# **Quantalic Behavioural Distances**

Functor Liftings, Predicate Liftings, Lax Extensions

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# Abstract

Behavioural distances measure the deviation between states in quantitative systems, such as probabilistic or weighted systems. There is growing interest in generic approaches to behavioural distances. In particular, coalgebraic methods capture variations in the system type (nondeterministic, probabilistic, game-based etc.), and the notion of quantale abstracts over actual values distances take, thus covering, e.g., two-valued equivalences, metrics, and probabilistic metrics. Coalgebraic behavioural distances have variously been based on liftings of Set-functors to categories of metric spaces; on modalities modeled as predicate liftings, via a generalised Kantorovich construction; and on lax extensions of Set-functors to categories of quantitative relations. Every lax extension induces a functor lifting in a straightforward manner. Moreover, it has recently been shown that every lax extension is Kantorovich, i.e. induced by a suitable choice of monotone predicate liftings. In the present work, we complete this picture by determining, in coalgebraic and quantalic generality, when a functor lifting is induced by a class of predicate liftings or by a lax extension. We subsequently show coincidence of the respective induced notions of behavioural distances, in a unified approach via double categories that applies even more widely, e.g. to (quasi)uniform spaces.

# 1 Introduction

Universal coalgebra [34] serves as a general framework for a wide range of transition systems of different types. It generalises (labelled) transition systems and encapsulates the emerging *transition types* as set functors. Coalgebras come equipped with a canonical notion of *behavioural equivalence* based on identifiability under coalgebra morphisms. However, it has long been realised that for quantitative systems more fine-grained comparisons are desirable, such as *behavioural distance* [4, 11, 12, 14, 17, 40]. A priori, notions of distance are less canonically defined than behavioural equivalence. One systematic approach to behavioural distances in coalgebraic generality uses liftings of set functors to categories of pseudometric spaces, briefly *metric liftings* [3]. The process of inducing a behavioural distance from such a lifting can roughly be summarized as follows. The forgetful functor from the category of pseudometric spaces is *topological* [1] (or, in alternative terminology, a  $CLat_{\Box}$ -*fibration* [24, 25]). It follows that if  $\overline{F}$  is a metric lifting of F, then the forgetful functor from  $\overline{F}$ -coalgebras to F-coalgebras is also topological [20], and as such has a right adjoint, which precisely equips an F-coalgebra with a behavioural distance. This idea is also behind categorical frameworks that connect *indistinguishability* and *games* [24], and *indistinguishability* and *expressivity of logic* [25].

Another coalgebraic approach to behavioural distance uses *lax extensions* [28, 36], which extend the action of a set functor to quantitative relations, adhering to suitable axioms. A lax extension gives rise to a notion of *quantitative bisimulation*, generalising the standard notion of a bisimulation as a relation that is compatible with the transition structure of a coalgebra [16, 33, 41, 43]; this induces a notion of behavioural distance as the greatest quantitative bisimulation.

It has been noted that both metric liftings and lax extensions can be constructed via *quantitative predicate liftings*, which encapsulate the semantics of *quantitative modalities* [5, 35], via respective generalised *Kantorovich* constructions [3, 41] (a special case of which is the standard Kantorovich lifting of distances to probability distributions). For lax extensions, it has been shown that indeed *every* lax extension *is Kantorovich*, i.e. induced by a suitable choice of *monotone* predicate liftings [18, 41]. Moreover, it is wellknown that every lax extension induces a metric lifting by restriction.

In the present paper we complete the existing picture of mutual connections between predicate liftings, metric liftings, monotone and unrestricted lax extensions, as summarized in Figure 1.

Specifically, we show that every functor lifting that preserves initial morphisms is *Kantorovich*, i.e. comes from a choice



**Figure 1.** Summary of connections. Indeed this diagram extends to a square of adjunctions (4.i).

of not necessarily monotone predicate liftings, namely the predicate liftings it *induces*; and that every functor lifting that preserves initial morphisms and induces only monotone predicate liftings is induced by a lax extension. A key ingredient of our general treatment is the use of *quantales* as the range of distance functions, and moreover we do not insist on distances being symmetric, thus covering also (quantitative) notions of *similarity*. (Recent work on quantitative modal characterization theorems [42] works at a similar level of generality.) In working with quantales we follow Worrell [43], who uses them to cover the classical two-valued setting, standard unit-interval-valued metrics, ultrametrics, and probabilistic metrics [15, 31] in a uniform way.

Finally, we propose to use double categories as a uniform framework to reason coalgebraically about *indistinguishability*. Our motivation stems from the fact that the *type of transition* and the *type of similarity* are naturally encapsulated by the single notion of a double functor. By applying well-known constructions in double categories, we illustrate how this point of view allows expressing the connection between *similarity* and *behavioural distance* beyond Set-based approaches.

In more detail, we proceed as follows. In the next section we briefly review the theory of quantale-enriched categories and relations, lax extensions and predicate liftings. In Sections 3 and 4 we describe how lax extensions, liftings and predicate liftings are related.

In Section 3 we relate liftings to predicate liftings. We review a simple construction to lift a functor  $F: A \rightarrow A$  along a topological functor  $B \rightarrow A -$  we call such liftings topological. We prove that for every lifting of a functor F: Set  $\rightarrow$  Set to the category  $\mathcal{V}$ -Cat of quantale-enriched categories and quantale-enriched functors there exists a *class* of predicate liftings of F such that the corresponding topological lifting differs from F *at most* at the empty  $\mathcal{V}$ -category. Therefore, from a coalgebraic point of view, one can always assume that a lifting is constructed from predicate liftings. Furthermore, we introduce the notion of predicate lifting – a natural generalisation of the notion introduced by Baldan et al. [3]. We characterise the Kantorovich liftings precisely as the liftings that preserve initial morphisms. This is a pleasant property

that is required in multiple results in the context of coalgebraic *indistinguishability* (e.g. [3, 25, 43]), and that now can be used, for example, to conclude that every Wasserstein lifting is Kantorovich, although not necessarily with respect to the same class of predicate liftings. On the other hand, this characterisation also shows that the notion of Kantorovich lifting is more restrictive than it seems at first sight. For instance, the discrete lifting of the identity functor on Set to the category Ord of preordered sets and monotone functions is not Kantorovich.

In Section 4 we relate lax extensions to liftings by building on the results of the previous section and a representation theorem for lax extensions [18]. We show that lax extensions, liftings and predicate liftings are linked by adjunctions, and that those liftings induced by lax extensions are precisely the Kantorovich liftings whose predicate liftings are monotone. This means, for example, that the lifting of the identity functor to Ord that maps each preordered set to its dual cannot be induced by a lax extension despite being Kantorovich.

Finally, in Section 5 we take advantage of several notions and constructions involving double categories to reason coalgebraically about indistinguishability. In particular, we show in this setting that the corresponding notions of what, for distinction, we call similarity (for lax extensions) and behavioural distance (for liftings), respectively, coincide. Taking into account the results of the previous sections, this implies that the notion of similarity provided by a lax extension corresponding to a class of monotone predicate liftings coincides with the notion of behavioural distance provided by the lifting associated with the same class of predicate liftings. This is the missing link mentioned by Komorida et al. [25] that makes is possible to incorporate the approach to similarity via lax extensions in the categorical frameworks that connect indistinguishability and games [24] and indistinguishability and *expressivity of logic* [25]. Furthermore, the point of view of double categories allows for expressing the connection between these concepts beyond Set-coalgebras and beyond V-relations as simulations. Indeed, by considering coalgebras over quantale-enriched categories and V-distributors as simulations, we recover Worrell's coinduction results [43], and by considering coalgebras over (quasi) uniform spaces and promodules [8] as simulations, we obtain a new result which complements the expressivity result for bisimulation uniformity presented by Komorida et al. [25].

**Related Work.** As indicated above, quantale-valued quantitative notions of bisimulation for functors that already live on generalised metric spaces (rather than being lifted from functors on sets) have been considered early on [43]. We have already mentioned previous work on coalgebraic behavioural metrics, for functors originally living on sets, via metric liftings [3] and via lax extensions [16, 41], where in particular it was shown that every lax extension is Kantorovich [41], a result that we complement here by showing,

among other things, that every metric lifting is Kantorovich. Existing work that combines coalgebraic and quantalic generality and accommodates asymmetric distances, like the present work, has so far concentrated on establishing socalled van Benthem theorems, concerned with characterizing (coalgebraic) quantitative modal logics by bisimulation invariance [42]. There is a line of work on Kantorovich-type coinductive predicates at the level of generality of topological categories [24, 25] (using fibrational terminology), with results including a game characterization and expressive logics for coinductive predicates already assumed to be Kantorovich in a general sense, i.e. induced by variants of predicate liftings. In this work, the condition of preserving initial morphisms already shows up as *fiberedness*, and indeed an instance of the condition already appears in work on metric liftings as preservation of isometries [3].

#### 2 Preliminaries

In this section we briefly review the theory of quantaleenriched categories, predicate liftings and lax extensions. Detailed information about these topics can be easily found in the literature (e.g. [7, 18, 21, 23, 29, 32, 36, 37, 39]).

#### 2.1 Quantale-enriched relations and categories

A *quantale*, more precisely a commutative and unital quantale, is a complete lattice  $\mathcal{V}$  that carries the structure of a commutative monoid  $(\mathcal{V}, \otimes, k)$  such that for every  $u \in \mathcal{V}$  the map  $u \otimes -: \mathcal{V} \to \mathcal{V}$  preserves suprema. Therefore, in a quantale every map  $u \otimes -: \mathcal{V} \to \mathcal{V}$  has a right adjoint  $hom(u, -): \mathcal{V} \to \mathcal{V}$  which is characterised by

$$u \otimes v \leq w \iff v \leq \hom(u, w),$$

for all  $v, w \in \mathcal{V}$ . A quantale is **non-trivial** whenever  $\bot$ , the least element of  $\mathcal{V}$ , does not coincide with  $\top$ , the greatest element of  $\mathcal{V}$ . Moreover, a quantale is **integral** if  $\top$  is the unit of the monoid operation  $\otimes$  of  $\mathcal{V}$ , which we refer to as tensor or multiplication.

**Examples 2.1.** Quantales are ubiquitous in mathematics and computer science.

1. Every frame is a quantale with  $\otimes = \wedge$  and  $k = \top$ .

2. Every commutative monoid  $(M, \cdot, e)$  generates a quantale structure on  $(PM, \bigcup)$ , the free quantale on a monoid. The tensor  $\otimes$  on PM is defined by

$$A \otimes B = \{a \cdot b \mid a \in A \text{ and } b \in B\},\$$

for all  $A, B \subseteq M$ , and  $\{e\}$  is the unit of  $\otimes$ .

3. The complete lattice  $[0, \infty]$  ordered by the "greater or equal" relation  $\geq$  and addition as tensor, denoted by  $[0, \infty]_{\oplus}$ , or maximum as tensor, denoted by  $[0, \infty]_{max}$ .

4. The complete lattice [0, 1] ordered by the "greater or equal" relation  $\geq$  and truncated addition as tensor, denoted by  $[0, 1]_{\oplus}$ .

For a quantale  $\mathcal{V}$  and sets X, Y, a  $\mathcal{V}$ -relation – an enriched relation – from X to Y is a map  $X \times Y \to \mathcal{V}$  and it is represented by  $X \to Y$ . As for ordinary relations,  $\mathcal{V}$ -relations can be composed via "matrix multiplication". That is, for  $r: X \to Y$  and  $s: Y \to Z$ , the composite  $s \cdot r: X \to Z$  is calculated pointwise by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

for every  $x \in X$  and  $z \in Z$ . The collection of all sets and  $\mathcal{V}$ -relations between them forms the category  $\mathcal{V}$ -Rel. For each set X, the identity morphism on X is the  $\mathcal{V}$ -relation  $1_X : X \rightarrow X$  that sends every diagonal element to k and all the others to  $\bot$ .

**Examples 2.2.** The category of 2-relations is the usual category Rel of sets and relations. Quantitative or "fuzzy" relations are usually defined as  $[0, 1]_{\oplus}$ -relations (e.g. [3, 41]).

We can compare  $\mathcal{V}$ -relations of type  $X \rightarrow Y$  using the *pointwise order induced by*  $\mathcal{V}$ ,

$$r \leq s \iff \forall (x, y) \in X \times Y, r(x, y) \leq s(x, y)$$

Every hom-set of  $\mathcal{V}$ -Rel becomes a complete lattice when equipped with this order, and an easy calculation reveals that  $\mathcal{V}$ -relational composition preserves suprema in each variable. Therefore,  $\mathcal{V}$ -Rel is a quantaloid and enjoys many properties inherited from  $\mathcal{V}$ . In particular, precomposition and postcomposition with a  $\mathcal{V}$ -relation  $r: X \rightarrow Y$  define maps with right adjoints that compute Kan lifts and extensions, respectively. Thus, the *lift of a*  $\mathcal{V}$ -*relation*  $s: Z \rightarrow Y$ along  $r: X \rightarrow Y$  is the  $\mathcal{V}$ -relation  $r \rightarrow s: Z \rightarrow X$  defined by the property  $r \cdot t \leq s \iff t \leq r \rightarrow s$ , for every  $t: Z \rightarrow X$ .

If  $\mathcal{V}$  is non-trivial, we can see  $\mathcal{V}$ -Rel as an extension of Set through the faithful functor  $(-)_{\circ}$ : Set  $\rightarrow \mathcal{V}$ -Rel that acts as identity on objects and interprets a function  $f: X \rightarrow Y$  as the  $\mathcal{V}$ -relation  $f_{\circ}: X \rightarrow Y$  that sends every element of the graph of f to k and all the others to  $\bot$ . To avoid unnecessary use of subscripts usually we write f instead of  $f_{\circ}$ .

The category  $\mathcal{V}$ -Rel comes equipped with a contravariant involution  $(-)^{\circ}$ :  $\mathcal{V}$ -Rel<sup>op</sup>  $\longrightarrow \mathcal{V}$ -Rel that maps objects identically and sends a  $\mathcal{V}$ -relation  $r: X \rightarrow Y$  to the  $\mathcal{V}$ -relation  $r^{\circ}: Y \rightarrow X$  defined by  $r^{\circ}(y, x) = r(x, y)$  and called the *converse* of r.

Category theory underlines preordered sets as the fundamental ordered structures. For an arbitrary quantale  $\mathcal{V}$ , the same role is taken by  $\mathcal{V}$ -categories. Analogously to the classical case, we say that a  $\mathcal{V}$ -relation  $r: X \rightarrow X$  is *reflexive* whenever  $1_X \leq r$ , and *transitive* whenever  $r \cdot r \leq r$ . A  $\mathcal{V}$ -category is a pair (X, a) consisting of a set X of objects and a reflexive and transitive  $\mathcal{V}$ -relation  $a: X \rightarrow X$ ; a  $\mathcal{V}$ -functor  $(X, a) \rightarrow (Y, b)$  is map  $f: X \rightarrow Y$  such that  $f \cdot a \leq b \cdot f$ . Clearly,  $\mathcal{V}$ -categories and  $\mathcal{V}$ -functors define a category, denoted as  $\mathcal{V}$ -Cat.

**Examples 2.3.** The following are some familiar examples of quantale-enriched categories.

1. The category 2-Cat is equivalent to the category Ord of preordered sets and monotone maps.

2. The category  $[0, \infty]_{\oplus}$ -Cat is the category Met of generalised *metric spaces* and non-expansive maps.

3. The category  $[0, \infty]_{max}$ -Cat is the category UMet of generalised *ultrametric spaces* and non-expansive maps.

4. The category  $[0, 1]_{\oplus}$ -Cat is the category BMet of generalised **bounded-by-1** metric spaces and non-expansive maps.

5. Categories enriched in a free quantale PM on a monoid M (such as  $M = \Sigma^*$  for some alphabet  $\Sigma$ ) can be interpreted as labelled transition systems with labels in M: in a PM-category (X, a) the objects represent the states of the system, and we can read  $m \in a(x, y)$  as an *m*-labelled transition from *x* to *y*.

There are several ways of constructing  $\mathcal{V}$ -categories from known ones. Of special importance in this work are the **dual** of a  $\mathcal{V}$ -category (X, a), which is the  $\mathcal{V}$ -category  $(X, a)^{\text{op}} =$  $(X, a^{\circ})$ ; and the  $\mathcal{V}$ -category composed by a set X equipped with the **initial structure**  $a: X \rightarrow X$  with respect to a structured cone  $(f_i: X \rightarrow (X_i, a_i))_I$ , that is,

$$a(x,y) = \bigwedge_{i \in I} a_i(f_i(x), f_i(y))$$

for all  $x, y \in X$ . The fact that the we can compute the initial structure of every structured cone of the forgetful functor  $\mathcal{V}$ -Cat  $\rightarrow$  Set means that the category  $\mathcal{V}$ -Cat is topological over Set [1]. Therefore, the category  $\mathcal{V}$ -Cat is complete and cocomplete, and the canonical forgetful functor preserves limits and colimits.

*Remark* 2.4. The quantale  $\mathcal{V}$  becomes a  $\mathcal{V}$ -category when equipped with structure hom:  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ .

**Proposition 2.5.** The  $\mathcal{V}$ -category  $\mathcal{V} = (\mathcal{V}, \text{hom})$  is injective with respect to initial morphisms and, for every  $\mathcal{V}$ -category X, the cone  $(f : X \to \mathcal{V})_f$  is initial with respect to the forgetful functor  $\mathcal{V}$ -Cat  $\to$  Set.

*Remark* 2.6. Since  $(-)^{\text{op}}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat is a concrete isomorphism, Proposition 2.5 applies also to the  $\mathcal{V}$ -category  $\mathcal{V}^{\text{op}}$  *in lieu* of  $\mathcal{V}$ .

Another useful way of organizing  $\mathcal{V}$ -categories in a category is to consider as morphisms the  $\mathcal{V}$ -relations that are compatible with the structure of  $\mathcal{V}$ -category. A  $\mathcal{V}$ -relation  $r: X \rightarrow Y$  between  $\mathcal{V}$ -categories (X, a) and (Y, b) is called a

 $\mathcal{V}$ -distributor<sup>1</sup> whenever  $r \cdot a \leq r$  and  $b \cdot r \leq r$ . Since the reverse inequalities always hold, these are in fact equalities. We distinguish a  $\mathcal{V}$ -distributor from (X, a) to (Y, b) with the notation  $(X, a) \Leftrightarrow (Y, b)$ .

The category  $\mathcal{V}$ -Dist has  $\mathcal{V}$ -categories as objects and  $\mathcal{V}$ distributors as morphisms; composition is inherited from  $\mathcal{V}$ -Rel, but the identity morphism on a  $\mathcal{V}$ -category (X, a)is the  $\mathcal{V}$ -distributor  $a: (X, a) \Leftrightarrow (X, a)$ . The fact that the structure of a  $\mathcal{V}$ -category is transitive implies that we have a structured version  $(-)_{\circ}: \mathcal{V}$ -Cat  $\to \mathcal{V}$ -Dist of the functor  $(-)_{\circ}:$  Set  $\to \mathcal{V}$ -Rel that maps objects identically, and sends a  $\mathcal{V}$ -functor  $f: (X, a) \to (Y, b)$  to the  $\mathcal{V}$ -distributor  $b \cdot$  $f: (X, a) \Leftrightarrow (Y, b)$ .

#### 2.2 Predicate liftings

Given a cardinal  $\kappa$ , a  $\kappa$ -ary **predicate lifting** of a functor F: Set  $\rightarrow$  Set is a natural transformation  $\mu$ :  $P_{V^{\kappa}} \rightarrow P_{V}F$ , where  $P_{V^{\kappa}}$  denotes the functor  $Set(-, V^{\kappa})$ :  $Set^{op} \rightarrow Set$ . We say that  $\mu$ :  $P_{V^{\kappa}} \rightarrow P_{V}F$  is **monotone** if every component of  $\mu$  is a monotone map with respect to the pointwise order.

*Remark* 2.7. The Yoneda lemma tells us that a natural transformation  $\mu: P_{V^{\kappa}} \to P_{V}F$  is completely determined by its action on the identity map on  $V^{\kappa}$ , in particular, the collection of all natural transformation of type  $P_{V^{\kappa}} \to P_{V}F$  is a set. With  $\mu$  denoting the resulting morphism of type  $FV^{\kappa} \to V$ , then  $\mu_X: P_{V^{\kappa}}X \to P_VFX$  is given by the map  $f \mapsto \mu \cdot Ff$ .

**Examples 2.8.** The paradigmatic examples of predicate liftings are motivated by Kripke semantics of modal logic. Coalgebras of the covariant powerset functor P: Set  $\rightarrow$  Set correspond precisely to Kripke frames. In coalgebraic modal logic, the Kripke semantics of the modal logic *K* is recovered by interpreting the modal operator  $\diamond$  as the predicate lifting  $\diamond$ : P<sub>2</sub>  $\rightarrow$  P<sub>2</sub>P whose *X*-component is defined by

$$A \longmapsto \{B \subseteq X \mid A \cap B \neq \emptyset\},\$$

or by interpreting the modal operator  $\Box$  as the predicate lifting  $\Box: P_2 \rightarrow P_2P$  whose *X*-component is defined by

$$A \longmapsto \{B \subseteq X \mid B \subseteq A\}.$$

#### 2.3 Lax extensions

A *lax extension* <sup>2</sup> of a functor F: Set  $\rightarrow$  Set to  $\mathcal{V}$ -Rel consists of a map  $(r: X \rightarrow Y) \longmapsto (\widehat{\mathsf{F}}r: \mathsf{F}X \rightarrow \mathsf{F}Y)$  such that:

(L1) 
$$r \le r' \implies \widehat{\mathsf{F}}r \le \widehat{\mathsf{F}}r'$$
,  
(L2)  $\widehat{\mathsf{F}}s \cdot \widehat{\mathsf{F}}r \le \widehat{\mathsf{F}}(s \cdot r)$ ,  
(L3)  $\mathsf{F}f \le \widehat{\mathsf{F}}f$  and  $(\mathsf{F}f)^{\circ} \le \widehat{\mathsf{F}}(f^{\circ})$ ,

for all  $r: X \rightarrow Y$ ,  $s: Y \rightarrow Z$  and  $f: X \rightarrow Y$ .

Every lax extension has a dual [37]. The *dual lax extension*  $\widehat{F}^{\circ}: \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel}$  of a lax extension  $\widehat{F}: \mathcal{V}\text{-Rel} \to \mathcal{V}\text{-Rel}$  is the lax extension of  $F: \text{Set} \to \text{Set}$  that is defined by the assignment  $r \longmapsto \widehat{F}(r^{\circ})^{\circ}$ . Notably, this means that we can symmetrise lax extensions. The *symmetrisation* 

<sup>&</sup>lt;sup>1</sup>Also called module, bimodule or profunctor in the literature.

<sup>&</sup>lt;sup>2</sup>It is also common to refer to some sort of extension of a Set-functor to Rel as "relator", "relational lifting" or "lax relational lifting".

 $\widehat{\mathsf{F}}^s \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  of a lax extension  $\widehat{\mathsf{F}} \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  is the lax extension given by the infimum between  $\widehat{\mathsf{F}}$  and  $\widehat{\mathsf{F}}^\circ$ .

Lax extensions are deeply connected with monotone predicate liftings. As advocated by [18], to realise this it is convenient to think of the *X*-component of a  $\kappa$ -ary predicate lifting as a map of type  $\mathcal{V}$ -Rel( $X, \kappa$ )  $\rightarrow \mathcal{V}$ -Rel(FX, 1). Hence, in the present work, we move freely between this point of view and the standard point of view.

**Definition 2.9.** A predicate lifting  $\mu: P_{\mathcal{V}^{\kappa}} \to P_{\mathcal{V}}F$  is *induced by a lax extension*  $\widehat{F}: \mathcal{V}$ -Rel  $\to \mathcal{V}$ -Rel whenever there exists a  $\mathcal{V}$ -relation  $\mathfrak{r}: F\kappa \to 1$  such that  $\mu(f) = \mathfrak{r} \cdot \widehat{F}f$ , for every  $\mathcal{V}$ -relation  $f: X \to \kappa$ . If  $\mathfrak{r}$  is the converse of an ele-

ment  $\mathfrak{k}: 1 \to F\kappa$ , then we say that  $\mu$  is a *Moss lifting* of  $\widehat{\mathsf{F}}$ , and we emphasise this by using the notation  $\mu^{\mathfrak{k}}: \mathsf{P}_{V^{\kappa}} \to \mathsf{P}_{V}\mathsf{F}$ .

*Remark* 2.10. It follows immediately from (L1) that every predicate lifting induced by a lax extension is monotone.

**Example 2.11.** Consider the lax extension of the covariant powerset functor P: Set  $\rightarrow$  Set to Rel given by

$$B(\Pr)C \iff \forall c \in C, \exists b \in B, b r c$$

The unary Moss lifting of  $\widehat{\mathsf{P}}$  determined by the element  $1 \in \mathsf{P1}$  is the predicate lifting  $\diamond : \mathsf{P}_{\mathcal{V}} \to \mathsf{P}_{\mathcal{V}}\mathsf{P}$  (see Examples 2.8). Similarly, the Moss lifting of the dual extension of  $\widehat{\mathsf{P}}$  determined by the element  $1 \in \mathsf{P1}$  is the predicate lifting  $\Box : \mathsf{P}_{\mathcal{V}} \to \mathsf{P}_{\mathcal{V}}\mathsf{P}$  (see Examples 2.8).

**Theorem 2.12** ([18]). Let  $\mu: \mathsf{P}_{\mathcal{V}^{\kappa}} \to \mathsf{P}_{\mathcal{V}}\mathsf{F}$  be a  $\kappa$ -ary predicate lifting. For every  $\mathcal{V}$ -relation  $r: X \twoheadrightarrow Y$ , consider the

 $\mathcal{V}$ -relation  $\widehat{\mathsf{F}}^{\mu}r \colon \mathsf{F}X \twoheadrightarrow \mathsf{F}Y$  given by

$$\widehat{\mathsf{F}}^{\mu}r = \bigwedge_{g: Y \to \kappa} \mu(g) \to \mu(g \cdot r).$$
(2.i)

If  $\mu$  is monotone, then the assignment  $r \mapsto \hat{F}r$  defines a lax extension  $\hat{F}: \mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel.

**Definition 2.13.** Let  $F: Set \rightarrow Set$  be a functor, and M a *class* of monotone predicate liftings. The *Kantorovich* lax extension of F with respect to M is the lax extension

$$\widehat{\mathsf{F}}^{M} = \bigwedge\nolimits_{\mu \in M} \widehat{\mathsf{F}}^{\mu}.$$

**Examples 2.14.** Let  $\mathcal{V}$  be a quantale.

1. The identity functor on  $\mathcal{V}$ -Rel is the Kantorovich extension of the identity functor on Set with respect to the identity natural transformation  $P_{\mathcal{V}} \rightarrow P_{\mathcal{V}}$ .

2. The largest extension of a functor  $F: Set \rightarrow Set$  to  $\mathcal{V}$ -Rel arises as the Kantorovich extension of F with respect to the natural transformation  $\top: P_{\mathcal{V}} \rightarrow P_{\mathcal{V}}F$  that sends every map to the constant map  $\top$ .

3. The Kantorovich extension of the powerset functor P: Set  $\rightarrow$  Set with respect to the predicate lifting  $\diamond$ : P<sub>V</sub>  $\rightarrow$ 

 $P_{\mathcal{V}}P$  (see Examples 2.8) is defined on a  $\mathcal{V}$ -relation  $r: X \rightarrow Y$  by the "Hausdorff formula",

$$\widehat{\mathsf{P}}^{\diamond}r(A,B) = \bigwedge_{b\in B} \bigvee_{a\in A} r(a,b),$$

for every  $A \subseteq X$  and  $B \subseteq Y$ . For  $\mathcal{V} = 2$  we obtain a generalisation of the upper-half of the Egli-Milner order; whereby for the quantales  $[0, \infty]_{\oplus}$ ,  $[0, \infty]_{\max}$  and  $[0, 1]_{\oplus}$  we obtain a generalisation of the upper-half of the corresponding Hausdorff metric. Therefore, the symmetrisation of these lax extensions are generalisations of the Egli-Milner order and the Hausdorff metric.

The Kantorovich extension leads to a representation theorem that plays an important role in Section 4.

**Theorem 2.15** ([18]). Let  $\widehat{\mathsf{F}} \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  be a lax extension, and M the class of all of its Moss liftings. Then,  $\widehat{\mathsf{F}} = \widehat{\mathsf{F}}^M$ .

# 3 Liftings versus predicate liftings

It is well-known that every lax extension  $\widehat{F} \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  gives rise to a functor  $\widehat{F} \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$  that commutes with the underlying forgetful functor to Set:



We say that a functor  $\overline{F} \colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$  with this property is a (strict) *lifting* of  $F \colon \text{Set} \to \text{Set}$  to  $\mathcal{V}\text{-Cat}$ , a lifting of  $F \colon \text{Set} \to \text{Set}$ , a functor lifting or simply a lifting. These functors are completely determined by their action on objects. In particular, the *lifting induced by a lax extension*  $\widehat{F} \colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$  sends a  $\mathcal{V}\text{-category}(X, a)$  to the  $\mathcal{V}\text{-category}(FX, \widehat{F}a)$ .

A less known fact is that predicate liftings also induce liftings. This is the outcome of a simple construction available on all topological categories. More in detail, to lift a functor  $G: A \rightarrow Y$  along a topological functor  $|-|: B \rightarrow Y$ , it is enough to give, for every object *A* in A, a |-|-structured cone

$$C(A) = (GA \xrightarrow{h} |B|)_{h,B}$$
(3.i)

so that, for every h in C(A) and every  $f: B \to A$ , the composite  $h \cdot Gf$  belongs to the cone C(B). Then, for an object A in A, one defines  $G^I A$  by equipping GA with the initial structure with respect to the structured cone (3.i). It is easy to see that the assignment  $X \mapsto G^I X$  indeed defines a functor  $G^I: A \to B$  so that  $|-| \cdot G^I = G$ . This technique has been recently applied to construct "codensity liftings" [24, 25] and "Kantorovich liftings" [3]. Applying this to our situation with  $G = F \cdot |-|$ , the idea is that to construct a lifting of a functor F: Set  $\to$  Set to V-Cat it suffices to provide a cocone indexed by a *class I* of natural transformations

$$\mu^{i} \colon \mathcal{V}\text{-}\mathsf{Cat}(-,A_{i}) \longrightarrow \mathsf{Set}(\mathsf{F}|-|,|B_{i}|).$$

Then, given a  $\mathcal{V}$ -category X, we consider the |-|-structured cone

$$\mu^{\iota}(f)\colon \mathsf{F}[X]\longrightarrow |B_i|$$

determined by every natural transformation  $\mu^i$  and every  $\mathcal{V}$ -functor  $f: X \to A_i$ , and, as described above, we obtain a  $\mathcal{V}$ -category  $\mathsf{F}^I X$  by equipping  $\mathsf{F} X$  with the initial structure with respect to this cone. We call a lifting constructed this way a *topological lifting*, although, as we will see next, it does not make much sense to talk about topological liftings without restricting the collection of the  $\mathcal{V}$ -categories that may take the role of  $A_i$ .

Note that for every lifting  $\overline{F} \colon \mathcal{V}\text{-}Cat \to \mathcal{V}\text{-}Cat$  of a functor  $F \colon Set \to Set$ , the forgetful functor  $|-| \colon \mathcal{V}\text{-}Cat \to Set$  induces a natural transformation

$$|-|_{-\mathcal{W}^{\mathrm{op}}} \colon \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}},\mathcal{V}^{\mathrm{op}}) \longrightarrow \mathsf{Set}(\mathsf{F}|-|,|\mathcal{V}^{\mathrm{op}}|).$$

**Lemma 3.1.** Let  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting of a functor F: Set  $\rightarrow$  Set and X a  $\mathcal{V}$ -category. Given a family of natural transformations

$$\mu^{i}: \mathcal{V}\text{-}Cat(-, A_{i}) \longrightarrow \mathcal{V}\text{-}Cat(\overline{\mathsf{F}}, \mathcal{V}^{\mathrm{op}}) \quad (for \ i \in I),$$

if the cocone

$$(\mu_X^i: \mathcal{V}\text{-}\operatorname{Cat}(X, A_i) \longrightarrow \mathcal{V}\text{-}\operatorname{Cat}(\overline{\mathsf{F}}X, \mathcal{V}^{\operatorname{op}}))_{i \in I}$$

is jointly epic, then the topological lifting with respect to the class of all natural transformations

 $\mu^i: \mathcal{V}\text{-}Cat(-, A_i) \to Set(\mathsf{F}|-|, |\mathcal{V}^{op}|)$ 

that factor as

$$\mathcal{V}\text{-}\mathsf{Cat}(-,A_i) \xrightarrow{\mu^i} \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}-,\mathcal{V}^{\mathrm{op}})$$

$$\downarrow^{|-|_{-,\mathcal{V}^{\mathrm{op}}}}$$

$$\mathsf{Set}(\mathsf{F}|-|,|\mathcal{V}^{\mathrm{op}}|)$$

coincides with  $\overline{F}$  on X.

#### Theorem 3.2. Every lifting is topological.

Constructing topological liftings from predicate liftings amounts to choosing possibly different  $\mathcal{V}$ -categories based on powers of the set  $\mathcal{V}$ . Given a  $\kappa$ -ary predicate lifting  $\mu: \mathsf{P}_{\mathcal{V}^{\kappa}} \to \mathsf{P}_{\mathcal{V}}\mathsf{F}$ , for each pair (A, B) of  $\mathcal{V}$ -categories based on the function set  $\mathcal{V}^{\kappa}$  and on the set  $\mathcal{V}$ , respectively, we use the natural transformation

$$\mathcal{V}\text{-}\mathsf{Cat}(-,A) \longrightarrow \mathsf{Set}(|-|,|A|)$$

defined by inclusion maps to obtain a natural transformation

$$\mu^{(A,B)}: \mathcal{V}\text{-}\mathsf{Cat}(-,A) \longrightarrow \mathsf{Set}(\mathsf{F}|-|,|B|)$$

as illustrated below.



The next results shows that in practice it is often reasonable to assume that every lifting is a topological lifting constructed from predicate liftings.

**Theorem 3.3.** For every lifting  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat there is a topological lifting constructed from a class of predicate liftings of F that agrees with  $\overline{F}$  on every non-empty  $\mathcal{V}$ -category.

**Corollary 3.4.** Let  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting such that one of the following conditions is satisfied:

1.  $\overline{\mathsf{F}}$  preserves the initial morphism  $\emptyset \to (1, 1_1)$ ;

2.  $\overline{\mathsf{F}}$  preserves the final monomorphism  $\emptyset \to (1, 1_1)$ .

Then,  $\overline{F}$  is a topological lifting constructed from predicate liftings.

**Corollary 3.5.** For every lifting  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat of a functor F: Set  $\rightarrow$  Set, there exists a topological lifting  $F^{I}: \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat of F constructed from predicate liftings such that  $CoAlg(\overline{F}) \simeq CoAlg(F^{I})$ .

*Remark* 3.6. The result above is already optimal with respect to the proof-strategy encapsulated in Lemma 3.1.

In the remaining of the paper, we will be primarily concerned with the situation where each  $\kappa$ -ary predicate lifting  $\mu$ :  $P_{V^{\kappa}} \rightarrow P_{V}F$  gives rise to the natural transformation:

$$\mu: \mathcal{V}\text{-}Cat(-, (\mathcal{V}^{\kappa})^{op}) \longrightarrow Set(\mathsf{F}|-|, |\mathcal{V}^{op}|),$$

where  $\mathcal{V}$  is the canonical  $\mathcal{V}$ -category induced by the internal hom of the quantale (see Remark 2.4). To simplify notation, in the sequel we often omit the forgetful functor to Set.

**Definition 3.7.** Let F: Set  $\rightarrow$  Set be a functor and *M* a class of predicate liftings of F with respect to a quantale  $\mathcal{V}$ . The *Kantorovich lifting* of F with respect to *M* is the topological lifting F<sup>M</sup>:  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat that sends a  $\mathcal{V}$ -category (*X*, *a*) to the  $\mathcal{V}$ -category (FX, F<sup>M</sup>a), where F<sup>M</sup>a denotes the initial structure on FX with respect to the structured cone of all functions

$$\mu(f) \colon \mathsf{F} X \longrightarrow \mathcal{V}^{\mathrm{op}}$$

where  $\mu: P_{\mathcal{V}^{\kappa}} \to P_{\mathcal{V}}F$  is a predicate lifting that belongs to M, and  $f: (X, a) \to (\mathcal{V}^{\kappa})^{\text{op}}$  is a  $\mathcal{V}$ -functor. A lifting  $\overline{F}: \mathcal{V}$ -Cat  $\to \mathcal{V}$ -Cat is said to be Kantorovich whenever it can be expressed as the Kantorovich lifting with respect to a class of predicate liftings of F.

As far as we know, the notion of Kantorovich lifting was first introduced in practice – without mentioning initial lifts at all – by Baldan et al. [2, 3] as a tool to study coalgebraically the notion of *behavioural distance*; more specifically, as a way of constructing liftings to (topological) categories of pseudometric spaces and non-expansive maps from unary predicate liftings.

In the remaining of this section we exploit the universal property of initial lifts of cones to obtain a pleasant characterisation of the class of Kantorovich liftings. To this end, in the following let  $F: Set \rightarrow Set$  be a functor, and  $\mathcal{V}$  be

a quantale. Consider the partially ordered *class* Pred(F) of *classes* of predicate liftings of F ordered by containment,

$$M \leq M' \iff M \supseteq M';$$

and the partially ordered *class* Lift(F) of liftings of F to V-Cat ordered by

$$\overline{\mathsf{F}} \le \overline{\mathsf{F}}' \iff \forall (X, a) \in \mathcal{V}\text{-}\mathsf{Cat}, \quad \overline{\mathsf{F}}a \le \overline{\mathsf{F}}'a$$

*Remark* 3.8. Alternatively,  $\overline{\mathsf{F}} \leq \overline{\mathsf{F}}'$  if  $(1_{\mathsf{F}X} : \mathsf{F}X \to \mathsf{F}X)_{(X,a) \in \mathcal{V}\text{-}\mathsf{Cat}}$  extends to a natural transformation from  $\overline{\mathsf{F}}$  to  $\overline{\mathsf{F}}'$ .

Motivated by Lemma 3.1 we introduce now the notion of predicate lifting induced by a functor lifting.

**Definition 3.9.** Let  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting. A *predicate lifting*  $\mu$ :  $\mathsf{P}_{\mathcal{V}^{\kappa}} \rightarrow \mathsf{P}_{\mathcal{V}}\mathsf{F}$  is *induced by*  $\overline{\mathsf{F}}$ , also said to be a predicate lifting of  $\overline{\mathsf{F}}$ , whenever  $\mu(1_{\mathcal{V}^{\kappa}})$ :  $\overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}} \rightarrow \mathcal{V}^{\mathrm{op}}$  is a  $\mathcal{V}$ -functor. The *class* of all predicate liftings induced by  $\overline{\mathsf{F}}$  is represented by  $\mathsf{P}(\overline{\mathsf{F}})$ .

*Remark* 3.10. In other words, a predicate lifting  $\mu: P_{\mathcal{V}^{\kappa}} \rightarrow P_{\mathcal{V}}F$  is induced by a lifting  $\overline{F}: \mathcal{V}\text{-}Cat \rightarrow \mathcal{V}\text{-}Cat$  whenever the corresponding natural transformation

$$\mu: \mathcal{V}\text{-}\mathsf{Cat}(-, (\mathcal{V}^{\kappa})^{\mathrm{op}}) \longrightarrow \mathsf{Set}(\mathsf{F}[-], |\mathcal{V}^{\mathrm{op}}|)$$

admits a lift

$$\begin{array}{c} \mathcal{V}\text{-}\mathsf{Cat}(-,(\mathcal{V}^{\kappa})^{\mathrm{op}}) \xrightarrow{\mu} \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}-,\mathcal{V}^{\mathrm{op}}) \\ & & \downarrow^{|-|_{-,\mathcal{V}^{\mathrm{op}}}} \\ & & \mathsf{Set}(\mathsf{F}|-|,|\mathcal{V}^{\mathrm{op}}|). \end{array}$$

**Theorem 3.11.** Let  $F: Set \to Set$  be a functor. Assigning to a class of predicate liftings of F the corresponding Kantorovich lifting yields a right adjoint  $F^{(-)}: Pred(F) \to Lift(F)$  whose left adjoint P: Lift(F)  $\to Pred(F)$  maps a lifting of F to its class of predicate liftings.

**Theorem 3.12.** Let  $F: Set \rightarrow Set$  be a functor. A lifting  $\overline{F}: \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat of F is Kantorovich if and only if it preserves initial morphisms (=fully faithfull  $\mathcal{V}$ -functors).

**Example 3.13.** The characterisation of Theorem 3.12 makes it easy to provide examples of Kantorovich liftings.

1. It is an elementary fact that every lifting induced by a lax extension preserves initial morphisms [20, Proposition 2.16]. In particular, the Wasserstein lifting [3] is Kantorovich.

2. Since we assume that a quantale  $\mathcal{V}$  is commutative, the involution  $(-)^{\circ}: \mathcal{V}\text{-Rel}^{\circ p} \to \mathcal{V}\text{-Rel}$  induces a lifting  $(-)^{\circ}: \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$  of the identity functor on Set that sends every  $\mathcal{V}$ -category to its dual. Clearly, this lifting preserves initial morphisms, and it can be shown that it is the Kantorovich lifting of the identity functor with respect to the set of all predicate liftings induced by the  $\mathcal{V}$ -functors  $\mathcal{V} \to \mathcal{V}^{\circ p}$ . *Remark* 3.14. The composite of  $\mathcal{V}$ -Cat-functors that preserve initial morphisms preserves initial morphisms. Consequently, the composite of Kantorovich liftings is Kantorovich. In particular, this entails that choosing the  $\mathcal{V}$ -categories  $\mathcal{V}$ or  $\mathcal{V}^{op}$  as the base  $\mathcal{V}$ -category to define the notion of Kantorovich lifting leads to the same class of functors. Our choice here prevents a mismatch in Section 4 when we compare the Kantorovich liftings and the liftings induced by the Kantorovich extension [18, 41].

**Example 3.15.** The following examples are topological liftings that do not preserve initial morphisms.

1. The discrete lifting of the identity functor on Set to  $\mathcal{V}$ -Cat does not preserve initial morphisms.

2. Let  $(3, \leq)$  denote the set  $\{0, 1, 2\}$  equipped with the preorder determined by  $2 \leq 0$  and  $2 \leq 1$ . Then, considering the preordered set  $(2 = \{0, 1\}, 1_2)$  – the symmetrisation of the quantale 2 – we have that the inclusion  $(2, 1_2) \rightarrow (3, \leq)$  is an order embedding. The topological lifting of the identity functor on Set to Ord with respect to the identity map  $(2, 1_2) \rightarrow (2, 1_2)$  acts as identity on  $(2, 1_2)$  but it sends  $(3, \leq)$  to the set  $\{0, 1, 2\}$  equipped with the indiscrete order. Therefore, this lifting does not preserve initial morphisms, hence, by Theorem 3.12, it is not Kantorovich.

## 4 Lax extensions versus liftings

In this section we show that lax extensions, liftings and predicate liftings are linked by adjunctions, and characterise the liftings induced by lax extensions. We begin by showing that the Kantorovich extension and the Kantorovich lifting are compatible. To achieve this, it is convenient to express the Kantorovich lifting in the language of  $\mathcal{V}$ -relations [18].

**Proposition 4.1.** Let  $\mu: \mathbb{P}_{V^{\kappa}} \to \mathbb{P}F$  be a  $\kappa$ -ary predicate lifting of a functor  $F: \text{Set} \to \text{Set}$ . The Kantorovich lifting  $F^{\mu}: \mathcal{V}\text{-}\text{Cat} \to \mathcal{V}\text{-}\text{Cat}$  of F with respect to  $\mu$  sends a  $\mathcal{V}$ category (X, a) to the  $\mathcal{V}$ -category  $(FX, F^{\mu}a)$ , where

$$\mathsf{F}^{\mu}a = \bigwedge_{r \colon (X,a) \Leftrightarrow (\kappa, \mathbf{1}_{\kappa})} \mu(r) \twoheadrightarrow \mu(r).$$

The Kantorovich lifting of a class of predicate liftings M sends a  $\mathcal{V}$ -category (X, a) to the  $\mathcal{V}$ -category  $(FX, F^M a)$ , where

$$\mathsf{F}^M a = \bigwedge_{\mu \in M} \mathsf{F}^\mu a.$$

**Theorem 4.2.** Let  $\widehat{F}: \mathscr{V}$ -Cat  $\rightarrow \mathscr{V}$ -Cat be a lifting of a functor F: Set  $\rightarrow$  Set induced by a lax extension  $\widehat{F}: \mathscr{V}$ -Rel  $\rightarrow \mathscr{V}$ -Rel. If  $\widehat{F}: \mathscr{V}$ -Rel  $\rightarrow \mathscr{V}$ -Rel is the Kantorovich extension with respect to a class M of predicate liftings, then the functor  $\widehat{F}: \mathscr{V}$ -Cat  $\rightarrow \mathscr{V}$ -Cat is the Kantorovich lifting of F: Set  $\rightarrow$  Set with respect to M.

Therefore, recalling that Theorem 2.15 states that every lax extension can be expressed as the Kantorovich extension with respect to its collection of Moss liftings, we obtain: **Corollary 4.3.** Let  $\widehat{F}: \mathcal{V}\text{-}Cat \to \mathcal{V}\text{-}Cat$  be a lifting of a functor  $F: Set \to Set$  induced by a lax extension  $\widehat{F}: \mathcal{V}\text{-}Rel \to \mathcal{V}\text{-}Rel$ . Then, the functor  $\widehat{F}: \mathcal{V}\text{-}Cat \to \mathcal{V}\text{-}Cat$  is the Kantorovich lifting of  $F: Set \to Set$  with respect to the class of Moss liftings of  $\widehat{F}: \mathcal{V}\text{-}Rel \to \mathcal{V}\text{-}Rel$ .

Remark 4.4. Taking as primitives the notions of bisimulation and behavioural equivalence would lead us to consider lax extensions to  $\mathcal{V}$ -Rel that preserve converses [41] and liftings to the category  $\mathcal{V}$ -Cat<sub>sym</sub> of symmetric quantale-enriched categories [3]. In this regard, we note that the symmetrisation of a lax extension of a Set-functor to  $\mathcal{V}$ -Rel yields a lifting of that functor to  $\mathcal{V}$ -Cat<sub>sym</sub>. Moreover, since the category  $\mathcal{V}$ -Cat<sub>sym</sub> is topological over Set, we also have a canonical notion of Kantorovich lifting to  $\mathcal{V}$ -Cat<sub>sym</sub> where  $\mathcal{V}_s$ , the symmetrisation of  $\mathcal{V}$ , takes the role of  $\mathcal{V}^{\text{op}}$ . Indeed, this is how the Kantorovich lifting (of a unary predicate lifting) to categories of pseudometrics is defined by Baldan et al. [3] . It follows from the fact that, for every symmetric  $\mathcal{V}$ category X,

$$\mathcal{V}$$
-Cat<sub>sym</sub> $(X, \mathcal{V}_s) = \mathcal{V}$ -Cat $(X, \mathcal{V}) = \mathcal{V}$ -Cat $(X, \mathcal{V}^{op})$ 

that the Kantorovich extension and the Kantorovich lifting of a functor are also compatible with respect to symmetrisation; that is, the lifting to  $\mathcal{V}$ -Cat<sub>sym</sub> of a functor F: Set  $\rightarrow$  Set induced by the symmetrisation of the Kantorovich extension of F to  $\mathcal{V}$ -Rel with respect to a class of monotone predicate liftings coincides with the Kantorovich lifting of F to  $\mathcal{V}$ -Cat<sub>sym</sub> with respect to the same class of predicate liftings.

Let Lax(F) denote the partially ordered *class* of lax extensions of a functor F: Set  $\rightarrow$  Set to  $\mathcal{V}$ -Rel ordered by the pointwise order,

$$\widehat{\mathsf{F}} \leq \widehat{\mathsf{F}}' \iff \forall r \in \mathcal{V}\text{-}\mathsf{Rel}, \quad \widehat{\mathsf{F}}r \leq \widehat{\mathsf{F}}'r;$$

and let  $Lift(F)_{I}$  represent the partially ordered *subclass* of Lift(F) consisting of the liftings that preserve initial morphisms. Furthermore,  $Pred(F)_{M}$  denotes the partially ordered *subclass* of Pred(F) of monotone predicate liftings. Clearly, the operations of taking Kantorovich extensions

$$\widehat{\mathsf{F}}^{(-)}$$
: Pred(F)<sub>M</sub>  $\rightarrow$  Lax(F),

and inducing liftings from lax extensions

$$I: Lax(F) \rightarrow Lift(F)_{I}$$

define monotone maps. Moreover, as we have seen in Theorem 3.12, the monotone map  $F^{(-)}$ :  $Pred(F) \rightarrow Lift(F)$  corestricts to  $Lift(F)_I$ . Therefore, our results so far tell us that lax extensions, liftings and predicate liftings are connected through a diagram of monotone maps

$$\begin{array}{c} \text{Lax}(F) & \stackrel{I}{\longrightarrow} \text{Lift}(F)_{I} \\ \widehat{F}^{(-)} & F^{(-)} \begin{pmatrix} A \\ F \end{pmatrix} \\ P \\ \text{Pred}(F)_{M} & \longleftrightarrow \\ \text{Pred}(F) \end{array}$$

which commutes if the left adjoint is ignored. We will see next that the monotone map  $\widehat{F}^{(-)}$ :  $Pred(F)_M \rightarrow Lax(F)$  is a right adjoint. This might not be immediately obvious without thinking in terms of liftings because the obvious guess – taking the predicate liftings induced by a lax extension [18] – does not always defines a monotone map  $Lax(F) \rightarrow Pred(F)_M$ , as it can be seen in Example 4.5. At a deeper level we can see that there is a slight mismatch between the predicate liftings of a lax extension and the predicate liftings of the functor lifting induced by that lax extension.

**Example 4.5.** The identity functor on Ord is the lifting induced by the identity functor on Rel which is a lax extension of identity functor on Set. The constant map into  $\top$  is a monotone map  $2^{\text{op}} \rightarrow 2^{\text{op}}$  and determines a monotone predicate lifting that is induced by the largest extension of the identity functor. However, it is easy to see that this predicate lifting is not a predicate lifting of the identity functor on Rel (for example, see [18, Example 3.12]).

**Proposition 4.6.** Every predicate lifting of a lax extension  $\widehat{F}: \mathcal{V}\text{-Rel} \rightarrow \mathcal{V}\text{-Rel}$  is also a predicate lifting of the corresponding lifting  $\widehat{F}: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cat}$ .

It should also be noted that even if a functor lifting preserves initial morphisms its predicate liftings might not be monotone. That is, the map P: Lift(F)<sub>I</sub>  $\rightarrow$  Pred(F) does not necessarily corestricts to Pred(F)<sub>M</sub>.

**Example 4.7.** Consider the lifting  $(-)^{\circ}$ : Ord  $\rightarrow$  Ord of the identity functor on Set to Ord that sends each preordered set to its dual. Then, the predicate lifting of  $(-)^{\circ}$  determined by the  $\mathcal{V}$ -functor hom(-, 0):  $(2, \text{hom}) \rightarrow (2, \text{hom})^{\text{op}}$  is not monotone since it sends the constant map  $0: 1 \rightarrow 2$  to the constant map  $1: 1 \rightarrow 2$ .

Accordingly, we need to "filter the monotone predicate liftings" first. This operation trivially defines the left adjoint

$$M: \operatorname{Pred}(F) \to \operatorname{Pred}(F)_M$$

of the inclusion map  $Pred(F)_M \hookrightarrow Pred(F)$ . In Proposition 4.11, we will observe that this procedure is redundant whenever applied to a collection of predicate liftings of a lifting induced by a lax extension.

**Lemma 4.8.** Let  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting induced by a lax extension  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel. Then, for every monotone predicate lifting  $\mu$ :  $P_{\mathcal{V}^{\kappa}} \rightarrow P_{\mathcal{V}}F$  of  $\widehat{F}$ ,  $\widehat{F} \leq \widehat{F}^{\mu}$ .

**Theorem 4.9.** Let  $F: Set \to Set$  be a functor. The monotone map MPI: Lax(F)  $\to$  Pred(F)<sub>M</sub> is left adjoint to the monotone map  $\widehat{F}^{(-)}: Pred(F)_M \to Lax(F)$ .

In the sequel, we show that the monotone maps  $\widehat{F}^{MP(-)}$ : Lift(F)<sub>1</sub>  $\rightarrow$  Lax(F) and I: Lift(F)<sub>1</sub>  $\rightarrow$  Lax(F) form an adjunction and characterise its fixed points. The following lemma does most of the remaining heavy lifting. **Lemma 4.10.** Let  $\overline{\mathsf{F}}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting. For every cardinal  $\kappa$ , let  $h_{\kappa}$  denote the structure of the  $\mathcal{V}$ -category  $(\mathcal{V}^{\kappa})^{\text{op}}$ . The following are equivalent:

- (i) every predicate lifting of  $\overline{F}$  is monotone;
- (ii) for every cardinal  $\kappa$  and every pair  $p, q: X \leftrightarrow \kappa$  of  $\mathcal{V}$ -relations,

$$p \le q \implies \mathsf{F} p^{\sharp} \le \overline{\mathsf{F}} h_{\kappa} \cdot \mathsf{F} q^{\sharp}$$

where  $r^{\sharp}: X \to \mathcal{V}^{Y}$  denotes the map corresponding to the  $\mathcal{V}$ -relation  $r: X \to Y$ .

**Proposition 4.11.** Let  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting induced by a lax extension  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel. Every predicate lifting of the functor lifting  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat is monotone.

**Theorem 4.12.** Let  $F: Set \to Set$  be a functor. The monotone  $map \widehat{F}^{MP(-)}: Lift(F)_{I} \to Lax(F)$  is left adjoint to the monotone  $map I: Lax(F) \to Lift(F)_{I}$ .

Therefore, the interplay between lax extensions, liftings and predicate liftings is captured by the diagram

$$Lax(F) \xrightarrow{\widehat{F}^{MP(-)}} Lift(F)_{I}$$

$$MPI \left( \overrightarrow{H} \right) \widehat{F}^{(-)} \qquad F^{(-)} \left( \overrightarrow{H} \right)_{P}$$

$$Pred(F)_{M} \xrightarrow{T} Pred(F)$$

$$(4.i)$$

which commutes when only the right adjoints or only the left adjoints are considered. Finally, let  $Lift(F)_{IM}$  denote the partially ordered subclass of  $Lift(F)_{I}$  consisting of the liftings that induce only monotone predicate liftings.

**Theorem 4.13.** Let  $F: Set \rightarrow Set$  be a functor. The partially ordered classes Lax(F) and Lift(F)<sub>IM</sub> are isomorphic.

Our characterisation of lax extensions makes it clear that even among Kantorovich liftings there are interesting liftings that are not induced by lax extensions. For instance, the lifting of the identity functor to Ord determined by the dual structure (3.13(2)) is Kantorovich, but as Example 4.7 shows it cannot be induced by lax extensions.

### 5 Quantale-enriched simulations

It is known that a lax extension of a Set-functor to  $\mathcal{V}$ -Rel corresponds precisely to a lax-double endofunctor on the standard double category of quantale-enriched relations [10]. This fact suggests that to reason coalgebraically about *in*-distinguishability, we should consider not just the category of coalgebras of an endofunctor but the double category of coalgebras of a lax-double endofunctor. In this section, after a brief review of basic facts about double categories, we show that double category of coalgebras inherits nice properties from the base double category. Then, for a given double

category of coalgebras, we show that the corresponding notions of *similarity* and *behavioural distance* coincide. This entails that the notion of similarity arising from a lax extension to  $\mathcal{V}$ -Rel coincides with the notion of behavioural distance arising from the lifting induced by the lax extension. Moreover, by considering coalgebras over quantale-enriched categories and  $\mathcal{V}$ -distributors as simulations, we recover Worrell's coinduction results [43], and by considering coalgebras over (quasi) uniform spaces and promodules [8] as simulations, we obtain a new result which complements the expressivity result for *bisimulation uniformity* presented by Komorida et al. [25].

We recall that a *double category* [13] A consists of objects and two types of arrows: horizontal and vertical ones, and cells in squares suggestively written as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ r \downarrow & \varepsilon & \downarrow s \\ A & \xrightarrow{q} & B. \end{array}$$

Cells can be composed horizontally and vertically, and these operations must satisfy the *middle-interchange law*. Furthermore, we write Horiz( $\mathcal{A}$ ) for the 2-category of objects of  $\mathcal{A}$ , horizontal arrows of  $\mathcal{A}$ , and with 2-cells  $\varepsilon : g \to h$  being

$$\begin{array}{c} A \xrightarrow{h} B \\ 1 \downarrow & \varepsilon & \downarrow 1 \\ A \xrightarrow{g} B \end{array}$$

from  $\mathcal{A}$ . Similarly, Ver( $\mathcal{A}$ ) denotes the bicategory of objects of  $\mathcal{A}$ , vertical arrows of  $\mathcal{A}$ , and with 2-cells  $\delta \colon r \to s$  given by cells in  $\mathcal{A}$  of type

$$\begin{array}{c} A \xrightarrow{1} A \\ r \downarrow & \delta & \downarrow s \\ B \xrightarrow{1} & B. \end{array}$$

To keep the presentation simple and to avoid coherence issues, *in this section we only consider flat double categories*, that is, we assume that each square has at most one cell, and in this case we write

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} Y \\ r \downarrow & \leq & \downarrow s \\ A & \stackrel{q}{\longrightarrow} B. \end{array}$$
(5.i)

**Examples 5.1.** Our paradigmatic example of a double category is the double category  $\mathcal{V}$ - $\mathcal{Rel}$  that has sets as objects, functions as horizontal arrows and  $\mathcal{V}$ -relations as vertical arrows. Then, (5.i) holds if and only if  $g \cdot r \leq s \cdot f$  in  $\mathcal{V}$ -Rel. Note that Horiz( $\mathcal{V}$ - $\mathcal{Rel}$ ) = Set and Ver( $\mathcal{V}$ - $\mathcal{Rel}$ ) =  $\mathcal{V}$ -Rel.

We will also make use of the notion of companion and conjoint. Given an horizontal arrow  $f: A \rightarrow B$  in a double category  $\mathcal{A}$ , a *companion* for f is a vertical arrow  $f_*: A \rightarrow B$ 

in  $\mathcal{A}$  so that

$$\begin{array}{cccc} A & \stackrel{1}{\longrightarrow} A & & & & A & \stackrel{f}{\longrightarrow} B \\ 1 & \downarrow & \leq & \downarrow f_* & \text{ and } & & f_* & \downarrow & \leq & \downarrow 1 \\ A & \stackrel{f}{\longrightarrow} B & & & & B & \stackrel{f}{\longrightarrow} B; \end{array}$$

somewhat dually, a conjoint for f is a vertical arrow  $f^*\colon B \nrightarrow A$  so that

$$\begin{array}{cccc} A & \xrightarrow{f} & B & & B & \xrightarrow{1} & B \\ 1 & \downarrow & \leq & \downarrow f^* & \text{and} & f^* & \downarrow & \leq & \downarrow 1 \\ A & \xrightarrow{1} & A & & A & \xrightarrow{f} & B. \end{array}$$

We note that companions and conjoints are unique up to equivalence. Following Shulman [38], we say that  $\mathcal{A}$  is a *framed bicategory* if every horizontal arrow has a companion and a conjoint.

**Examples 5.2.** The double category  $\mathcal{V}$ - $\mathcal{Rel}$  is a framed bicategory. The companion of a function  $f: A \to B$  is the  $\mathcal{V}$ -relation  $f_{\circ}: A \to B$  and the conjoint is the  $\mathcal{V}$ -relation  $f^{\circ}: B \to A$ .

As observed by Clementino et al. [9], Cruttwell and Shulman [10], for a framed bicategory  $\mathcal{A}$  there is the framed bicategory  $\operatorname{Pro}(\mathcal{A})$  of the *procompletion* of  $\operatorname{Ver}(\mathcal{A})$ . The objects and the horizontal arrows of  $\operatorname{Pro}(\mathcal{A})$  are the same of  $\mathcal{A}$ , and a vertical arrow from an object X to an object Y in  $\operatorname{Pro}(\mathcal{A})$  is a down-directed set of vertical arrows from X to Y in  $\mathcal{A}$ . Moreover,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ R & \downarrow & \leq & \downarrow S \\ A & \xrightarrow{g} & B \end{array}$$

in  $Pro(\mathcal{A})$  if for all  $s \in S$  there is  $r \in R$  such that in  $\mathcal{A}$ ,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ r & \downarrow & \leq & \downarrow s \\ A & \xrightarrow{g} & B. \end{array}$$

Horizontal composition and identities are the same of  $\mathcal{A}$ , and the composite of composable vertical arrows is defined as

$$R' \circ R = \{r' \cdot r \mid r' \in R', r \in R\}$$

with the identity arrow on X being  $\{1_X\}$ . For every horizontal arrow f in  $Pro(\mathcal{A})$ , the set  $\{f_*\}$  is a companion for f, while the set  $\{f^*\}$  is a conjoint for f.

**Example 5.3.** The framed bicategory  $Pro(\mathcal{V}-Rel)$  allows modelling uniform structures [9].

We also recall that, given a framed bicategory  $\mathcal{A}$ , there is the framed bicategory Mon( $\mathcal{A}$ ) of monoids, monoid homomorphisms and bimodules in  $\mathcal{A}$  [38]. More concretely,

• a *monoid* in  $\mathcal{A}$  consists of an object A in  $\mathcal{A}$  together with a vertical arrow  $a: A \rightarrow A$  so that  $1 \leq a$  and  $a \circ a \leq a$ , • a *monoid homomorphism*  $f: (A, a) \to (B, b)$  consists of a horizontal arrow  $f: A \to B \in \mathcal{A}$  so that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & \leq & \downarrow b \\ A & \xrightarrow{f} & B. \end{array}$$

- a *bimodule* φ: (A, a) → (B, b) consists of a vertical arrow φ: A → B in A so that φ ∘ a ≤ φ and b ∘ φ ≤ φ.
- The cells of Mon( $\mathcal{A}$ ) and their compositions are the same as in  $\mathcal{A}$ .

For every horizontal arrow  $f: (A, a) \to (B, b)$  in Mon( $\mathcal{A}$ ), the vertical arrow  $b \circ f_*$  in  $\mathcal{A}$  is a companion for f, while the vertical arrow  $f^* \circ b$  in  $\mathcal{A}$  is a conjoint for f.

**Examples 5.4.** The framed bicategory  $Mon(\mathcal{V-Rel})$ , which we denote by  $\mathcal{V}$ -Dist, consists of  $\mathcal{V}$ -categories as objects,  $\mathcal{V}$ -functors as horizontal arrows and  $\mathcal{V}$ -distributors as vertical arrows. The framed bicategory  $Mon(Pro(\mathcal{V-Rel}))$ , which we denote by  $\mathcal{V}$ -qUnif, consists of  $\mathcal{V}$ -enriched quasiuniform spaces as objects,  $\mathcal{V}$ -uniformly continuous maps as horizontal arrows, and  $\mathcal{V}$ -promodules as vertical arrows [8]. In particular 2-qUnif is composed by quasiuniform spaces, uniformly continuous maps, and promodules. We also note that  $[0, \infty]_{\oplus}$ -qUnif is closely related to the notion of approach uniform space [30].

A *lax-double functor*  $\mathcal{F} \colon \mathcal{A} \to \mathcal{X}$  sends

$$\begin{array}{cccc} X & \xrightarrow{f} & Y & & \mathcal{F}X & \xrightarrow{\mathcal{F}f} & \mathcal{F}Y \\ r \downarrow & \leq & \downarrow s & \text{to} & \mathcal{F}r \downarrow & \leq & \downarrow \mathcal{F}s \\ A & \xrightarrow{g} & B & & \mathcal{F}A & \xrightarrow{\mathcal{F}g} & \mathcal{F}B \end{array}$$

and preserves horizontal composition and identities strictly and vertical composition and identities laxly. Therefore  $\mathcal{F}$ induces a 2-functor  $F = \text{Horiz}(\mathcal{F})$ :  $\text{Horiz}(\mathcal{A}) \rightarrow \text{Horiz}(\mathcal{X})$ and a lax functor  $\widehat{F} = \text{Ver}(\mathcal{F})$ :  $\text{Ver}(\mathcal{A}) \rightarrow \text{Ver}(\mathcal{X})$ . Following Shulman [38] again, a *lax-framed functor* is a lax-double functor between framed bicategories.

**Theorem 5.5.** A lax-framed functor  $\mathcal{F} : \mathcal{A} \to X$  corresponds precisely to a pair (F,  $\widehat{F}$ ), where F: Horiz( $\mathcal{A}$ )  $\to$  Horiz( $\mathcal{X}$ ) is a 2-functor and  $\widehat{F} : \operatorname{Ver}(\mathcal{A}) \to \operatorname{Ver}(\mathcal{X})$  is a lax functor, such that for every  $f : X \to Y \in \mathcal{A}$ ,  $F(f)_* \leq \widehat{F}(f_*)$  and  $F(f)^* \leq \widehat{F}(f^*)$ .

**Corollary 5.6.** Lax-double functors  $\mathcal{F} \colon \mathcal{V}\text{-}\mathcal{Rel} \to \mathcal{V}\text{-}\mathcal{Rel}$  correspond precisely to lax extensions  $\widehat{F} \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  of functors  $F \colon$  Set  $\to$  Set.

From a lax-framed functor  $\mathcal{F} \colon \mathcal{A} \to X$ , we obtain the lax-framed functor  $\operatorname{Pro}(\mathcal{F}) \colon \operatorname{Pro}(\mathcal{A}) \to \operatorname{Pro}(X)$ , which acts as  $\mathcal{F}$  on objects and horizonal arrows, and sends a vertical arrow  $R \colon X \to Y$  to the vertical arrow  $\{\mathcal{F}r \mid r \in R\}$ ; and the lax-framed functor  $\operatorname{Mon}(\mathcal{F}) \colon \operatorname{Mon}(\mathcal{A}) \to \operatorname{Mon}(X)$  given

by

$$\begin{array}{ll} (A,a) \stackrel{f}{\longrightarrow} (B,b) & (\mathcal{F}A,\mathcal{F}a) \stackrel{\mathcal{F}f}{\longrightarrow} (\mathcal{F}B,\mathcal{F}b) \\ \varphi \downarrow & \leq & \downarrow \psi & \longmapsto & \mathcal{F}\varphi \downarrow & \leq & \downarrow \mathcal{F}\psi \\ (X,c) \stackrel{}{\longrightarrow} (Y,d) & (\mathcal{F}X,\mathcal{F}c) \stackrel{}{\longrightarrow} (\mathcal{F}Y,\mathcal{F}d). \end{array}$$

**Examples 5.7.** A lax-framed functor  $(F, \widehat{F})$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel gives rise to the lax-framed functors:

1.  $Pro(F, \widehat{F})$ :  $Pro(\mathcal{V}\text{-Rel}) \rightarrow Pro(\mathcal{V}\text{-Rel})$  that applies F to sets and functions and  $\widehat{F}$ :  $\mathcal{V}\text{-Rel} \rightarrow \mathcal{V}\text{-Rel}$  to every  $\mathcal{V}\text{-}$  relation in a down-directed set of  $\mathcal{V}\text{-}$  relations;

2.  $Mon(F, \widehat{F})$ :  $Mon(\mathcal{V}\text{-}Rel) \rightarrow Mon(\mathcal{V}\text{-}Rel)$  that applies the lifting to  $\mathcal{V}\text{-}Cat$  induced by  $\widehat{F}: \mathcal{V}\text{-}Rel \rightarrow \mathcal{V}\text{-}Rel$  on  $\mathcal{V}\text{-}$ categories and  $\mathcal{V}\text{-}$ functors, and  $\widehat{F}$  on  $\mathcal{V}\text{-}$ distributors.

Now, we introduce the central notion of category of coalgebras of a lax-double functor.

**Definition 5.8.** Let  $\mathcal{F} : \mathcal{A} \to \mathcal{A}$  be a lax-double functor. The *double category of (horizontal) coalgebras* CoAlg( $\mathcal{F}$ ) is defined as follows:

- the objects of CoAlg(𝒫) are the coalgebras (𝐴, α) for
   F: Horiz(𝐴) → Horiz(𝐴),
- the horizontal arrows in  $CoAlg(\mathcal{F})$  between objects  $(A, \alpha)$  and  $(B, \beta)$  are coalgebra homomorphisms for F: Horiz $(\mathcal{A}) \rightarrow$  Horiz $(\mathcal{A})$  from  $(A, \alpha)$  to  $(B, \beta)$ ,
- the vertical arrows in CoAlg(F) between objects (A, α) and (B, β) are F-simulations, that is, vertical arrows s: X → Y in A so that

$$\begin{array}{ccc}
A & \stackrel{\alpha}{\longrightarrow} & \mathsf{F}A \\
s & \downarrow & \leq & \downarrow \widehat{\mathsf{F}s} \\
B & \stackrel{\beta}{\longrightarrow} & \mathsf{F}B
\end{array}$$
(5.ii)

in  $\mathcal{A}$  [33], and finally

• the cells in CoAlg( $\mathcal{F}$ ) and their compositions are the same as in  $\mathcal{A}$ .

*Remark* 5.9. If  $\alpha : A \to FA$  and  $\beta : B \to FB$  have companions in  $\mathcal{A}$ , then (5.ii) is equivalent to  $\beta_* \circ s \leq \widehat{Fs} \circ \alpha_*$  in Ver( $\mathcal{A}$ ) [19, 1.6. Ortogonal flipping].

**Example 5.10.** Double categories of coalgebras of a laxframed endofunctor on  $\mathcal{V}$ - $\mathcal{Rel}$  provide a framework to reason about *indistinguishability* in Set-coalgebras via  $\mathcal{V}$ -relations (e.g. [16, 22, 41]).

**Example 5.11.** Double categories of coalgebras of a laxframed endofunctor on  $\mathcal{V}$ - $\mathcal{D}ist$  provide a framework to reason about *indistinguishability* in  $\mathcal{V}$ -Cat-coalgebras via  $\mathcal{V}$ distributors (e.g. [33, 43]).

**Example 5.12.** Double categories of coalgebras of a laxframed endofunctor on 2-*ProRel* provide a framework to reason about *indistinguishability* in Set-coalgebras via quasiuniformities. For instance, in the double category of coalgebras of the lax-framed functor  $\operatorname{Pro}(\mathsf{F}, \widehat{\mathsf{F}}) : 2\operatorname{-}\operatorname{Pro}\operatorname{Rel} \to 2\operatorname{-}\operatorname{Pro}\operatorname{Rel}$ , a down-directed set *R* of  $\mathcal{V}$ -relations from *A* to *B* is a  $\operatorname{Pro}(\mathsf{F}, \widehat{\mathsf{F}})$ -simulation from an F-coalgebra  $(A, \alpha)$  to a F-coalgebra  $(B, \beta)$  whenever *R* is "jointly an  $\widehat{\mathsf{F}}$ -simulation"; that is, for every  $r \in R$ , there is  $s \in R$  such that  $\beta \cdot s \leq \widehat{\mathsf{F}}r \cdot \alpha$  in Rel.

**Theorem 5.13.** Let  $\mathcal{F} \colon \mathcal{A} \to \mathcal{A}$  be a lax-framed functor. Then  $\operatorname{CoAlg}(\mathcal{F})$  is a framed bicategory.

**Definition 5.14.** We call a framed bicategory  $\mathcal{A}$  *locally complete* whenever the ordered category Ver( $\mathcal{A}$ ) has complete hom-sets.

**Example 5.15.** For every quantale  $\mathcal{V}$ , the double category  $\mathcal{V}$ - $\mathcal{Rel}$  is locally complete. Moreover, the framed bicategories  $\operatorname{Pro}(\mathcal{A})$  and  $\operatorname{Mon}(\mathcal{A})$  are locally complete whenever the framed bicategory  $\mathcal{A}$  is locally complete. In particular, the framed bicategories  $\mathcal{V}$ - $\mathcal{ProRel}$  and  $\mathcal{V}$ - $\mathcal{Dist}$  are locally complete.

**Theorem 5.16.** Let  $\mathcal{F} \colon \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. Then  $\operatorname{CoAlg}(\mathcal{F})$  is locally complete.

**Definition 5.17.** Let  $\mathcal{F}: \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. Let  $(A, \alpha)$  and  $(B, \beta)$  be  $\mathcal{F}$ -coalgebras. The  $\mathcal{F}$ -similarity from  $(A, \alpha)$  to  $(B, \beta)$  is the greatest  $\mathcal{F}$ -simulation from  $(A, \alpha)$  to  $(B, \beta)$ , and we denote it by  $\top_{\alpha,\beta}$ , or by  $\top_{\alpha}$ , if  $\alpha = \beta$ .

**Example 5.18.** For a lax-framed functor  $\mathcal{F}: \mathcal{V}-\mathcal{Rel} \to \mathcal{V}-\mathcal{Rel}, \mathcal{F}$ -similarity recovers the notions of  $\widehat{\mathsf{F}}$ -similarity and  $\widehat{\mathsf{F}}$ -behavioural distance considered by Hughes and Jacobs [22] and Wild and Schröder [41] when  $\mathcal{V}$  is equal to 2 or  $[0,1]_{\oplus}$ , respectively.

The following result, which is an immediate consequence of the fact that right adjoints preserve limits, show that  $\mathcal{F}$ -similarity is compatible with coalgebra homomorphisms.

**Theorem 5.19.** Let  $\mathcal{F} \colon \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. For every pair of horizontal arrows

$$(A, \alpha) \xrightarrow{f} (C, \gamma) \qquad (B, \beta) \xrightarrow{g} (D, \delta)$$

 $in \operatorname{CoAlg}(\mathcal{F}), \, \top_{\alpha,\beta} = g^* \circ \top_{\gamma,\delta} \circ f_*.$ 

Next we explain how the previous results allow connecting the coalgebraic treatments of similarity and behavioural distance via lax extensions and via liftings, respectively. We recall that there is a canonical lax-framed functor  $Mon(\mathcal{A}) \rightarrow \mathcal{A}$  that forgets the monoid structures. Furthermore, for every lax-framed functor  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}$  on a locally complete framed bicategory, the forgetful functor

 $\operatorname{Horiz}(\operatorname{CoAlg}(\operatorname{Mon}(\mathcal{F}))) \longrightarrow \operatorname{Horiz}(\operatorname{CoAlg}(\mathcal{F}))$ 

is topological [6, 20] and therefore has a right adjoint

 $\mathsf{gfp}\colon \mathsf{Horiz}(\mathsf{CoAlg}(\mathcal{F})) \to \mathsf{CoAlg}(\mathsf{Mon}(\mathcal{F}))$ 

that sends an  $\mathcal{F}$ -coalgebra  $(A, \alpha)$  to the Mon $(\mathcal{F})$ -coalgebra  $(A, a_{\alpha}, \alpha)$  where  $a_{\alpha}$  is given by

$$\bigvee \{A \xrightarrow{a} A \text{ in } \mathcal{A} \mid \alpha \colon (A, a) \to \operatorname{Mon}(\mathcal{F})(A, a) \text{ in } \operatorname{Mon}(\mathcal{A})\}$$
(5.iii)

**Definition 5.20.** Let  $\mathcal{F}$  be a lax-framed endofunctor on a locally complete framed bicategory. The  $\mathcal{F}$ -*behavioural distance* on a  $\mathcal{F}$ -coalgebra (A,  $\alpha$ ), denoted by  $bd_{\alpha}$ , is the monoid structure on A given by gfp(A,  $\alpha$ ).

**Example 5.21.** For a lax-framed functor  $\mathcal{F}: 2\text{-}ProRel \rightarrow 2\text{-}ProRel$ , the notion of  $\mathcal{F}$ -behavioural distance corresponds essentially to the notion of *bisimilarity uniformity* [25].

**Example 5.22.** For  $\mathcal{V} = [0, 1]_{\oplus}$ , it follows from the fact that every lifting induced by a lax extension is Kantorovich (see Example 3.13(1)), that the notion of  $\mathcal{F}$ -behavioural distance for a lax-framed functor  $(F, \widehat{F}) : \mathcal{V}-\mathcal{Rel} \to \mathcal{V}-\mathcal{Rel}$  corresponds to the asymmetric version of the notion of *behavioural distance* [26] – generalised to predicate liftings of arbitrary arity – with respect to the lifting of F: Set  $\to$  Set to  $\mathcal{V}$ -Cat induced by the lax extension  $\widehat{F} : \mathcal{V}$ -Rel  $\to \mathcal{V}$ -Rel.

The following result, in particular, connects the approaches to *indistinguishability* via lax extensions and via liftings.

**Theorem 5.23.** Let  $\mathcal{F}$  be a lax-framed endofunctor on a locally complete framed bicategory. Then,  $\mathcal{F}$ -similarity and  $\mathcal{F}$ -behavioural distance coincide on every  $\mathcal{F}$ -coalgebra.

*Remark* 5.24. As a consequence of Theorems 5.23 and 4.2 we obtain that the notion of  $\mathcal{F}$ -similarity provided by a lax extension corresponding to a class of monotone predicate liftings coincides with the notion of behavioural distance provided by the lifting associated with the same class of predicate liftings.

Remark 5.25. Baldan et al. [3] restricted the definition of bisimilarity pseudometric (= behavioural distance) to liftings of Set-functors that admit a terminal coalgebra. Arguably, the reason behind this restriction is that if we allow an arbitrary lifting  $\overline{F} : \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat of an arbitrary functor  $F : Set \rightarrow$ Set, then it is easy to see that the notion of behavioural distance arising from the functor gfp:  $CoAlg(F) \rightarrow CoAlg(\overline{F})$ as described above is not necessarily invariant under coalgebra homomorphisms. In this regard, Theorem 5.19 and Theorem 5.23 show that for liftings induced by lax extensions the notion of  $\mathcal{F}$ -behavioural distance is invariant under coalgebra homomorphisms. Therefore, whenever a functor  $F: Set \rightarrow Set$  admits a terminal coalgebra, the definition of  $(F, \widehat{F})$ -behavioural distance coincides with the one considered by Baldan et al. [3] for the lifting  $\widehat{F}$ :  $[0,1]_{\oplus}$ -Cat  $\rightarrow$ [0, 1]<sub>⊕</sub>-Cat.

We conclude this section by showing how we can use the framed bicategory  $\mathcal{V}$ - $\mathcal{D}ist$  to recover coinduction results presented by Worrell [43]. In particular, this is as an example of a context where, by choosing a double category different from  $\mathcal{V}$ - $\mathcal{Rel}$ , we can reason about liftings not necessarily induced by a lax extension. Recall that Worrell considers a functor F:  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat – not necessarily a lifting – that admits a terminal coalgebra and preserves initial morphisms, and shows, for example, that the  $\mathcal{V}$ -category structure on the terminal coalgebra coincides with the greatest  $\mathcal{F}$ -simulation with respect to a lax-framed endofunctor on  $\mathcal{V}$ - $\mathcal{Dist}$ . This result seems to be outside the scope of our setting, however, as we show next, we are able to include it by applying the *Mon* construction twice.

For a double category  $\mathcal{A}$ , an object of  $Mon(Mon(\mathcal{A}))$  is given by an  $\mathcal{A}$ -object A together with vertical  $\mathcal{A}$ -arrows  $a_0 \leq a_1 \colon A \to A$  making  $(A, a_0)$  and  $(A, a_1)$  monoids. Moreover, given also an  $Mon(Mon(\mathcal{A}))$ -object  $(B, b_0, b_1)$ , a horizontal arrow  $f \colon A \to B$  in  $\mathcal{A}$  is a horizontal arrow  $f \colon (A, a_0, a_1) \to (B, b_0, b_1)$  in  $Mon(Mon(\mathcal{A}))$  if and only if  $f \colon (A, a_0) \to (B, b_0)$ and  $f \colon (A, a_1) \to (B, b_1)$  are horizontal arrows in  $Mon(\mathcal{A})$ .

In the light of Theorem 5.23, the following result corresponds to [43, Theorem 5.7].

**Proposition 5.26.** Let  $\mathcal{A}$  be a framed bicategory and let  $\mathcal{F} \colon \operatorname{Mon}(\mathcal{A}) \to \operatorname{Mon}(\mathcal{A})$  be a lax-double functor preserving vertical identities. If  $(C, c, \gamma)$  is a terminal  $\mathcal{F}$ -coalgebra, then  $(C, c, c, \gamma)$  is a terminal  $\operatorname{Mon}(\mathcal{F})$ -coalgebra.

**Example 5.27.** A 2-functor  $F: \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat preserving initial morphisms extends to a identity preserving lax functor  $\widehat{F}: \mathcal{V}$ -Dist  $\rightarrow \mathcal{V}$ -Dist [43], and these data gives rise to a vertical identity preserving lax-double functor  $\mathcal{F}: \mathcal{V}$ -Dist  $\rightarrow \mathcal{V}$ -Dist. Let  $(C, c, \gamma)$  be a terminal F-coalgebra and  $(A, a, \alpha)$  be an F-coalgebra. Applying Theorem 5.23 and Proposition 5.26 to this situation, we obtain that the  $\mathcal{F}$ -similarity and the  $\mathcal{F}$ -behavioural distance of  $(A, a, \alpha)$  coincide, moreover, bd<sub> $\alpha$ </sub> = !\*  $\circ c \circ$ !\* [43, Theorem 5.10]. Considering the case of a discrete  $\mathcal{V}$ -category  $(A, 1_A, \alpha)$ , this specialises to [3, Theorem 6.7].

### 6 Conclusions and Future Work

We have completed the diagram of known correspondences between three systematic ways of inducing behavioural distances on quantitative state-based systems: via functor liftings, via predicate liftings, and via lax extensions. Specifically, we have shown that every functor lifting that preserves initial morphisms is *Kantorovich*, i.e. it is canonically induced by a suitable choice of – not necessarily monotone – predicate liftings; and that essentially we can always assume that every functor lifting is induced by a suitable choice of predicate liftings, although not in a canonical way. Moreover, we have shown that a functor lifting  $\overline{F}$  that preserves initial morphisms is induced by a lax extension if and only if every predicate lifting induced by  $\overline{F}$  is monotone. Finally, we have introduced the double category of coalgebras of a lax-double functor as a flexible framework to reason about *indistinguishability*. In particular, we have shown that the notions of similarity and behavioural distance induced by a lax-double functor coincide. This result applies simultaneously to quantitative relations, distributors, and uniformities, for instance. One important issue for future research is to investigate the modal logics of functor liftings that arise from our results, in particular showing expressiveness in the style of Hennessy-Milner, complementing existing results that rely on monotonicity of predicate liftings [27, 41].

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#### **A** Appendix: Omitted Proofs

#### A.1 Omitted proofs and details for Section 3

**Lemma** (3.1). Let  $\overline{\mathsf{F}}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting of a functor  $\mathsf{F}$ : Set  $\rightarrow$  Set and X a  $\mathcal{V}$ -category. Given a family of natural transformations

$$\mu^{i}: \mathcal{V}\text{-}Cat(-, A_{i}) \longrightarrow \mathcal{V}\text{-}Cat(\overline{\mathsf{F}}, \mathcal{V}^{\mathrm{op}}) \quad (for \ i \in I)$$

if the cocone

$$(\mu_X^i: \mathcal{V}\text{-}\operatorname{Cat}(X, A_i) \longrightarrow \mathcal{V}\text{-}\operatorname{Cat}(\overline{\mathsf{F}}X, \mathcal{V}^{\operatorname{op}}))_{i \in I}$$

is jointly epic, then the topological lifting with respect to the class of all natural transformations

$$\mu^i : \mathcal{V}\text{-}Cat(-, A_i) \to Set(\mathsf{F}|-|, |\mathcal{V}^{op}|)$$

that factor as

$$\mathcal{V}\text{-}\mathsf{Cat}(-,A_i) \xrightarrow{\mu^i} \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}-,\mathcal{V}^{\mathrm{op}})$$

$$\downarrow^{|-|_{-,\mathcal{V}^{\mathrm{op}}}}$$

$$\mathsf{Set}(\mathsf{F}|-|,|\mathcal{V}^{\mathrm{op}}|)$$

coincides with  $\overline{F}$  on X.

*Proof.* By hypothesis, the topological lifting maps the  $\mathcal{V}$ -category X to the domain of the initial lift of the structured cone

$$(|f|: \mathsf{F}|X| \longrightarrow |\mathcal{V}^{\mathrm{op}}|)_{f \in \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}X, \mathcal{V}^{\mathrm{op}})}.$$

Therefore, the claim follows from the fact that the cone  $\mathcal{V}$ -Cat( $\overline{\mathsf{F}}X, \mathcal{V}^{\mathrm{op}}$ ) is initial (see Remark 2.6).

**Theorem** (3.2). Every lifting  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat is topological.

Proof. The Yoneda lemma guarantees that there is an epi-cocone of natural transformations

$$\mu_i: \mathcal{V}\text{-}Cat(-, A_i) \longrightarrow \mathcal{V}\text{-}Cat(\overline{\mathsf{F}}, \mathcal{V}^{\mathrm{op}})$$

indexed by the class *I* of pairs (A, f) consisting of a  $\mathcal{V}$ -category *A* and a  $\mathcal{V}$ -functor  $\overline{\mathsf{F}}A \to \mathcal{V}^{\mathrm{op}}$ .

**Theorem** (3.3). For every lifting  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat there is a topological lifting constructed from a class of predicate liftings of F that agrees with  $\overline{F}$  on every non-empty  $\mathcal{V}$ -category.

*Proof.* Let  $\overline{F}: \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting. If  $\mathcal{V}$  is the trivial quantale then there is only one lifting of F to  $\mathcal{V}$ -Cat  $\simeq$  Set. Therefore, the lifting with respect to the empty class coincides with  $\overline{F}$ . Now suppose that  $\mathcal{V}$  is non-trivial. Consider the class of all natural transformations of type

$$\mathcal{V}$$
-Cat $(-, (\mathcal{V}^{\kappa}, a)) \longrightarrow \mathcal{V}$ -Cat $(\overline{\mathsf{F}}, \mathcal{V}^{\mathrm{op}})$ 

where, for some cardinal  $\kappa$ , ( $\mathcal{V}^{\kappa}$ , a) is an arbitrary  $\mathcal{V}$ -category based on the set  $\mathcal{V}^{\kappa}$ . In light of Lemma 3.1, we claim that for every non-empty  $\mathcal{V}$ -category X the cocone formed by the X-components of the natural transformations described above is epi.

Let (X, a) be a non-empty  $\mathcal{V}$ -category of cardinality  $\kappa$  and  $f : \overline{\mathsf{F}}(X, a) \to \mathcal{V}^{\mathrm{op}}$  a  $\mathcal{V}$ -functor. Then, since  $\mathcal{V}$  is non-trivial, there is a monomorphism  $m : X \to \mathcal{V}^{\kappa}$ . Consequently, as X is non-empty, there is a map  $g : \mathcal{V}^{\kappa} \to X$  such that  $1_X = g \cdot m$ . This factorisation lifts to a factorisation of the identity  $\mathcal{V}$ -functor on (X, a) via the  $\mathcal{V}$ -category  $(\mathcal{V}^{\kappa}, m_a)$  given by the final structure with respect to the structured map  $m : (X, a) \to \mathcal{V}^{\kappa}$ . Therefore, the (X, a)-component of the natural transformation

$$\begin{array}{c} \mathcal{V}\text{-}\mathsf{Cat}(-,(\mathcal{V}^{\kappa},m_{a})) \\ g \cdot - \downarrow & & \downarrow \\ \mathcal{V}\text{-}\mathsf{Cat}(-,(X,a)) \xrightarrow{f \cdot \overline{\mathsf{F}}(g \cdot -)} \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}-,\mathcal{V}^{\mathrm{op}}) \end{array}$$

sends the  $\mathcal{V}$ -functor  $m: (X, a) \to (\mathcal{V}^{\kappa}, m_a)$  to the  $\mathcal{V}$ -functor  $f: (X, a) \to \mathcal{V}^{\text{op}}$ .

The following example illustrates Remark 3.6.

**Example A.1.** Let  $\overline{F}$ : Ord  $\rightarrow$  Ord be the discrete lifting of the Set-functor that sends the empty set to 2 and all other sets to 1. Then, the constant maps are the only monotone functions  $(2, 1_2) \rightarrow 2^{\text{op}}$  that factor via  $(1, 1_1)$ . Therefore, the topological lifting corresponding to the class of natural transformations of the proof of Theorem 3.3 sends the empty ordered set to the indiscrete ordered set with two elements.

**Theorem** (3.11). Let  $F: Set \to Set$  be a functor. Assigning to a class of predicate liftings of F the corresponding Kantorovich lifting yields a right adjoint  $F^{(-)}: Pred(F) \to Lift(F)$  whose left adjoint  $P: Lift(F) \to Pred(F)$  maps a lifting of F to its class of predicate liftings.

*Proof.* It is clear that the assignments  $F^{(-)}$ :  $Pred(F) \rightarrow Lift(F)$  and P:  $Lift(F) \rightarrow Pred(F)$  define functors. The first by definition of Kantorovich lifting, and the latter because for all liftings  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat and  $\overline{F}'$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat such that  $\overline{F} \leq \overline{F}'$ , and every  $\mathcal{V}$ -functor  $f: \overline{F}'(\mathcal{V}^{\kappa})^{op} \rightarrow \mathcal{V}^{op}$ , the map

$$\overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}} \xrightarrow{f} \overline{\mathsf{F}}'(\mathcal{V}^{\kappa})^{\mathrm{op}} \xrightarrow{f} \mathcal{V}^{\mathrm{op}}$$

is a  $\mathcal{V}$ -functor.

Furthermore, by definition of Kantorovich lifting, for every predicate lifting  $\mu: P_{\mathcal{V}^{\kappa}} \to P_{\mathcal{V}}F$  in a class of predicate liftings M of F, the map  $\mu(1_{\mathcal{V}^{\kappa}}): F^{M}(\mathcal{V}^{\kappa})^{\text{op}} \to \mathcal{V}^{\text{op}}$  is a  $\mathcal{V}$ -functor. Therefore,  $P(F^{M}) \supseteq M$ . On the other hand, for every predicate lifting  $\mu: P_{\mathcal{V}^{\kappa}} \to P_{\mathcal{V}}F$  of a lifting  $\overline{F}: \mathcal{V}$ -Cat and every  $\mathcal{V}$ -functor  $f: (X, a) \to (\mathcal{V}^{\kappa})^{\text{op}}$ ,

$$\overline{\mathsf{F}}(X,a) \xrightarrow[\mathbf{F}_{f}]{} \overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}} \xrightarrow[\mu(1_{\mathcal{V}^{\kappa}})]{} \mathcal{V}^{\mathrm{op}}$$

is a  $\mathcal{V}$ -functor. Therefore,  $\overline{\mathsf{F}}(X, a) \leq \mathsf{F}^{\mathsf{P}(\overline{\mathsf{F}})}(X, a)$ .

**Theorem** (3.12). Let  $F: \text{Set} \to \text{Set}$  be a functor. A lifting  $\overline{F}: \mathcal{V}\text{-}\text{Cat} \to \mathcal{V}\text{-}\text{Cat}$  of F is Kantorovich if and only if it preserves initial morphisms (=fully faithfull  $\mathcal{V}$ -functors).

*Proof.* Firstly, we show that every Kantorovich lifting preserves initial morphisms.

Let  $i: (X, a) \to (Y, b)$  be an initial  $\mathcal{V}$ -functor, M a class of predicate liftings of  $\mathsf{F}, j: (Z, c) \to \mathsf{F}^M(Y, b)$  a  $\mathcal{V}$ -functor, and  $h: Z \to \mathsf{F}X$  a map such that  $j = \mathsf{F}i \cdot h$ .

By definition of  $\mathsf{F}^M$  it is sufficient to show that for every  $\mu: \mathsf{P}_{\mathcal{V}^{\kappa}} \to \mathsf{P}_{\mathcal{V}}\mathsf{F} \in M$  and every  $\mathcal{V}$ -functor  $f: (X, a) \to (\mathcal{V}^{\kappa})^{\mathrm{op}}$ ,  $\mu(f) \cdot h$  is a  $\mathcal{V}$ -functor. Since  $\mathcal{V}^{\mathrm{op}}$  is injective in  $\mathcal{V}$ -Cat with respect to initial morphisms,  $(\mathcal{V}^{\kappa})^{\mathrm{op}}$  is also injective in  $\mathcal{V}$ -Cat with respect to initial morphisms. Hence, for every  $\mathcal{V}$ -functor  $f: (X, a) \to (\mathcal{V}^{\kappa})^{\mathrm{op}}$  there is a  $\mathcal{V}$ -functor  $g: (Y, b) \to (\mathcal{V}^{\kappa})^{\mathrm{op}}$  such that  $f = g \cdot i$ . Consequently,

$$\mu(f) \cdot h = \mu(g \cdot i) \cdot h = \mu(g) \cdot \mathsf{F}i \cdot h = \mu(g) \cdot j$$

is a  $\mathcal V\text{-}\mathsf{functor}.$ 

Secondly, we show that the converse statement holds. Suppose that  $\overline{\mathsf{F}}$  is a lifting that preserves initial morphisms. We already know from Theorem 3.11 that  $\overline{\mathsf{F}} \leq \mathsf{F}^{\mathsf{P}(\overline{\mathsf{F}})}$ . To prove that under our assumption the reverse inequality also holds, let (X, a) be a  $\mathcal{V}$ -category and  $\kappa = |X|$ . Then, the (co)yoneda embedding  $(X, a) \rightarrow [(X, a), \mathcal{V}]^{\mathrm{op}}$  gives us an initial  $\mathcal{V}$ -functor



Hence, since  $\overline{\mathsf{F}}$  preserves initial morphisms,  $\overline{\mathsf{F}} h : \overline{\mathsf{F}}(X, a) \to \overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}}$  is initial. Now, let  $\mathsf{P}_{\kappa}(\overline{\mathsf{F}})$  denote the set of all  $\kappa$ -ary predicate liftings in  $\mathsf{P}(\overline{\mathsf{F}})$ . Given that the cone of all  $\mathcal{V}$ -functors

$$\overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}} \longrightarrow \mathcal{V}^{\mathrm{op}}$$

is initial and the composition of initial cones is initial, the cone

$$(\mu(h): \mathsf{F}(X, a) \longrightarrow \mathcal{V}^{\mathrm{op}})_{\mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})}$$

is initial. Therefore,  $F^{P(\overline{F})}(X, a) \leq \overline{F}(X, a)$ .



**Remark** (3.14). The composite of  $\mathcal{V}$ -Cat-functors that preserve initial morphisms preserves initial morphisms. Consequently, the composite of Kantorovich liftings is Kantorovich. In particular, for every Kantorovich lifting  $F^I: \mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat, by precomposing and postcomposing

$$(-)^{\circ} \colon \mathcal{V}\text{-}Cat \to \mathcal{V}\text{-}Cat$$

with  $F^{I}$  we obtain a lifting defined by the assignment

$$(X, a) \longmapsto \mathsf{F}^{I}(X, a^{\circ})^{\operatorname{op}}.$$

This is the topologial lifting constructed from the class of predicate liftings I where every  $\kappa$ -ary predicate lifting gives rise to the natural transformation

$$\mathcal{V}$$
-Cat $(-, (\mathcal{V}^{\kappa})) \longrightarrow$  Set $(\mathsf{F}|-|, |\mathcal{V}|)$ .

Therefore, choosing  $\mathcal{V}$  or  $\mathcal{V}^{\text{op}}$  as the base category to define the notion of Kantorovich lifting leads to the same class of functors. Our choice here prevents a mismatch in Section 4 when we compare the Kantorovich liftings and the liftings induced by the Kantorovich extension [18, 41].

#### A.2 Omitted proofs and details for Section 4

In the sequel we denote the  $\mathcal{V}$ -relation associated with a function  $f: X \to \mathcal{V}^{\kappa}$  by  $f^{\flat}: X \to \kappa$ ,

**Proposition A.2.** Let (X, a) be a  $\mathcal{V}$ -category,  $\kappa$  a cardinal and  $f: X \to \mathcal{V}^{\kappa}$  a function. The following propositions are equivalent: (i)  $f: (X, a) \to (\mathcal{V}^{\kappa})^{\text{op}}$  is a  $\mathcal{V}$ -functor;

(i)  $f: (X, a) \rightarrow (V) \rightarrow is a V - functor,$ (ii)  $a \leq f^{\flat} \rightarrow f^{\flat};$ (iii)  $f^{\flat} \cdot a = f^{\flat};$ (iv)  $f^{\flat}: (X, a) \rightarrow (\kappa, 1_{\kappa})$  is a V-distributor.

**Corollary A.3.** Let  $C = (f_i \colon X \to (\mathcal{V}^{\kappa})^{\mathrm{op}})_{i \in I}$  be a structured cone. The initial structure of  $\mathcal{V}$ -category on X with respect to C is given by

$$\bigwedge_{i\in I} f_i^{\flat} \bullet f_i^{\flat}.$$

**Proposition** (4.1). Let  $\mu: P_{\mathcal{V}^{\kappa}} \to PF$  be a  $\kappa$ -ary predicate lifting of a functor  $F: Set \to Set$ . The Kantorovich lifting  $F^{\mu}: \mathcal{V}$ -Cat  $\to \mathcal{V}$ -Cat of F with respect to  $\mu$  sends a  $\mathcal{V}$ -category (X, a) to the  $\mathcal{V}$ -category  $(FX, F^{\mu}a)$ , where

$$\mathsf{F}^{\mu}a = \bigwedge_{r \colon (X,a) \Leftrightarrow (\kappa, 1_{\kappa})} \mu(r) \twoheadrightarrow \mu(r).$$

**Lemma A.4.** Let  $\mu: \mathsf{P}_{\mathcal{V}^{\kappa}} \to \mathsf{P}_{\mathcal{V}}\mathsf{F}$  be a  $\kappa$ -ary monotone predicate lifting and (X, a) a  $\mathcal{V}$ -category. Then,

$$\bigwedge_{g: X \to \kappa} \mu(g) \to \mu(g \cdot a) = \bigwedge_{r: (X,a) \Leftrightarrow (\kappa, 1_{\kappa})} \mu(r) \to \mu(r).$$

*Proof.* Let  $g: X \to \kappa$  be a  $\mathcal{V}$ -relation. Since  $1_X \leq a$ , we have  $g \leq g \cdot a$ . Also, given that  $a \cdot a \leq a, g \cdot a: (X, a) \Leftrightarrow (\kappa, 1_{\kappa})$  is a  $\mathcal{V}$ -distributor. Therefore, because  $\mu$  is monotone,

 $\mu(g \cdot a) \twoheadrightarrow \mu(g \cdot a) \le \mu(g) \twoheadrightarrow \mu(g \cdot a).$ 

The other inequality is an immediate consequence of Proposition A.2((iii)).

**Theorem** (4.2). Let  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting of a functor F: Set  $\rightarrow$  Set induced by a lax extension  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel. If  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel is the Kantorovich extension with respect to a class M of predicate liftings, then the functor  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat is the Kantorovich lifting of F: Set  $\rightarrow$  Set with respect to M.

**Proposition** (4.6). Every predicate lifting of a lax extension  $\widehat{F} \colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$  is also a predicate lifting of the corresponding lifting  $\widehat{F} \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ .

*Proof.* Let  $\mu: \mathbb{P}_{\mathcal{V}^{\kappa}} \to \mathbb{P}_{\mathcal{V}} \mathbb{F}$  be a predicate lifting of a lax extension  $\widehat{\mathbb{F}}: \mathcal{V}$ -Rel  $\to \mathcal{V}$ -Rel and  $h: \mathcal{V}^{\kappa} \to \mathcal{V}^{\kappa}$  the structure of the  $\mathcal{V}$ -category  $(\mathcal{V}^{\kappa})^{\mathrm{op}}$ . Then,

$$\mu(\operatorname{ev}_{\kappa}) \cdot \widehat{\mathsf{F}}h = \mu(1_{\kappa}) \cdot \mathsf{F}(1_{\kappa}^{\sharp})^{\circ} \cdot \widehat{\mathsf{F}}h \cdot \widehat{\mathsf{F}}h$$
$$= \mu(1_{\kappa}) \cdot \mathsf{F}(1_{\kappa}^{\sharp})^{\circ} \cdot \widehat{\mathsf{F}}h$$
$$= \mu(\operatorname{ev}_{\kappa}).$$

Therefore, the claim follows from Proposition A.2.

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**Lemma** (4.8). Let  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting induced by a lax extension  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel. Then, for every monotone predicate lifting  $\mu$ :  $P_{\mathcal{V}^{\kappa}} \rightarrow P_{\mathcal{V}}F$  of  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat,  $\widehat{F} \leq \widehat{F}^{\mu}$ .

*Proof.* Let  $r: X \to Y$  and  $g: Y \to \kappa$  be  $\mathcal{V}$ -relations and  $h: \mathcal{V}^{\kappa} \to \mathcal{V}^{\kappa}$  the structure of the  $\mathcal{V}$ -category  $(\mathcal{V}^{\kappa})^{\text{op}}$ . The fact that  $\mu(1_{\mathcal{V}^{\kappa}}): \overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\text{op}} \to \mathcal{V}^{\text{op}}$  is a  $\mathcal{V}$ -functor, by Proposition A.2, translates to

$$\mu(\mathrm{ev}_{\kappa}) \cdot Fh = \mu(\mathrm{ev}_{\kappa})$$

in the language of  $\mathcal V$ -relations. Therefore, it follows from [18] and (L2) that

$$\begin{split} \widehat{\mathsf{F}}r &\leq \widehat{\mathsf{F}}(h \cdot g^{\sharp}) \twoheadrightarrow \widehat{\mathsf{F}}(h \cdot g^{\sharp} \cdot r) \\ &\leq \mu(\mathrm{ev}_{\kappa}) \cdot \widehat{\mathsf{F}}(h \cdot g^{\sharp}) \twoheadrightarrow \mu(\mathrm{ev}_{\kappa}) \cdot \widehat{\mathsf{F}}(h \cdot (g \cdot r)^{\sharp}) \\ &= \mu(\mathrm{ev}_{\kappa}) \cdot \widehat{\mathsf{F}}h \cdot \mathsf{F}g^{\sharp} \twoheadrightarrow \mu(\mathrm{ev}_{\kappa}) \cdot \widehat{\mathsf{F}}h \cdot \mathsf{F}(g \cdot r)^{\sharp} \\ &= \mu(\mathrm{ev}_{\kappa}) \cdot \mathsf{F}g^{\sharp} \twoheadrightarrow \mu(\mathrm{ev}_{\kappa}) \cdot \mathsf{F}(g \cdot r)^{\sharp} \\ &= \mu(g) \twoheadrightarrow \mu(g \cdot r). \end{split}$$

**Theorem** (4.9). Let  $F: \text{Set} \to \text{Set}$  be a functor. The monotone map MPI:  $\text{Lax}(F) \to \text{Pred}(F)_M$  is left adjoint to the monotone map  $\widehat{F}^{(-)}: \text{Pred}(F)_M \to \text{Lax}(F)$ .

*Proof.* Let *M* be a class of monotone predicate liftings of F. Then, by Theorem 4.2 and the fact that MP:  $Lift(F)_{I} \rightarrow Pred(F)_{M}$  is left adjoint to  $F^{(-)}$ :  $Pred(F)_{M} \rightarrow Lift(F)_{I}$ ,

$$MPI(\widehat{F}^M) = MP(F^M) \le M.$$

The other inequality follows immediately from Lemma 4.8.

**Lemma** (4.10). Let  $\overline{\mathsf{F}}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting. For every cardinal  $\kappa$ , let  $h_{\kappa}$  denote the structure of the  $\mathcal{V}$ -category  $(\mathcal{V}^{\kappa})^{\mathrm{op}}$ . The following are equivalent:

- (*i*) every predicate lifting of  $\overline{F}$  is monotone;
- (ii) for every cardinal  $\kappa$  and every pair  $p, q: X \leftrightarrow \kappa$  of V-relations,

$$p \le q \implies \mathsf{F}p^{\sharp} \le \overline{\mathsf{F}}h_{\kappa} \cdot \mathsf{F}q^{\sharp},$$

where  $r^{\sharp} \colon X \to \mathcal{V}^Y$  denotes the map corresponding to the  $\mathcal{V}$ -relation  $r \colon X \to Y$ .

*Proof.* We begin by proving  $(i) \implies (ii)$ . Let  $\kappa$  be a cardinal and  $p, q: X \rightarrow \kappa \mathcal{V}$ -relations such that  $p \leq q$ . Let  $\mathsf{P}_{\kappa}(\overline{\mathsf{F}})$  denote the set of  $\kappa$ -ary predicate liftings of  $\overline{\mathsf{F}}$ . Then, by hypothesis,

$$1_{\mathsf{F}X} \leq \bigwedge_{\mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})} \mu(p) \twoheadrightarrow \mu(q)$$
$$= \bigwedge_{\mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})} (\mu(\mathsf{ev}_{\kappa}) \cdot \mathsf{F}p^{\sharp}) \twoheadrightarrow (\mu(\mathsf{ev}_{\kappa}) \cdot \mathsf{F}q^{\sharp}).$$

Hence,

$$1_{\mathsf{F}X} \leq (\mathsf{F}p^{\sharp})^{\circ} \cdot \left( \bigwedge_{\mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})} \mu(\mathrm{ev}_{\kappa}) \twoheadrightarrow \mu(\mathrm{ev}_{\kappa}) \right) \cdot \mathsf{F}q^{\sharp}.$$

Moreover, by hypothesis,

$$\{\mu(1_{\mathcal{V}^{\kappa}}) \mid \mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})\} = \mathcal{V}\text{-}\mathsf{Cat}(\overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}}, \mathcal{V}^{\mathrm{op}}).$$

Thus, since the cone  $\mathcal{V}$ -Cat( $\overline{\mathsf{F}}(\mathcal{V}^{\kappa})^{\mathrm{op}}, \mathcal{V}^{\mathrm{op}}$ ) is initial (see Remark 2.6 and Corollary A.3),

$$\overline{\mathsf{F}}h_{\kappa} = \bigwedge_{\mu \in \mathsf{P}_{\kappa}(\overline{\mathsf{F}})} \mu(\mathrm{ev}_{\kappa}) \twoheadrightarrow \mu(\mathrm{ev}_{\kappa}).$$

Therefore,  $\mathsf{F}p^{\sharp} \leq \overline{\mathsf{F}}h_{\kappa} \cdot \mathsf{F}q^{\sharp}$ .

Now we show (*ii*)  $\implies$  (*i*). Let  $\mu: \mathsf{P}_{\mathcal{V}^{\kappa}} \to \mathsf{P}_{\mathcal{V}}\mathsf{F}$  be a predicate lifting of  $\overline{\mathsf{F}}$  and  $p, q: X \to \kappa \mathcal{V}$ -relations such that  $p \leq q$ . Then, by hypothesis,  $\mu(\mathsf{ev}_{\kappa}) \cdot \overline{\mathsf{F}}h_{\kappa} = \mu(\mathsf{ev}_{\kappa})$  and  $\mathsf{F}p^{\sharp} \leq \mathsf{F}h_{\kappa} \cdot \mathsf{F}p^{\sharp}$ . Therefore,

$$\mu(p) = \mu(ev_{\kappa}) \cdot Fp^{\sharp}$$

$$\leq \mu(ev_{\kappa}) \cdot \overline{F}h_{\kappa} \cdot Fq^{\sharp}$$

$$= \mu(ev_{\kappa}) \cdot Fq^{\sharp}$$

$$= \mu(q).$$

**Proposition** (4.11). Let  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting induced by a lax extension  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel. Every predicate lifting of the functor lifting  $\widehat{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat is monotone.

*Proof.* Let  $\kappa$  be a cardinal,  $h: \mathcal{V}^{\kappa} \to \mathcal{V}^{\kappa}$  the structure of  $\mathcal{V}$ -relation of  $(\mathcal{V}^{\kappa})^{\text{op}}$  and  $p, q: X \to \kappa \mathcal{V}$ -relations such that  $p \leq q$ . Note that

$$p \leq q \iff 1_X \leq (p^{\sharp})^{\circ} \cdot h \cdot q^{\sharp}.$$

Therefore,

$$\begin{split} \mathbf{1}_{\mathsf{F}X} &\leq \widehat{\mathsf{F}}\mathbf{1}_X \\ &\leq \widehat{\mathsf{F}}(\left(p^{\sharp}\right)^{\circ} \cdot h \cdot q^{\sharp}) \\ &= \mathsf{F}(p^{\sharp})^{\circ} \cdot \widehat{\mathsf{F}}h \cdot \mathsf{F}q^{\sharp}. \end{split}$$

**Theorem** (4.12). Let  $F: \text{Set} \to \text{Set}$  be a functor. The monotone map  $\widehat{F}^{MP(-)}: \text{Lift}(F)_1 \to \text{Lax}(F)$  is left adjoint to the monotone map  $I: \text{Lax}(F) \to \text{Lift}(F)_1$ .

*Proof.* Let  $\widehat{F}$ :  $\mathcal{V}$ -Rel  $\rightarrow \mathcal{V}$ -Rel be a lax extension of F. Then, by Propositions 4.6 and 2.10, MPI( $\widehat{F}$ ) contains the Moss liftings of  $\widehat{F}$ . Hence, by Theorem 2.15,

$$\widehat{\mathsf{F}}^{\mathsf{MPI}(\widehat{\mathsf{F}})} < \widehat{\mathsf{F}}.$$

On the other hand, let  $\overline{F}$ :  $\mathcal{V}$ -Cat  $\rightarrow \mathcal{V}$ -Cat be a lifting of F that preserves initial morphisms. Then, by Theorem 4.2 and the fact that MP: Lift(F)<sub>I</sub>  $\rightarrow$  Pred(F)<sub>M</sub> is left adjoint to F<sup>(-)</sup>: Pred(F)<sub>M</sub>  $\rightarrow$  Lift(F)<sub>I</sub>,

$$\overline{F} \le F^{MP(\overline{F})} = I(\widehat{F}^{MP(\overline{F})}).$$

**Theorem.** Let  $F: Set \rightarrow Set$  be a functor. The partially ordered classes Lax(F) and  $Lift(F)_{IM}$  are isomorphic.

Proof. By Proposition 4.11, the adjunction

 $\widehat{F}^{MP(-)} \dashv I: Lax(F) \rightarrow Lift(F)_{I}$ 

(co)restricts to an adjunction

$$F^{P(-)} \dashv I: Lax(F) \rightarrow Lift(F)_{IM}.$$

The fact that I: Lax(F)  $\rightarrow$  Lift(F)<sub>IM</sub> is also right adjoint to  $\widehat{F}^{P(-)}$ : Lift(F)<sub>IM</sub>  $\rightarrow$  Lax(F) follows from the adjunction of Theorem 4.9, Theorem 4.2 and the proof of Theorem 3.12.

#### A.3 Omitted proofs and details for Section 5

**Theorem** (5.5). A lax-framed functor  $\mathcal{F} \colon \mathcal{A} \to X$  corresponds precisely to a pair  $(F, \widehat{F})$ , where  $F \colon \text{Horiz}(\mathcal{A}) \to \text{Horiz}(X)$  is a 2-functor and  $\widehat{F} \colon \text{Ver}(\mathcal{A}) \to \text{Ver}(X)$  is a lax functor, such that for every  $f \colon X \to Y \in \mathcal{A}$ ,  $F(f)_* \leq \widehat{F}(f_*)$  and  $F(f)^* \leq \widehat{F}(f^*)$ .

Proof. Analogous to [21, Proposition III.1.13.1].

**Theorem** (5.13). Let  $\mathcal{F} \colon \mathcal{A} \to \mathcal{A}$  be a lax-framed functor. Then  $\operatorname{CoAlg}(\mathcal{F})$  is a framed bicategory.

*Proof.* Let  $f: (A, \alpha) \to (B, \beta)$  be a horizontal arrow in the double category  $\text{CoAlg}(\mathcal{F})$ . We show that the companion  $f_*$  and the conjoint  $f^*$  of  $f: A \to B$  in  $\mathcal{A}$  are  $\widehat{\mathsf{F}}$ -simulations.

Using [19, 1.6. Ortogonal flipping], from

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathsf{F}A & \xrightarrow{\mathsf{F}f} & \mathsf{F}B \\ \downarrow^{1} & = & \downarrow^{1} \\ A & \xrightarrow{f} & B & \xrightarrow{\beta} & \mathsf{F}B \end{array}$$

- -

we obtain

and

 $A \xrightarrow{\alpha} FA \xrightarrow{Ff} FB$   $f_* \downarrow \leq \qquad \downarrow 1$   $B \xrightarrow{\beta} FB$   $A \xrightarrow{\alpha} FB$   $f_* \downarrow \leq \qquad \downarrow (Ff)_*$   $B \xrightarrow{\beta} FB.$ 

With  $(Ff)_* \leq \widehat{F}(f_*)$  we conclude that  $f_*$  is a  $\widehat{F}$ -simulation. Similarly, from

 $\begin{array}{ccc} A & \stackrel{f}{\longrightarrow} B & \stackrel{\beta}{\longrightarrow} \mathsf{F}B \\ \downarrow^{1} & = & \downarrow^{1} \\ A & \stackrel{\alpha}{\longrightarrow} \mathsf{F}A & \stackrel{Ff}{\longrightarrow} \mathsf{F}B \end{array}$ 

and

we obtain

 $A \xrightarrow[]{\alpha} \mathsf{F}A.$ 

Finally, with  $(Ff)^* \leq \widehat{F}(f^*)$  we conclude that  $f^*$  is a  $\widehat{F}$ -simulation.

**Theorem** (5.16). Let  $\mathcal{F} : \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. Then  $\operatorname{CoAlg}(\mathcal{F})$  is locally complete. *Proof.* Let  $(s_i : (\mathcal{A}, \alpha) \to (\mathcal{B}, \beta))_{i \in I}$  be a family of  $\mathcal{F}$ -simulations. Then the supremum of this family, taken in  $\operatorname{Ver}(\mathcal{A})$ , is again an  $\mathcal{F}$ -simulation:

$$\beta_* \circ \bigvee_{i \in I} s_i = \bigvee_{i \in I} \beta_* \circ s_i \le \bigvee_{i \in I} \widehat{\mathsf{F}} s_i \circ \alpha_* \le \widehat{\mathsf{F}}(\bigvee_{i \in I} s_i) \circ \alpha_*.$$

**Theorem** (5.19). Let  $\mathcal{F} \colon \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. For every pair of horizontal arrows

$$(A, \alpha) \xrightarrow{f} (C, \gamma) \qquad (B, \beta) \xrightarrow{g} (D, \delta)$$

 $in \operatorname{CoAlg}(\mathcal{F}), \, \top_{\alpha,\beta} = g^* \circ \top_{\gamma,\delta} \circ f_*.$ 

*Proof.* Follows from the fact that  $g^* \circ - \circ f_*$  is a right adjoint.

**Corollary A.5.** Let  $\mathcal{F}: \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete and the category Horiz( $\mathcal{A}$ ) has binary coproducts. Then, for all coalgebras ( $A, \alpha$ ) and ( $B, \beta$ ),

$$\top_{\alpha,\beta} = i_{\beta}^* \circ \top_{\alpha+\beta} \circ i_{\alpha*},$$

where  $i_{\alpha}$  and  $i_{\beta}$  denote the morphisms from  $(A, \alpha)$  and  $(B, \beta)$  to the coproduct  $(A + B, \alpha + \beta)$ .

**Corollary A.6.** Let  $\mathcal{F} : \mathcal{A} \to \mathcal{A}$  be a lax-framed functor where  $\mathcal{A}$  is locally complete. If the functor  $F : \operatorname{Horiz}(\mathcal{A}) \to \operatorname{Horiz}(\mathcal{A})$  admits a terminal coalgebra  $(C, \gamma)$ , then, for all  $\mathcal{F}$ -coalgebras  $(A, \alpha)$  and  $(B, \beta)$ ,

$$\top_{\alpha,\beta} = !_{\beta}^* \circ \top_{\gamma} \circ !_{\alpha*}$$

where  $!_{\alpha}$  and  $!_{\beta}$  denote the unique morphisms from  $(A, \alpha)$  and  $(B, \beta)$  to  $(C, \gamma)$ .

**Theorem** (5.23). Let  $\mathcal{F}$  be a lax-framed endofunctor on a locally complete framed bicategory. Then,  $\mathcal{F}$ -similarity and  $\mathcal{F}$ -behavioural distance coincide on every  $\mathcal{F}$ -coalgebra.

$$B \xrightarrow{\beta} \mathsf{F}B$$
$$f^* \downarrow \leq \downarrow (\mathsf{F}f)^*$$



*Proof.* Follows from the fact that  $\mathcal{F}$ -similarity  $\top_{\alpha}$  on  $(A, \alpha)$  is a monoid on A.

**Proposition** (5.26). Let  $\mathcal{A}$  be a double category and  $\mathcal{F}$ : Mon $(\mathcal{A}) \to$  Mon $(\mathcal{A})$  be a lax-double functor preserving vertical identities. If  $(C, c, \gamma)$  is a terminal  $\mathcal{F}$ -coalgebra, then  $(C, c, c, \gamma)$  is a terminal Mon $(\mathcal{F})$ -coalgebra.

*Proof.* Let  $(A, a_0, a_1, \alpha)$  be a Mon( $\mathcal{F}$ )-coalgebra. First observe that  $f : A \to C$  in  $\mathcal{A}$  is a morphism of Mon( $\mathcal{F}$ )-coalgebras if and only if  $f : (A, a_0, \alpha) \to (C, c, \gamma)$  is a morphism of  $\mathcal{F}$ -coalgebras and  $f : (A, a_1) \to (C, c)$  is a horizontal arrow in Mon( $\mathcal{A}$ ). Therefore there is at most one Mon( $\mathcal{F}$ )-coalgebra homomorphism  $(A, a_0, a_1, \alpha) \to (C, c, c, \gamma)$ . Consider now  $i : (A, a_0, a_1) \to (A, a_1, a_1)$  in Mon(Mon( $\mathcal{A}$ )) and



Note that  $Mon(\mathcal{F})(A, a_1, a_1) = (\mathcal{F}(A, a_1), \mathcal{F}a_1)$  and, since  $\mathcal{F}$  preserves vertical identities,  $\mathcal{F}a_1$  coincides with the monoid structure of  $\mathcal{F}(A, a_1)$ . Therefore also  $\alpha_1 \colon (A, a_1) \to \mathcal{F}(A, a_1)$  in  $Mon(\mathcal{A})$  and the identity on A is an  $\mathcal{F}$ -coalgebra homomorphism  $(A, a_0, \alpha) \to (A, a_1, \alpha_1)$ . Since  $(C, c, \gamma)$  is the terminal  $\mathcal{F}$ -coalgebra, there is  $\mathcal{F}$ -coalgebra homomorphism  $f \colon (A, a_1, \alpha) \to (C, c, \gamma)$ , which proves the claim.