner(\mathfrak{z}), \ulcorner\varphi\urcorner(y)]\!] = (\varphi \multimapinv \psi)(\mathfrak{z}, y).$$

For each $\mathcal{T}$-category $X = (X, a)$, $a : X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ is a $\mathcal{T}$-distributor which gives us the ***Yoneda functor***

$$y_X = \ulcorner a \urcorner : X \to P_{\mathcal{T}}X.$$

As a special case of Theorem III.4.2 we obtain:

Corollary III.4.3. *For each* $\psi \in P_{\mathcal{T}}X$ *and each* $\mathfrak{x} \in TX$, $\psi(\mathfrak{x}) = [\![T\, y_X(\mathfrak{x}), \psi]\!]$. *Consequently,* $y_X : X \to P_{\mathcal{T}}X$ *is fully faithful.*

The Yoneda functor $y_X = \ulcorner a \urcorner : X \to P_{\mathcal{T}}X$ is the X-component of a natural transformation $y : 1 \to P_{\mathcal{T}}$. Furthermore, for every $\mathcal{T}$-category $X = (X, a)$, the $\mathcal{V}$-category structure $\hat{a} = T_\xi a \cdot m_X^\circ : TX \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} TX$ on TX is actually a $\mathcal{T}$-distributor $\hat{a} : X \mathrel{-\!\!\circ\!\!\rightharpoonup} TX$. Hence, $\hat{a}$ corresponds to a $\mathcal{T}$-functor $\mathcal{Y}_X : TX \to P_{\mathcal{T}}X$ and this construction yields a natural transformation $\mathcal{Y} : T \to P_{\mathcal{T}}$.

By definition, the underlying ordered set of $P_{\mathcal{T}}X$ can be identified with $\mathcal{T}\text{-}\mathsf{Dist}(X, G)$. Every $\mathcal{T}$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ induces monotone maps $- \circ \varphi : P_{\mathcal{T}}Y \to P_{\mathcal{T}}X$ and $- \mathrel{\circ\!\!-} \varphi : P_{\mathcal{T}}X \to P_{\mathcal{T}}Y$ which are both $\mathcal{T}$-functors as $- \circ \varphi$ is the mate of the $\mathcal{T}$-distributor $(y_Y)_\circledast \circ \varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} P_{\mathcal{T}}Y$, and $- \mathrel{\circ\!\!-} \varphi$ is the mate of $(\ulcorner \varphi \urcorner)_\circledast : Y \mathrel{-\!\!\circ\!\!\rightharpoonup} P_{\mathcal{T}}X$. In the sequel we write $P_{\mathcal{T}}\varphi$ to denote $- \circ \varphi$, note that we have an adjunction $P_{\mathcal{T}}\varphi \dashv - \mathrel{\circ\!\!-} \varphi$ in $\mathcal{T}\text{-}\mathsf{Cat}$. Clearly, this construction defines a 2-functor $P_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Dist}^{\mathrm{op}} \to \mathcal{T}\text{-}\mathsf{Cat}$.

Theorem III.4.4. *The functor* $P_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Dist}^{\mathrm{op}} \to \mathcal{T}\text{-}\mathsf{Cat}$ *is right adjoint to* $(-)^\circledast : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Dist}^{\mathrm{op}}$. *The unit and counit of this adjunction are given by* $(y_X)_X$ *and* $(y_{X^\circledast})_X$ *respectively.*

The adjunction of the theorem above induces a monad $\mathbb{P}_{\mathcal{T}} = (P_{\mathcal{T}}, m, y)$ on $\mathcal{T}\text{-}\mathsf{Cat}$ where $P_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Cat}$ sends a $\mathcal{T}$-functor f to $P_{\mathcal{T}}f := - \circ f^\circledast$, hence $P_{\mathcal{T}}$ is a 2-functor. Moreover, for every $f : X \to Y$ in $\mathcal{T}\text{-}\mathsf{Cat}$ one has $\mathcal{T}$-functors

$$\begin{tikzcd}
P_{\mathcal{T}}X \arrow[r, bend left=40, "P_{\mathcal{T}}f"] \arrow[r, bend right=40, "(- \mathrel{\circ\!\!-} f_\circledast)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & P_{\mathcal{T}}Y, \arrow[l, "(-\circ f_\circledast)" description]
\end{tikzcd}$$

hence $P_{\mathcal{T}}f \leq (- \mathrel{\circ\!\!-} f_\circledast)$. Also note that $\psi \mathrel{\circ\!\!-} (y_X)_\circledast = [\![-, \psi]\!] = \psi^\circledast$, hence $- \mathrel{\circ\!\!-} (y_X)_\circledast = y_{P_{\mathcal{T}}X}$ and therefore $P_{\mathcal{T}}\, y_X \leq y_{P_{\mathcal{T}}X}$. Consequentely:

Theorem III.4.5. *The monad* $\mathbb{P}_{\mathcal{T}} = (P_{\mathcal{T}}, m, y)$ *on* $\mathcal{T}\text{-}\mathsf{Cat}$ *is of Kock-Zöberlein type. The natural transformation* $\mathcal{Y} : T \to P_{\mathcal{T}}$ *is actually a monad morphism* $\mathcal{Y} : \mathbb{T} \to \mathbb{P}_{\mathcal{T}}$.

Example III.4.6. For $\mathcal{T} = \mathcal{U}_2$, it is shown in [Hofmann and Tholen, 2010] that the topological space $P_{\mathcal{U}_2}X$ is homeomorphic to the space $F_o(X)$ of all filters (including the improper one) on the lattice $\mathcal{O}X$ of open sets of X, where the topology on $F_o(X)$ has

$$\{\mathfrak{f} \in F_o(X) \mid A \in \mathfrak{f}\} \qquad\qquad (A \subseteq X \text{ open})$$

as basic open sets (see [Escardó, 1997]). Here we can identify an element $\psi \in P_{\mathcal{U}_2}X = 2^{(UX)^{\mathrm{op}}}$ with a closed subset $\mathcal{A}$ of UX and, with this identification, the maps

$$P_{\mathcal{U}_2}X \xrightarrow{\Phi_X} F_o(X),\ \mathcal{A} \mapsto (\textstyle\bigcap \mathcal{A}) \cap \mathcal{O}X \qquad \text{and} \qquad F_o(X) \xrightarrow{\Pi_X} P_{\mathcal{U}_2}X,\ \mathfrak{f} \mapsto \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$$

are indeed continuous and inverse to each other. Furthermore, $\Phi_X \cdot y_X(x)$ is the open neighbourhood filter, for every topological space X and every $x \in X$; and $\Phi = (\Phi_X)_X$ is a natural transformation. Therefore, since both monads are of Kock-Zöberlein type, the monad $\mathbb{P}_{\mathcal{U}_2}$ on $\mathsf{Top} \simeq \mathcal{U}_2\text{-}\mathsf{Cat}$ is isomorphic to the filter-of-opens monad $\mathbb{F}_o$ on Top.

A $\mathcal{U}_2$-distributor $\psi : X \multimap G$ is right adjoint if and only if the corresponding filter of opens is completely prime. A $\mathcal{U}_2$-distributor $\varphi : G \multimap X$ corresponds to a continuous map $\varphi : X \to 2$ which we can identify with a closed subset A of X; here $x_\circledast$ corresponds to $\overline{\{x\}}$. Furthermore, $\varphi : G \multimap X$ is left adjoint if and only if A is irreducible. Hence, a topological space X is weakly sober if and only if every left adjoint $\mathcal{U}_2$-distributor $\varphi : G \multimap X$ is of the from $\varphi = x_\circledast$, for some $x \in X$. This description of weakly sober spaces is obtained in [Clementino and Hofmann, 2009a] and, with Example II.3.6 in mind, suggests to think of these spaces as the "Cauchy complete topological spaces".

Since $(-)^\circledast : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{T}\text{-}\mathsf{Dist}^{\mathrm{op}}$ is surjective on objects, $\mathcal{T}\text{-}\mathsf{Dist}^{\mathrm{op}}$ is equivalent to the Kleisli category of $\mathbb{P}_\mathcal{T}$. In the next chapter we will analyse the Eilenberg–Moore category of $\mathbb{P}_\mathcal{T} = (P_\mathcal{T}, m, y)$.

III.5. Compactness

In Section III.1 we have seen that the syntactic object $\mathcal{V}$ becomes naturally a $\mathcal{T}$-category. Furthermore, several maps induced by the components of a topological theory are indeed $\mathcal{T}$-functors:

PROPOSITION III.5.1. (1). *For each set I, the map $\bigwedge : \mathcal{V}^I \to \mathcal{V}$ is a $\mathcal{T}$-functor.*
(2). $\hom_\xi : (T\mathcal{V})^{\mathrm{op}} \otimes \mathcal{V} \to \mathcal{V}$ *is a $\mathcal{T}$-functor.*
(3). $\otimes : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ *is a $\mathcal{T}$-functor.*

A special role plays the map induced by suprema in $\mathcal{V}$: the study of its functoriality leads us to a study of compact $\mathcal{T}$-categories. Motivated by the notion of 0-compactness in approach theory (see [Lowen, 1997, Section 6.1]), the ***degree of compactness*** of a $\mathcal{T}$-category $X = (X, a)$ is defined as

$$\mathrm{comp}(X) = \bigwedge_{\mathfrak{x} \in TX} \bigvee_{x \in X} a(\mathfrak{x}, x),$$

and we call X ***compact*** if $\mathrm{comp}(X) \geq k$. Before stating the main result, we recall that $(-)^X$ denotes the right adjoint to $- \otimes X$; and note that this functor takes values in $\mathcal{T}\text{-}\mathsf{Gph}$.

THEOREM III.5.2. *A $\mathcal{T}$-category X is compact if and only if the map $\bigvee : \mathcal{V}^X \to \mathcal{V}$ is actually a $\mathcal{T}$-graph morphism.*

For topological spaces, the result above is essentially due to [Nachbin, 1992] and [Escardó, 2004]. It implies the rather surprising fact that not only finite but also compact union of closed sets is closed. It also provides an elegant argument for the Kuratowski-Mrówka characterisation of compactness (see [Escardó, 2004, Theorem 9.15], [Kuratowski, 1931] and [Mrówka, 1959]), as we explain next.

A map $f : X \to Y$ between topological spaces is closed if it sends closed subsets to closed subsets, or, in terms of characteristic maps, if $Pf : 2^X \to 2^Y$ sends continuous maps to continuous maps. With this in mind, we call a map $f : X \to Y$ between $\mathcal{T}$-categories ***closed*** if $P_\mathcal{V}(f) : P_\mathcal{V}(X) \to P_\mathcal{V}(Y)$ (see Definition II.1.1) sends $\mathcal{T}$-functors to $\mathcal{T}$-functors.

PROPOSITION III.5.3. *A $\mathcal{T}$-category X is compact if and only if, for each $\mathcal{T}$-category Y, the projection map $\pi_2 : X \otimes Y \to Y$ is closed.*

To prove the left-to-right implication, the crucial observation is that

$$P_\mathcal{V}(\pi_2)(\varphi) = \bigvee \cdot \ulcorner \varphi \urcorner,$$

for every $\mathcal{T}$-functor $\varphi : X \otimes Y \to \mathcal{V}$; and, if X is compact, the composite

$$Y \xrightarrow{\ulcorner \varphi \urcorner} \mathcal{V}^X \xrightarrow{\bigvee} \mathcal{V}$$

is a $\mathcal{T}$-functor.

Remark III.5.4. A Kuratowski-Mrówka type characterisation of compact approach spaces is given in [Colebunders *et al.*, 2005]; however, their result differers from the approach version of our result as it uses the categorical product $X \times Y$ instead of the tensor product $X \otimes Y$.

IV

Cocomplete categories

In this chapter we explain the notions of weighted colimit and cocompleteness for $\mathcal{T}$-categories and develop some aspects of their theory. This was first done in Hofmann [2011]; Clementino and Hofmann [2009b] following closely the theory of $\mathcal{V}$-categories as pioneered in [Eilenberg and Kelly, 1966]. The classical theory of enriched categories is extensively described in the monograph [Kelly, 1982]; unfortunately, without making use of the powerful notion of distributor. This is in our opinion a pity as the "distributor viewpoint" adds considerable clarity to the subject. In fact, the notion of cocompleteness can be described roughly as follows: move from $\mathcal{V}\text{-}\mathsf{Cat}$ to $\mathcal{V}\text{-}\mathsf{Dist}$ via $(-)_* : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Dist}$, solve the problem in $\mathcal{V}\text{-}\mathsf{Dist}$ using the quantaloid structure; a $\mathcal{V}$-category is cocomplete if the "solution distributor" comes from a $\mathcal{V}$-functor, i.e. is of the form f_*. This idea is formalised in the context of quantaloid-enriched categories in [Stubbe, 2005, 2006] which also influenced our definitions.

Furthermore, we stress that the $\mathcal{T}$-case presents a fundamental difference. Whereby in the classical case of enriched categories it is enough to consider distributors with codomain the generator G, for $\mathcal{T}$-categories it is necessary to consider weights with arbitrary codomain. This fact forces us to distinguish between cocomplete $\mathcal{T}$-categories and totally cocomplete $\mathcal{T}$-categories. Below we present various results about totally cocomplete $\mathcal{T}$-categories and expose in particular the tight connection with the theory of continuous lattices.

IV.1. Weighted colimits

We start by introducing the pertinent notions. A ***weighted colimit diagram*** in a $\mathcal{T}$-category X is given by a $\mathcal{T}$-functor $d : D \to X$ and a $\mathcal{T}$-distributor $\psi : D \,{-\!\!\circ\!\!\rightharpoonup}\, G$.

$$\begin{array}{ccc} D & \xrightarrow{d} & X \\ {\scriptstyle\psi}\big\downarrow\!\circ & & \\ G & & \end{array}$$

A ***colimit*** of such a weighted diagram is an element $x \in X$ which represents $d_\circledast \multimap \psi : G \,{-\!\!\circ\!\!\rightharpoonup}\, X$, that is, $x_\circledast = d_\circledast \multimap \psi$. If such an element x exists, it is unique up to equivalence, and we call x a ***ψ-weighted colimit of*** d and write $x \simeq \operatorname{colim}(d, \psi)$. We say that a $\mathcal{T}$-functor $f : X \to Y$ ***preserves*** the ψ-weighted colimit x of d if $f(x)$ is the ψ-weighted colimit of $f \cdot d$, that is, if $f(x)_\circledast = (f \cdot d)_\circledast \multimap \psi$. A $\mathcal{T}$-functor $f : X \to Y$ is called ***cocontinuous*** if it preserves all weighted colimits which exist in X, and a $\mathcal{T}$-category X is ***cocomplete*** if every weighted colimit diagram in X has a colimit in X.

Cocompleteness of X follows from the existence of colimits along identities: for any weight $\psi : D \mathrel{-\!\!\circ\!\!\rightharpoonup} G$, the ψ-weighted colimit of d exists if and only if the $(\psi \circ d^{\circledast})$-weighted colimit of $1_X : X \to X$ exists, and in that case one has $\operatorname{colim}(d, \psi) \simeq \operatorname{colim}(1_X, \psi \circ d^{\circledast})$. Moreover, a $\mathcal{T}$-functor $f : X \to Y$ preserves the ψ-weighted colimit of d if and only if it preserves the $(\psi \circ d^{\circledast})$-weighted colimit of 1_X. In the sequel we will write $\operatorname{Sup}_X(\psi)$ or simply $\operatorname{Sup}(\psi)$ instead of $\operatorname{colim}(1_X, \psi)$.

REMARK IV.1.1. If $\psi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} G$ is right adjoint, then $1_\circledast \mathrel{\circ\!\!-} \psi : G \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ is necessarily the left adjoint of ψ. Hence, by Example II.3.6, a metric space X is Cauchy complete if and only if every right adjoint distributor $\psi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} G$ has a supremum in X.

EXAMPLE IV.1.2. For $\mathcal{T}$-distributors $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ and $\psi : Y \mathrel{-\!\!\circ\!\!\rightharpoonup} Z$, with the help of Theorem III.4.2 one shows that the mate $\ulcorner \psi \circ \varphi \urcorner$ of the composite of ψ and φ is the colimit of the diagram

$$\begin{tikzcd} Y \arrow[r, "\ulcorner \varphi \urcorner"] \arrow[d, "\psi"', "\circ" marking] & P_{\mathcal{T}}X. \\ Z \end{tikzcd}$$

Hence, for every $\mathcal{T}$-category $X = (X, a)$, every $\mathcal{T}$-distributor $\psi \in PX$ is the colimit of the weighted diagram

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "\psi"', "\circ" marking] & P_{\mathcal{T}}X \\ G \end{tikzcd}$$

since $\ulcorner a \urcorner = y_X$.

EXAMPLE IV.1.3. For a topological space X, a $\mathcal{U}_2$-distributor $\psi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} G$ corresponds to a filter $\mathfrak{f}$ in the lattice of opens of X (see Example III.4.6), and $x \in X$ is a supremum of ψ if and only if x is a smallest convergence point of $\mathfrak{f}$ with respect to the underlying order of X. Hence, X is cocomplete if and only if every filter of opens of X has a smallest convergence point in X.

IV.2. Totally cocomplete categories

For a cocomplete $\mathcal{T}$-category X, the map defined by $\psi \mapsto \operatorname{Sup}_X(\psi)$ turns out to be left adjoint to the $y_X : X_0 \to (P_{\mathcal{T}}X)_0$ in $\mathcal{V}$-Cat; and, if $T1 = 1$, the existence of such a left adjoint guarantees also cocompleteness of X. However, Example 5.7 of [Hofmann and Waszkiewicz, 2011] shows that Sup_X does not need to be a $\mathcal{T}$-functor.

DEFINITION IV.2.1. A $\mathcal{T}$-category X is called ***totally cocomplete*** whenever $y_X : X \to P_{\mathcal{T}}X$ has a left adjoint $\operatorname{Sup}_X : P_{\mathcal{T}}X \to X$ in $\mathcal{T}$-Cat; that is, if X is a pseudo-algebra for $\mathbb{P}_{\mathcal{T}}$.

It follows from the general theory of Kock–Zöberlein monad that the totally cocomplete $\mathcal{T}$-categories are precisely the injective ones, where a $\mathcal{T}$-category X is called ***injective*** if, for all $\mathcal{T}$-functors $f : A \to X$ and fully faithful $\mathcal{T}$-functors $i : A \to B$, there exists a $\mathcal{T}$-functor $g : B \to X$ such that $g \cdot i \simeq f$. Clearly, for a separated $\mathcal{T}$-category X we have then $g \cdot i = f$.

Curiously, total cocompleteness can be characterised by the existence of a slighly more general type of colimits, as we explain next. From now on we let in a *weighted colimit diagram the weight* $\psi : D \mathrel{-\!\!\circ\!\!\rightharpoonup} A$ *be a* $\mathcal{T}$*-distributor with arbitrary codomain.* A colimit of such a diagram is a $\mathcal{T}$-functor $g : A \to X$ which represents $d_\circledast \mathrel{\circ\!\!-} \psi : A \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ in the sense that $g_\circledast = d_\circledast \mathrel{\circ\!\!-} \psi$, and we write $g \simeq \operatorname{colim}(d, \psi)$. We note that one still has $\operatorname{colim}(d, \psi) \simeq \operatorname{colim}(1_X, \psi \circ d^{\circledast})$. A $\mathcal{T}$-functor

$f : X \to Y$ preserves the ψ-weighted colimit g of d if $f \cdot g$ is the ψ-weighted colimit of $f \cdot d$, that is, if $(f \cdot g)_{\circledast} = (f \cdot d)_{\circledast} \multimap \psi$. We write

$$\mathcal{T}\text{-}\mathsf{CoCts}$$

to denote the category of totally cocomplete $\mathcal{T}$-categories and weighted colimit preserving $\mathcal{T}$-functors, and $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}}$ for its full subcategory defined by the separated $\mathcal{T}$-categories. As before, in the $\mathcal{V}$-case we use the designations $\mathcal{V}\text{-}\mathsf{CoCts}$ and $\mathcal{V}\text{-}\mathsf{CoCts}_{\mathrm{sep}}$.

THEOREM IV.2.2. *The following assertions are equivalent, for a $\mathcal{T}$-category X.*

(i). *X is injective.*
(ii). *$y_X : X \to P_{\mathcal{T}}X$ has a left inverse $\mathrm{Sup}_X : P_{\mathcal{T}}X \to X$ in $\mathcal{T}\text{-}\mathsf{Cat}$, that is, $\mathrm{Sup}_X \cdot y_X \simeq 1_X$.*
(iii). *$y_X : X \to P_{\mathcal{T}}X$ has a left adjoint $\mathrm{Sup}_X : P_{\mathcal{T}}X \to X$ in $\mathcal{T}\text{-}\mathsf{Cat}$.*
(iv). *X has all weighted colimits (in the generalised sense).*

THEOREM IV.2.3. *Let $f : X \to Y$ be a $\mathcal{T}$-functor between totally cocomplete $\mathcal{T}$-categories. Then the following assertions are equivalent.*

(i). *f preserves all weighted colimits (in the generalised sense).*
(ii). *f preserves all weighted colimits with weight of type $D \,{-\!\!\circ\!\!\rightharpoonup}\, G$.*
(iii). *The diagram*

$$\begin{tikzcd} P_{\mathcal{T}}X \arrow[r, "P_{\mathcal{T}}f"] \arrow[d, "\mathrm{Sup}_X"'] \arrow[dr, phantom, "\simeq"] & P_{\mathcal{T}}Y \arrow[d, "\mathrm{Sup}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes up to equivalence.

From the two results above we infer immediately:

THEOREM IV.2.4. *The category $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}}$ is precisely the category $(\mathcal{T}\text{-}\mathsf{Cat})^{\mathbb{P}_{\mathcal{T}}}$ of Eilenberg–Moore algebras for the monad $\mathbb{P}_{\mathcal{T}} = (P_{\mathcal{T}}, m, y)$ on $\mathcal{T}\text{-}\mathsf{Cat}$. In particular, $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}}$ is complete.*

We also have the expected connection with left adjoint $\mathcal{T}$-functors.

PROPOSITION IV.2.5. *Every left adjoint $\mathcal{T}$-functor is cocontinuous. A $\mathcal{T}$-functor between totally cocomplete $\mathcal{T}$-categories is left adjoint if and only if it is cocontinuous.*

From Theorem III.4.5 we infer that every totally cocomplete $\mathcal{T}$-category is representable and that every cocontinuous $\mathcal{T}$-functor between representable $\mathcal{T}$-categories is a pseudo-homomorphism. Therefore we obtain forgetful functors

$$\mathcal{T}\text{-}\mathsf{CoCts} \to \mathcal{T}\text{-}\mathsf{ReprCat} \qquad \text{and} \qquad \mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathcal{T}\text{-}\mathsf{ReprCat}_{\mathrm{sep}}.$$

IV.3. Monadicity results

We have seen in the previous section that injective separated $\mathcal{T}$-categories are algebras over $\mathcal{T}\text{-}\mathsf{Cat}$. In Example III.4.6 we have seen that the monad $\mathbb{P}_{\mathcal{U}_2}$ on Top is isomorphic to the filter-of-opens monad $\mathbb{F}_o$ on Top, and it is shown in [Day, 1975, Theorem 4.5] that $\mathsf{Top}^{\mathbb{F}_o}$ "is precisely the category of continuous lattices with maps preserving directed suprema and all infima". The objects of the latter category where earlier introduced in [Scott, 1972] in order to give a mathematical model for the λ-*calculus* of Church and Curry. Scott actually considers injective T0-spaces and shows that their specialisation order determines their topology and characterises them as particular complete lattices, namely the continuous ones. Since in this report the underlying order of a space is dual to the specialisation order (see Remark II.4.4), it is more convenient to consider the isomorphic category of op-continuous lattices (i.e. those lattices whose dual is a continuous lattice) and maps preserving directed infima and all suprema. By the results of the previous section, this category is concretely isomorphic to the category of T0-spaces admitting all weighted colimits and colimit-preserving continuous maps, denoted as

$$\mathsf{CoCtsTop}_{\text{sep}}.$$

Another important result of [Day, 1975] is the fact that continuous lattices are the Eilenberg–Moore algebras for the filter monad on Set. In this section we provide a similar result for $\mathcal{T}$-categories and show that the forgetful functor

$$G : \mathcal{T}\text{-}\mathsf{CoCts}_{\text{sep}} \to \mathsf{Set}$$

is monadic (see [Hofmann, 2011]). Clearly, G has a left adjoint, namely the composite

$$\mathsf{Set} \xrightarrow{\text{discrete}} \mathcal{T}\text{-}\mathsf{Cat} \xrightarrow{P} \mathcal{T}\text{-}\mathsf{CoCts}_{\text{sep}};$$

and one also easily verifies that $G : \mathcal{T}\text{-}\mathsf{CoCts}_{\text{sep}} \to \mathsf{Set}$ reflects isomorphisms. In order to verify Duskin's criterion (see [MacDonald and Sobral, 2004, Corollary 2.7]), the main difficulty lies in the proof of the following result.

PROPOSITION IV.3.1. *$\mathcal{T}$-$\mathsf{CoCts}_{\text{sep}}$ has and $G : \mathcal{T}\text{-}\mathsf{CoCts}_{\text{sep}} \to \mathsf{Set}$ preserves coequaliser of G-equivalence relations.*

To see this, consider morphisms $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-$\mathsf{CoCts}_{\text{sep}}$ which define an equivalence relation on the set X. Let $q : X \to Q$ the coequaliser of π_1, π_2 in $\mathcal{T}$-Cat. The crucial observation here is that

$$P_{\mathcal{T}}R \underset{P_{\mathcal{T}}\pi_2}{\overset{P_{\mathcal{T}}\pi_1}{\rightrightarrows}} P_{\mathcal{T}}X \xrightarrow{Pq} P_{\mathcal{T}}Q \tag{$\dagger$}$$

is a split fork in $\mathcal{T}$-Cat; the splitting maps are given by $q^{\circledast}\circ - : P_{\mathcal{T}}Q \to P_{\mathcal{T}}X$ and $\pi_1^{\circledast}\circ - : P_{\mathcal{T}}X \to P_{\mathcal{T}}R$. Therefore ($\dagger$) is a coequaliser diagram in $\mathcal{T}$-Cat , which implies that $q : X \to Q$ is also the coequaliser of π_1, π_2 in $\mathcal{T}$-$\mathsf{CoCts}_{\text{sep}}$ as indicated in the diagram below:

$$\begin{tikzcd}
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "y_R"'] \arrow[dd, bend right=60, "1_R"'] & X \arrow[r, "q"] \arrow[d, "y_X"'] & Q \arrow[d, "y_Q"] \arrow[dd, bend left=60, "1_Q"] \\
P_{\mathcal{T}}R \arrow[r, shift left, "P_{\mathcal{T}}\pi_1"] \arrow[r, shift right, "P_{\mathcal{T}}\pi_2"'] \arrow[d, "\mathrm{Sup}_R"'] & P_{\mathcal{T}}X \arrow[r, "P_{\mathcal{T}}q"] \arrow[d, "\mathrm{Sup}_X"'] & P_{\mathcal{T}}Q \arrow[d, dotted, "\mathrm{Sup}_Q"] \\
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "q"] & Q.
\end{tikzcd}$$

All told:

THEOREM IV.3.2. *The forgetful functor $G : \mathcal{T}\text{-}\mathsf{CoCts}_{\text{sep}} \to \mathsf{Set}$ is monadic. Hence, $\mathcal{T}$-$\mathsf{CoCts}_{\text{sep}}$ is also cocomplete.*

The induced monad on Set we also write as $\mathbb{P}_{\mathcal{T}} = (P_{\mathcal{T}}, m, y)$. Once monadicity over Set is established, one easily obtains:

Corollary IV.3.3. *The canonical forgetful functors* $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathcal{V}\text{-}\mathsf{Cat}$ *and* $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathsf{Ord}$ *are monadic.*

Recall from Section III.2 hat there is a canonical functor $\mathcal{T}\text{-}\mathsf{ReprCat}_{\mathrm{sep}} \to \mathsf{Set}^{\mathbb{T}}$ which, when composed with $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathcal{T}\text{-}\mathsf{ReprCat}_{\mathrm{sep}}$ of Section IV.2, yields a functor

$$\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathsf{Set}^{\mathbb{T}}.$$

Since both $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}}$ and $\mathsf{Set}^{\mathbb{T}}$ are monadic over Set, general results about Set-monads imply (see [Linton, 1969]):

Corollary IV.3.4. *The functor* $\mathcal{T}\text{-}\mathsf{CoCts}_{\mathrm{sep}} \to \mathsf{Set}^{\mathbb{T}}$ *is monadic.*

Manes proved in 1969 that $\mathsf{Set}^{\mathbb{U}} \simeq \mathsf{CompHaus}$, and monadicity of $\mathsf{CoCtsTop}_{\mathrm{sep}}$ over $\mathsf{CompHaus}$ is shown in [Wyler, 1981]. The monad on $\mathsf{CompHaus}$ induced by the functor $\mathsf{CoCtsTop}_{\mathrm{sep}} \to \mathsf{CompHaus}$ corresponding to the one of Corollary IV.3.4 is the *Vietoris monad* whose functor part was originally studied in [Vietoris, 1922]. We have explored this theme in [Hofmann, 2014].

We believe that one of the virtues of our approach is the fact that it automatically provides us with metric and other variants of results originally proven for topological spaces and ordered sets. In the next section we have a closer look at injective approach spaces which in our opinion define an interesting metric counterpart to (op-)continuous lattices.

IV.4. Continuous metric spaces

Lawvere's presentation of metric spaces as enriched categories has motivated much work on the reconciliation of order, metric and category theory. In fact, many order theoretic notions can be appropriately translated into the metric context, for instance

- a non-empty up-closed subset of X can be identified with a monotone map $\varphi : X \to 2$ satisfying $\exists x \in X \,.\, \varphi(x)$; in a metric space we could now talk about a non-expansive map $\varphi : X \to [0,\infty]$ with $0 \geqslant \inf_{x\in\varphi} \varphi(x)$;
- a subset with characteristic map $\varphi : X \to 2$ is directed if it is non-empty and, for all $x, y \in X$,
$$\varphi(x) \,\&\, \varphi(y) \;\Rightarrow\; \exists z \in X \,.\, (x \leq z \,\&\, y \leq z \,\&\, \varphi(z));$$
which in the metric world could be written as
$$\varphi(x) \,+\, \varphi(y) \;\geqslant\; \inf_{z\in X}(d(x,z) + d(y,z) + \varphi(z)).$$

Hence, the notion of order ideal and eventually the order theoretic definition of continuous lattice can be brought into the realm of metric spaces. These analogies led indeed to many interesting results, see for instance [Waszkiewicz, 2009, 2002], [Flagg *et al.*, 1996] and [Wagner, 1994]. But continuous lattices live at the border between Order Theory, Topology and Algebra, here we take injectivity as primitive notion and define:

Definition IV.4.1. A metric space is called ***continuous*** if it underlies an injective approach space.

We hasten to remark that the use of approach spaces in Quantitative Domain Theory was already advocated in [Windels, 2001, 2000]. Our first goal is to show that this is indeed a property rather then an additional structure in the sense that there is at most one such approach space.

Proposition IV.4.2. *For a metric space* $X = (X, d)$*, there exists at most one injective approach space* $X = (X, a)$ *so that* $d(x, y) = a(\dot{x}, y)$*, for all* $x, y \in X$*.*

More precise, each injective approach space is a metric compact Hausdorff space whose compact Hausdorff topology is the Lawson topology of the underlying order of the metric. The full subcategory of App consisting of all injective approach spaces we denote as

$$\mathsf{ContMet}.$$

By the proposition above, $\mathsf{ContMet}$ can be also viewed as a (non-full) subcategory of Met which justifies our notation. Clearly, $\mathsf{ContMet}$ inherits products from App, we are now going to show that $\mathsf{ContMet}$ is Cartesian closed which resembles the fact that the full subcategory of Top defined by all injective topological spaces is Cartesian closed (see [Scott, 1972]). A first step towards this result is the following characterisation of exponentiable approach spaces obtained in [Hofmann, 2006; Hofmann and Seal, 2013].

THEOREM IV.4.3. *The following assertions are equivalent for an approach space* (X,a).

(i). (X,a) *is exponentiable in* App.

(ii). *For* $\mathfrak{X} \in UUX$, $x \in X$ *and* $v,u \in [0,\infty]$ *with* $v + u = a(m_X(\mathfrak{X}),x)$,

$$a(m_X(\mathfrak{X}),x) \geqslant \inf\{\max\{Ua(\mathfrak{X},\mathfrak{x}),v\} + \max\{a(\mathfrak{x},x),u\} \mid \mathfrak{x} \in UX\}.$$

(iii). *For* $\mathfrak{X} \in UUX$, $x \in X$ *with* $a(m_X(\mathfrak{X}),x) < \infty$ *and* $v,u \in [0,\infty)$ *with* $v+u = a(m_X(\mathfrak{X}),x)$, *for each* $\varepsilon > 0$ *there exists an ultrafilter* $\mathfrak{x} \in UX$ *such that*

$$v + \varepsilon \geqslant Ua(\mathfrak{X},\mathfrak{x}) \qquad \text{and} \qquad u + \varepsilon \geqslant a(\mathfrak{x},x).$$

COROLLARY IV.4.4. *Every exponentiable approach space is core-compact.*

It is shown in [Hofmann, 2013, Theorem 5.14] that every injective approach space satisfies the conditions of Theorem IV.4.3. That is:

COROLLARY IV.4.5. *Every injective approach space is exponentiable in* App.

For an exponentiable approach space X, we write $[X,-]$ for the right adjoint of $- \times X$. A categorical standard argument (see [Johnstone, 1986, Lemma 4.10]) shows that with Y also $[X,Y]$ is injective. Therefore:

THEOREM IV.4.6. $\mathsf{ContMet}$ *is Cartesian closed.*

We turn now our attention to the category

$$\mathsf{CoCtsApp}_{\mathrm{sep}}$$

of separated totally cocomplete (=injective) approach spaces and cocontinuous non-expansive maps, hence $\mathsf{CoCtsApp}_{\mathrm{sep}} = \mathsf{App}^{\mathbb{P}_{\mathcal{U}_{\mathsf{P}_+}}}$. By the results of the previous section, $\mathsf{CoCtsApp}_{\mathrm{sep}}$ is also monadic over $\mathsf{CompHaus}$, Met, Ord and Set. Furthermore, the canonical forgetful functor $\mathsf{App} \to \mathsf{Top}$ preserves injectives since its left adjoint preserves fully faithfulness, and therefore restricts to a (necessarily monadic) functor

$$\mathsf{CoCtsApp}_{\mathrm{sep}} \to \mathsf{CoCtsTop}_{\mathrm{sep}}.$$

Consequently, separated continuous metric spaces can be also seen as op-continuous lattices with an algebra structure. More precise, as we explain now, they are the op-continuous lattices equipped with an unitary and associative action of $[0,\infty]$.

Proposition IV.4.7. *The category* $\mathsf{CoCtsTop}_{\mathrm{sep}}$ *admits a tensor product which represents bimorphisms. That is, for all* X, Y *in* $\mathsf{CoCtsTop}_{\mathrm{sep}}$*, the functor*

$$\mathrm{Bimorph}(X \times Y, -) : \mathsf{CoCtsTop}_{\mathrm{sep}} \to \mathsf{Set}$$

is representable by some object $X \boxtimes Y$ *in* $\mathsf{CoCtsTop}_{\mathrm{sep}}$*.*

We write

$$\mathsf{CoCtsTop}_{\mathrm{sep}}^{[0,\infty]}$$

for the category whose objects are op-continuous lattices X equipped with a unitary and associative action $+ : X \boxtimes [0, \infty] \to X$ in $\mathsf{CoCtsTop}_{\mathrm{sep}}$, and whose morphisms are those $\mathsf{CoCtsTop}_{\mathrm{sep}}$-morphisms $f : X \to Y$ which satisfy $f(x + u) = f(x) + u$, for all $x \in X$ and $u \in [0, \infty]$. As shown in [Gutierres and Hofmann, 2013]:

Theorem IV.4.8. $\mathsf{CoCtsApp}_{\mathrm{sep}}$ *is concretely isomorphic to* $\mathsf{CoCtsTop}_{\mathrm{sep}}^{[0,\infty]}$.

V Epilog

In the aforegoing pages we have described some "micro-aspects of Categorical Topology" based on viewing an individual space as a category. Our picture is by no means complete, and below we list further topics and connections with other work.

- A radically different approach to some topics of this report is developed in [Seal, 2009, 2010]. Seal considers monads $\mathbb{T}$ on Set equipped with a power-enrichment, that is, a monad morphism $\tau : \mathbb{P} \to \mathbb{T}$ from the powerset monad $\mathbb{P}$ to $\mathbb{T}$. This additional structure allows for an extension of $\mathbb{T}$ to Rel (called *Kleisli extension*) and consequently for the definition of lax Eilenberg–Moore algebras. These lax algebras can be equivalently described as monoids (called *Kleisli monoids*) in the Kleisli 2-category of $\mathbb{T}$, and the category of injective objects and left adjoints is concretely isomorphic to $\mathsf{Set}^{\mathbb{T}}$. For example, the monad $\mathbb{P}_{\mathcal{T}}$ on Set turns out to be power-enriched, and the associated Kleisli monoids are precisely the $\mathcal{T}$-categories. Therefore Seal's approach gives an alternative route to our Theorem IV.3.2 which, however, needs the knowledge of the monad $\mathbb{P}_{\mathcal{T}}$ beforehand. As a side-effect we see that $\mathcal{T}$-categories can be always presented as lax Eilenberg–Moore algebras for a (possibly complicated) Set-monad extended to Rel. For approach spaces, this fact was first observed in [Lowen and Vroegrijk, 2008] and further studied in [Colebunders *et al.*, 2011].
- In [Hofmann, 2007] we introduced a notion of morphism between topological theories which, however, we neglected in the subsequent development. These morphisms induce functors between the categories of models of these theories, along our way we have encountered some of them like the canonical forgetful functors

$$\widetilde{O}_{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Cat} \to \mathsf{Ord}, \qquad (-)_0 : \mathcal{T}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat} \qquad \text{and} \qquad \mathsf{App} \to \mathsf{Top}.$$

  So far there is no systematic treatment of these functors and their properties.
- In this report we have only considered $\mathcal{T}$-categories admitting *all* weighted colimits. Further important categories of spaces are captured when requiring only the existence of weighted colimits where the weight belongs to a certain *choice* of distributors. Many results of Chapter IV remain true in this context, as shown in [Clementino and Hofmann, 2009b].
- Along the way we talked about sober spaces which are identified in Example III.4.6 as the "Cauchy complete topological T0-spaces" since, in the language of distributors, they satisfy the same condition as Cauchy complete metric spaces (see Example II.3.6). However, sober

spaces typically appear as fixed points of the adjunction

$$\begin{tikzcd} \mathsf{Top}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{O}"] \arrow[r, phantom, "\top"] & \mathsf{Frm} \arrow[l, bend left, "\mathrm{Spec}"] \end{tikzcd}$$

between the dual of the category Top and the category Frm of frames and homomorphisms (see [Isbell, 1972]). Similarly, sober approach spaces are introduced in [Banaschewski *et al.*, 2006] via an adjunction

$$\begin{tikzcd} \mathsf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathsf{AFrm} \arrow[l, bend left] \end{tikzcd}$$

between the dual of App and the category of approach frames and homomorphisms; and are extensively studied in the Ph.D. thesis [Van Olmen, 2005]. With these examples in mind, we studied duality theory for $\mathcal{T}$-categories in [Hofmann and Stubbe, 2011]. Since the definition of frame involves finite infima, one major difficulty here is to understand what "finite" should mean depending on the theory $\mathcal{T}$. The solution offered in [Hofmann and Stubbe, 2011] is motivated by the results presented in Section III.5: $\mathcal{T}$-finite means compact. A rather different approach to duality theory for $\mathcal{T}$-categories inspired by [Rosebrugh and Wood, 1994, 2004] is presented in [Hofmann, 2013].

- The main theme of this report is the study of cocomplete $\mathcal{T}$-categories. Their designation suggests that there should be a dual(?) concept of completeness. Of course, complete $\mathcal{V}$-categories are precisely the duals of cocomplete ones, but the general $\mathcal{T}$-case needs further elaboration. We have started this task in [Hofmann, 2014] under the assumption $T1 = 1$; in particular it is shown there that a $\mathcal{T}$-category X is
  - cocomplete if and only if X is complete, and
  - totally cocomplete if and only if X^{op} is totally complete.

  We also note that a totally complete $\mathcal{T}$-category need not be totally cocomplete.

Bibliography

Banaschewski, B., Lowen, R. and Van Olmen, C. (2006), Sober approach spaces, *Topology Appl.* **153** (16), 3059–3070.

Barr, M. (1970), Relational algebras, in *Reports of the Midwest Category Seminar, IV*, pages 39–55, Lecture Notes in Mathematics, Vol. 137. Springer, Berlin.

Bénabou, J. (2000), Distributors at work, lecture notes by Thomas Streicher, `http://www.mathematik.tu-darmstadt.de/~streicher/`.

Betti, R., Carboni, A., Street, R. and Walters, R. (1983), Variation through enrichment, *J. Pure Appl. Algebra* **29** (2), 109–127.

Bonsangue, M. M., van Breugel, F. and Rutten, J. J. M. M. (1998), Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding, *Theoret. Comput. Sci.* **193** (1-2), 1–51.

Choquet, G. (1948), Convergences, *Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.)* **23**, 57–112.

Clementino, M. M. and Hofmann, D. (2003), Topological features of lax algebras, *Appl. Categ. Structures* **11** (3), 267–286.

Clementino, M. M. and Hofmann, D. (2009a), Lawvere completeness in topology, *Appl. Categ. Structures* **17** (2), 175–210, `arXiv:math.CT/0704.3976`.

Clementino, M. M. and Hofmann, D. (2009b), Relative injectivity as cocompleteness for a class of distributors, *Theory Appl. Categ.* **21**, No. 12, 210–230, `arXiv:math.CT/0807.4123`.

Clementino, M. M., Hofmann, D. and Tholen, W. (2003), Exponentiability in categories of lax algebras, *Theory Appl. Categ.* **11**, No. 15, 337–352.

Clementino, M. M. and Tholen, W. (1997), A characterization of the Vietoris topology, in *Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997)*, volume 22, pages 71–95.

Clementino, M. M. and Tholen, W. (2003), Metric, topology and multicategory—a common approach, *J. Pure Appl. Algebra* **179** (1-2), 13–47.

Colebunders, E., Lowen, R. and Rosiers, W. (2011), Lax algebras via initial monad morphisms: APP, TOP, MET and ORD, *Topology Appl.* **158** (7), 882–903.

Colebunders, E., Lowen, R. and Wuyts, P. (2005), A Kuratowski-Mrówka theorem in approach theory, *Topology Appl.* **153** (5-6), 756–766.

DAY, A. (1975), Filter monads, continuous lattices and closure systems, *Canad. J. Math.* **27**, 50–59.

EILENBERG, S. and KELLY, G. M. (1966), Closed categories, in *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562, Springer, New York.

ESCARDÓ, M. H. (1997), Injective spaces via the filter monad, in *Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997)*, volume 22, pages 97–100.

ESCARDÓ, M. H. (2004), Synthetic topology of data types and classical spaces, *Electronic Notes in Theoretical Computer Science* **87**, 21–156.

FLAGG, R. C., SÜNDERHAUF, P. and WAGNER, K. (1996), A logical approach to quantitative domain theory, *Topology Atlas Preprint # 23* `http://at.yorku.ca/e/a/p/p/23.htm`.

GIERZ, G., HOFMANN, K. H., KEIMEL, K., LAWSON, J. D., MISLOVE, M. W. and SCOTT, D. S. (1980), *A compendium of continuous lattices*, Springer-Verlag, Berlin, xx+371 pages.

GUTIERRES, G. and HOFMANN, D. (2013), Approaching metric domains, *Appl. Categ. Structures* **21** (6), 617–650, `arXiv:math.GN/1103.4744`.

HERMIDA, C. (2000), Representable multicategories, *Adv. Math.* **151** (2), 164–225.

HERMIDA, C. (2001), From coherent structures to universal properties, *J. Pure Appl. Algebra* **165** (1), 7–61.

HOFMANN, D. (2005), An algebraic description of regular epimorphisms in topology, *J. Pure Appl. Algebra* **199** (1-3), 71–86.

HOFMANN, D. (2006), Exponentiation for unitary structures, *Topology Appl.* **153** (16), 3180–3202.

HOFMANN, D. (2007), Topological theories and closed objects, *Adv. Math.* **215** (2), 789–824.

HOFMANN, D. (2011), Injective spaces via adjunction, *J. Pure Appl. Algebra* **215** (3), 283–302, `arXiv:math.CT/0804.0326`.

HOFMANN, D. (2013), Duality for distributive spaces, *Theory Appl. Categ.* **28** (3), 66–122, `arXiv:math.CT/1009.3892`.

HOFMANN, D. (2014), The enriched Vietoris monad on representable spaces, *accepted for publication in J. Pure Appl. Algebra*, `arXiv:math.CT/1212.5539`.

HOFMANN, D. and SEAL, G. J. (2013), Exponentiable approach spaces, *accepted for publication in Houston J. Math.*, `arXiv:math.GN/1304.6862`.

HOFMANN, D. and STUBBE, I. (2011), Towards Stone duality for topological theories, *Topology Appl.* **158** (7), 913–925, `arXiv:math.CT/1004.0160`.

HOFMANN, D. and THOLEN, W. (2010), Lawvere completion and separation via closure, *Appl. Categ. Structures* **18** (3), 259–287, `arXiv:math.CT/0801.0199`.

HOFMANN, D. and WASZKIEWICZ, P. (2011), Approximation in quantale-enriched categories, *Topology Appl.* **158** (8), 963–977, `arXiv:math.CT/1004.2228`.

ISBELL, J. R. (1972), Atomless parts of spaces, *Math. Scand.* **31**, 5–32.

ISBELL, J. R. (1986), General function spaces, products and continuous lattices, *Math. Proc. Cambridge Philos. Soc.* **100** (2), 193–205.

JOHNSTONE, P. T. (1986), *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, xxii+370 pages, reprint of the 1982 edition.

JUNG, A. (2004), Stably compact spaces and the probabilistic powerspace construction, in J. Desharnais and P. Panangaden, editors, *Domain-theoretic Methods in Probabilistic Processes*, volume 87, 15pp.

KELLY, G. M. (1982), *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 245 pages, also in: Repr. Theory Appl. Categ. No. 10 (2005), 1–136.

KOSTANEK, M. and WASZKIEWICZ, P. (2011), The formal ball model for $\mathcal{Q}$-categories, *Math. Structures Comput. Sci.* **21** (1), 41–64.

KURATOWSKI, C. (1931), Evaluation de la classe borelienne ou projective d'un ensemble de points à l'aide des symboles logiques, *Fundam. Math.* **17**, 249–272.

LAWSON, J. (2011), Stably compact spaces, *Math. Structures Comput. Sci.* **21** (1), 125–169.

LAWVERE, F. W. (1973), Metric spaces, generalized logic, and closed categories, *Rendiconti del Seminario Matematico e Fisico di Milano* **43**, 135–166, Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002) pp 1–37, `http://www.tac.mta.ca/tac/reprints/articles/1/tr1.pdf`.

LINTON, F. E. J. (1969), Coequalizers in categories of algebras, in *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 75–90, Springer, Berlin, Republished in: Reprints in Theory and Applications of Categories, No. 18 (2008) pp. 61-72.

LOWEN, E. and LOWEN, R. (1988), A quasitopos containing CONV and MET as full subcategories, *Internat. J. Math. Math. Sci.* **11** (3), 417–438.

LOWEN, R. (1989), Approach spaces: a common supercategory of TOP and MET, *Math. Nachr.* **141**, 183–226.

LOWEN, R. (1997), *Approach spaces*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, x+253 pages, the missing link in the topology-uniformity-metric triad, Oxford Science Publications.

LOWEN, R. and VROEGRIJK, T. (2008), A new lax algebraic characterization of approach spaces, in *Theory and applications of proximity, nearness and uniformity*, volume 22 of *Quad. Mat.*, pages 199–232, Dept. Math., Seconda Univ. Napoli, Caserta.

MACDONALD, J. and SOBRAL, M. (2004), Aspects of monads, in *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 213–268, Cambridge Univ. Press, Cambridge.

MACLANE, S. (1971), *Categories for the working mathematician*, Springer-Verlag, New York, ix+262 pages, graduate Texts in Mathematics, Vol. 5.

MANES, E. G. (1969), A triple theoretic construction of compact algebras, Sem. on Triples and Categorical Homology Theory, ETH Zürich 1966/67, Lect. Notes Math. 80, 91-118 (1969).

MANES, E. G. (1974), Compact Hausdorff objects, *General Topology and Appl.* **4**, 341–360.

MÖBUS, A. (1983), Alexandrov compactification of relational algebras, *Arch. Math. (Basel)* **40** (6), 526–537.

MRÓWKA, S. (1959), Compactness and product spaces, *Colloq. Math.* **7**, 19–22.

NACHBIN, L. (1950), *Topologia e Ordem*, Univ. of Chicago Press, in Portuguese, English translation: Topology and Order, Van Nostrand, Princeton (1965).

NACHBIN, L. (1992), Compact unions of closed subsets are closed and compact intersections of open subsets are open, *Portugal. Math.* **49** (4), 403–409.

PEDICCHIO, M. C. and THOLEN, W., editors (2004), *Categorical foundations: special topics in order, topology, algebra, and sheaf theory*, volume 97 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, xii+417 pages.

PISANI, C. (1999), Convergence in exponentiable spaces, *Theory Appl. Categ.* **5**, No. 6, 148–162.

ROSEBRUGH, R. and WOOD, R. J. (1994), Constructive complete distributivity IV, *Appl. Categ. Structures* **2** (2), 119–144.

ROSEBRUGH, R. and WOOD, R. J. (2004), Split structures, *Theory Appl. Categ.* **13**, No. 12, 172–183.

RUTTEN, J. J. M. M. (1998), Weighted colimits and formal balls in generalized metric spaces, *Topology Appl.* **89** (1-2), 179–202, domain theory.

SCHWARZ, F. (1984), Product compatible reflectors and exponentiability, in *Categorical topology (Toledo, Ohio, 1983)*, volume 5 of *Sigma Ser. Pure Math.*, pages 505–522, Heldermann, Berlin.

SCOTT, D. (1972), Continuous lattices, in *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*, pages 97–136. Lecture Notes in Math., Vol. 274, Springer, Berlin.

SEAL, G. J. (2009), A Kleisli-based approach to lax algebras, *Appl. Categ. Structures* **17** (1), 75–89.

SEAL, G. J. (2010), Order-adjoint monads and injective objects, *J. Pure Appl. Algebra* **214** (6), 778–796.

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

STUBBE, I. (2005), Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory Appl. Categ.* **14**, No. 1, 1–45.

STUBBE, I. (2006), Categorical structures enriched in a quantaloid: tensored and cotensored categories, *Theory Appl. Categ.* **16**, No. 14, 283–306.

THOLEN, W. (2009), Ordered topological structures, *Topology Appl.* **156** (12), 2148–2157.

VAN OLMEN, C. (2005), *A study of the interaction between frame theory and approach theory*, Ph.D. thesis, University of Antwerp.

VICKERS, S. (2005), Localic completion of generalized metric spaces. I, *Theory Appl. Categ.* **14**, No. 15, 328–356.

VIETORIS, L. (1922), Bereiche zweiter Ordnung, *Monatsh. Math. Phys.* **32** (1), 258–280.

WAGNER, K. R. (1994), *Solving Recursive Domain Equations with Enriched Categories*, Ph.D. thesis, Carnegie Mellon University, `ftp://ftp.risc.uni-linz.ac.at/pub/techreports/1994/94-62.ps.gz`.

WASZKIEWICZ, P. (2002), *Quantitative Continuous Domains*, Ph.D. thesis, School of Computer Science, The University of Birmingham, Edgbaston, Birmingham B15 2TT.

WASZKIEWICZ, P. (2009), On domain theory over Girard quantales, *Fund. Inform.* **92** (1-2), 169–192.

WINDELS, B. (2000), The Scott approach structure: an extension of the Scott topology for quantitative domain theory, *Acta Math. Hungar.* **88** (1-2), 35–44, Third Iberoamerican Conference on Topology and its Applications (Valencia, 1999).

WINDELS, B. (2001), Solve[order/topology == quasi-metric/x, x], *Electr. Notes Theor. Comput. Sci.* **40**, 410, MFCSIT2000, The First Irish Conference on the Mathematical Foundations of Computer Science and Information Technology.

WOOD, R. (2004), Ordered sets via adjunction, in *Categorical foundations: special topics in order, topology, algebra, and sheaf theory*, volume 97 of *Encyclopedia Math. Appl.*, pages 5–47, Cambridge Univ. Press, Cambridge.

WYLER, O. (1981), Algebraic theories of continuous lattices, Continuous lattices, Proc. Conf., Bremen 1979, Lect. Notes Math. 871, 390-413 (1981).