Duality theory for enriched Priestley spaces

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The term Stone-type duality often refers to a dual equivalence between a category of lattices or other partially ordered structures on one side and a category of topological structures on the other. This paper is part of a larger endeavour that aims to extend a web of Stone-type dualities from ordered to metric structures and, more generally, to quantale-enriched categories. In particular, we improve our previous work and show how certain duality results for categories of $[0,1]$-enriched Priestley spaces and $[0,1]$-enriched relations can be restricted to functions. In a broader context, we investigate the category of quantale-enriched Priestley spaces and continuous functors, with emphasis on those properties which identify the algebraic nature of the dual of this category.

Keywords: Stone duality, metric space, Priestley space, quantale-enriched category, variety, quasivariety.

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1 Introduction

Naturally, the starting point of our investigation of Stone-type dualities is Stone's classical 1936 duality result

$$\mathsf{BooSp} \sim \mathsf{BA}^{\mathrm{op}} \tag{1.i}$$

for Boolean algebras and homomorphisms together with its generalisation

$$\mathsf{Spec} \sim \mathsf{DL}^{\mathrm{op}}$$

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to distributive lattices and homomorphisms obtained shortly afterwards in [Stone, 1938]. Here BooSp denotes the category of Boolean spaces[1] and continuous maps, and Spec the category of spectral spaces and spectral maps (see also [Hochster, 1969]). In this paper we will often work with *Priestley spaces* rather than with *spectral spaces*, and therefore consider the "equivalent equivalence"

$$\mathsf{Priest} \sim \mathsf{DL}^{\mathrm{op}} \tag{1.ii}$$

discovered in [Priestley, 1970, 1972]. There are many ways to deduce the duality result (1.i) from (1.ii), we mention here one possibly lesser-known argument: in [Brümmer *et al.*, 1992] it is observed that BA is the only epi-mono-firm epireflective full subcategory of DL, and, using that in both BooSp and Priest the epimorphisms are precisely the surjective morphisms, an easy calculation shows that BooSp is the only mono-epi-firm mono-coreflective full subcategory of Priest.

Exactly 20 years later, Halmos gave an extension of (1.i) to categories of *continuous relations* between Boolean spaces and *hemimorphisms* between Boolean algebras, and a similar generalisation of $\mathsf{Priest} \sim \mathsf{DL}^{\mathrm{op}}$ is described in [Cignoli *et al.*, 1991]. Denoting by

- $\mathsf{PriestDist}$ the category of Priestley spaces and continuous monotone relations, by
- FinSup the category of finitely cocomplete partially ordered sets and finite suprema preserving maps, and by
- $\mathsf{FinSup}_{\mathrm{DL}}$ the full subcategory of FinSup defined by all distributive lattices,

this result is expressed as

$$\mathsf{PriestDist} \sim \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}. \tag{1.iii}$$

We note that $\mathsf{PriestDist}$ is precisely the Kleisli category of the Vietoris monad $\mathbb{H} = (\mathsf{H}, w, h)$ on Priest, and that the functor $\mathsf{PriestDist} \longrightarrow \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$ is a lifting of the hom-functor $\mathsf{PriestDist}(-, 1)$ into the one-element space. Furthermore, the two structures of a Priestley space — the partial order and the compact Hausdorff topology — can be combined into a single topology: the so-called downwards topology (see [Jung, 2004], for instance). In particular, the two-element Priestley space $2 = \{0 \le 1\}$ produces the Sierpiński space 2 with $\{1\}$ closed, whereby the dual space 2^{op} of 2 induces the topology on $\{0, 1\}$ with $\{1\}$ being the only non-trivial open subset. With this notation, the elements of the Vietoris hyperspace $\mathsf{H}X$ of a Priestley space X can be identified with continuous maps $\varphi\colon X \longrightarrow 2$, whereby arrows of type $X \longrightarrow\!\!\!\!\!\circ\;\, 1$ in $\mathsf{PriestDist}$ correspond to spectral maps $\psi\colon X \longrightarrow \mathsf{H}1 \simeq 2^{\mathrm{op}}$. In order to deduce the equivalence (1.iii), it is important to establish that there are "enough" spectral maps $\psi\colon X \longrightarrow 2^{\mathrm{op}}$; in fact, by definition, a partially ordered compact Hausdorff space X is Priestley whenever the cone $(\psi\colon X \longrightarrow 2^{\mathrm{op}})_\psi$ is point-separating and initial. Here it does not matter if we use 2 or 2^{op} since $2 \simeq 2^{\mathrm{op}}$ in Priest; however, when moving to the quantale-enriched setting, the corresponding property does not necessarily hold and therefore we must identify carefully if we refer to 2 or to 2^{op}.

[1] Also designated as Stone spaces in the literature, see [Johnstone, 1986], for instance.

Under the equivalence (1.iii), continuous monotone *functions* correspond precisely to *homomorphisms* of distributive lattices, therefore the equivalence $\mathsf{Priest} \sim \mathsf{DL}^{\mathrm{op}}$ is a direct consequence of (1.iii). Furthermore, other well-known duality results can be obtained from (1.iii) in a categorical way, we mention here the following examples.

- As (1.i) can be deduced from $\mathsf{Priest} \sim \mathsf{DL}^{\mathrm{op}}$, Halmos's duality

  $$\mathsf{BooSpRel} \sim \mathsf{FinSup}_{\mathrm{BA}}^{\mathrm{op}}$$

  between the category $\mathsf{BooSpRel}$ of Boolean spaces and Boolean relations and the category $\mathsf{FinSup}_{\mathrm{BA}}$ of Boolean algebras and hemimorphisms (that is, the full subcategory of FinSup defined by all Boolean algebras) can be deduced from (1.iii).

- Combining $\mathsf{PriestDist} \sim \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$ and $\mathsf{Priest} \sim \mathsf{DL}^{\mathrm{op}}$ gives immediately the duality result for distributive lattices with an operator (see [Petrovich, 1996; Bonsangue *et al.*, 2007]).

- The equivalence $\mathsf{PriestDist} \sim \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$ has the surprising(?) consequence that $\mathsf{PriestDist}$ is idempotent split complete. Hence, the idempotent split completion of $\mathsf{BooSpRel}$ can be calculated as the full subcategory of $\mathsf{PriestDist}$ defined by all split subobjects of Boolean spaces in $\mathsf{PriestDist}$; likewise, the idempotent split completion of $\mathsf{FinSup}_{\mathrm{BA}}$ can be taken as the full subcategory of $\mathsf{FinSup}_{\mathrm{DL}}$ defined by all split subobjects of Boolean algebras. Now, in the former case, these split subobjects are precisely the so-called Esakia spaces (see [Esakia, 1974]), and in the latter case precisely the co-Heyting algebras (see [McKinsey and Tarski, 1946]). Putting these facts together, we obtain a relational version of Esakia duality as described in [Hofmann and Nora, 2014].

The situation is depicted in Figure 1.

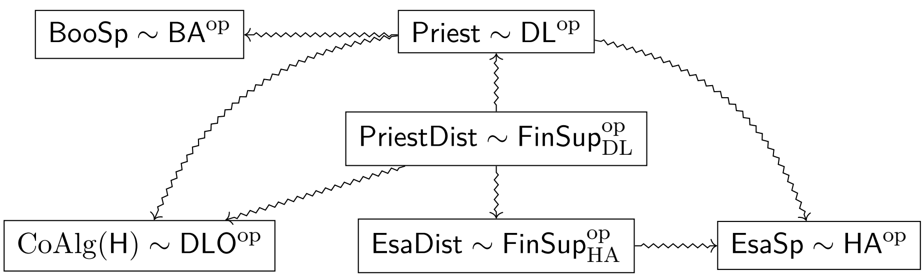


Figure 1: Stone type dualities

One might wish to consider all compact Hausdorff spaces in (1.i) instead of only the totally disconnected ones. Then the two-element space and the two-element Boolean algebra still induce naturally an adjunction

$$\mathsf{CompHaus} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\rightleftarrows}} \mathsf{BA}^{\mathrm{op}} \quad (\bot);$$

however, its restriction to the *fixed* subcategories is precisely (1.i) (for the pertinent notions of duality theory we refer to [Dimov and Tholen, 1989; Porst and Tholen, 1991]). In fact, by definition, a compact Hausdorff space X is Boolean whenever the cone $(f\colon X \longrightarrow 2)_f$ is point-separating and initial with respect to the forgetful $\mathsf{CompHaus} \longrightarrow \mathsf{Set}$.

In order to obtain a duality result for all compact Hausdorff spaces this way, one needs to substitute the dualising object 2 by a cogenerator in $\mathsf{CompHaus}$, for instance, by the unit interval $[0,1]$ with the Euclidean topology. Accordingly, one typically considers other types of algebras on the dual side; *i.e.* C^*-algebras instead of Boolean algebras. In contrast, our aim is to develop a duality theory where one actually keeps the "type of algebras" in Figure 1 but substitutes *order* by *metric* everywhere; that is, one considers $[0,\infty]$-enriched categories instead of 2-enriched categories (see [Lawvere, 1973]). Therefore one might attempt to create a network of dual equivalences

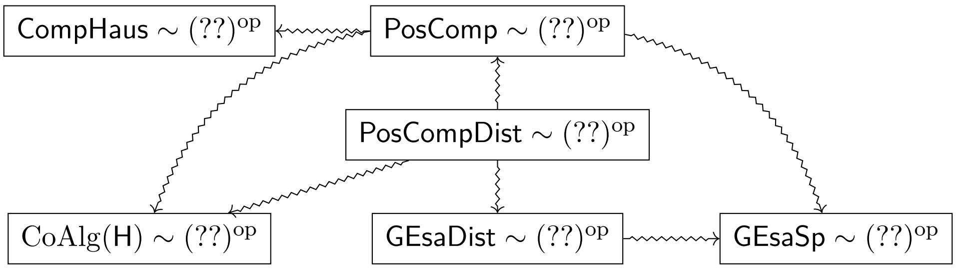


Figure 2: Metric Stone type dualities

where each "question mark category" should be substituted by its metric counterpart of Figure 1, or even better, a quantale-enriched counterpart. For instance, for a quantale $\mathcal{V}$, instead of DL one would expect a category of $\mathcal{V}$-categories with all "finite" weighted limits and colimits and satisfying some sort of "distributivity" condition. Moreover, these results should have the property that, when choosing the quantale $\mathcal{V} = 2$, we get the original picture of Figure 1 back.

Unfortunately, the last requirement does not make much sense ... since the picture of Figure 2 is somehow inconsequential: both sides of the equivalences should be generalised to corresponding metric or even quantale-enriched versions, in particular, partially ordered compact spaces should be substituted by their metric versions. This requirement brings a new class of categories into play: as shown in [Tholen, 2009], the ultrafilter monad extends naturally to $\mathcal{V}$-Cat, and we consider the category $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$ of Eilenberg–Moore algebras and homomorphisms for the ultrafilter monad $\mathbb{U}$ on $\mathcal{V}$-Cat. For $\mathcal{V} = 2$, these Eilenberg–Moore algebras are precisely Nachbin's (pre)ordered compact Hausdorff spaces. In this paper we denote the category of partially ordered compact Hausdorff spaces and monotone continuous maps by $\mathsf{PosComp}$. For the Lawvere quantale $\overleftarrow{[0,\infty]}_+$ we obtain metric spaces equipped with a (somehow compatible) compact Hausdorff topology, these spaces should be thought of as natural generalisations of compact metric spaces. We believe that these spaces are interesting in their own right as they allow to generalise arguments from the theory of compact metric spaces (see also [Hofmann and Reis, 2018]). Furthermore, the enriched Vietoris monad encodes at the same time the classical Hausdorff metric and Vietoris topology (see [Hofmann and Nora, 2020]).

In analogy with the ordered case, we call an $\mathbb{U}$-algebra *Priestley* whenever the cone of all homomorphisms $X \longrightarrow \mathcal{V}^{\mathrm{op}}$ in $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$ is point-separating and initial. In [Hofmann and Nora, 2018] we made an attempt to create at least parts of this picture, for continuous quantale structures on the lattice $\mathcal{V} = [0,1]$. In particular, we showed that the Kleisli category of the

Vietoris monad on PosComp can be fully embedded into a category of monoids of finitely cocomplete $[0,1]$-categories and lax homomorphisms, and we were able to identify the image of this embedding. For the enriched Vietoris monad, the situation is even better, we obtained a full embedding of its Kleisli category into a category of finitely cocomplete $[0,1]$-categories — no monoid structure needed anymore. In Section 3 we build upon this result and show how this embedding can be restricted to the category of enriched Priestley spaces and homomorphisms.

The classical duality results of Stone and Priestley tell us in particular that $\mathsf{BooSp}^{\mathrm{op}}$ and $\mathsf{Priest}^{\mathrm{op}}$ are finitary varieties. It is known since the late 1960's that $\mathsf{CompHaus}^{\mathrm{op}}$ is also a variety, not finitary but with rank $\aleph_1$ (see [Duskin, 1969; Gabriel and Ulmer, 1971]); however, this fact might not be obvious from the classical Gelfand duality result

$$\mathsf{CompHaus}^{\mathrm{op}} \sim C^*\text{-}\mathsf{Alg}$$

stating the equivalence between $\mathsf{CompHaus}^{\mathrm{op}}$ and the category C^*-Alg of commutative $C^\star$-algebras and homomorphisms. Nonetheless, it can be deduced "abstractly" from the following well-known results.

Theorem 1.1. *A cocomplete category is equivalent to a quasivariety if and only if it has a regular projective regular generator.*

Proof. See, for instance, [Adámek, 2004, Theorem 3.6]. □

Theorem 1.2. *A category is a variety if and only if it is a quasivariety and has effective equivalence relations.*

Proof. See, for instance, [Borceux, 1994, Theorem 4.4.5] □

Surprisingly, a similar investigation of $\mathsf{PosComp}^{\mathrm{op}}$ was initiated only recently: in [Hofmann *et al.*, 2018] we show that $\mathsf{PosComp}^{\mathrm{op}}$ is a $\aleph_1$-ary quasivariety, and in [Abbadini, 2019; Abbadini and Reggio, 2020] it is shown that $\mathsf{PosComp}^{\mathrm{op}}$ is indeed a $\aleph_1$-ary variety. In Section 4 we investigate the category $\mathcal{V}$-Priest of $\mathcal{V}$-enriched Priestley spaces and morphisms, with emphasis on those properties which identify $\mathcal{V}$-$\mathsf{Priest}^{\mathrm{op}}$ as some kind of algebraic category. In particular, for certain quantales $\mathcal{V}$, we characterise the $\aleph_1$-copresentable objects in $\mathcal{V}$-Priest and show that $\mathcal{V}$-Priest is locally $\aleph_1$-copresentable.

2 Quantale-enriched Priestley spaces

In this section we recall the notions of quantale-enriched category and its generalisation to compact Hausdorff spaces, which eventually leads to the notion of *quantale-enriched Priestley space* already studied in [Hofmann and Nora, 2018, 2020]. We recall some of the basic definitions and properties, for more information we refer to [Kelly, 1982; Lawvere, 1973; Tholen, 2009]. For a nice introduction to quantale and quantaloid-enriched categories we refer to [Stubbe, 2014].

Definition 2.1. A ***quantale*** $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$ equipped with a commutative monoid structure $\otimes$, with identity k, so that, for each $u \in \mathcal{V}$,

$$u \otimes - \colon \mathcal{V} \longrightarrow \mathcal{V} \quad \text{has a right adjoint} \quad \hom(u, -) \colon \mathcal{V} \longrightarrow \mathcal{V}.$$

Definition 2.2. Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale.

1. A ***$\mathcal{V}$-category*** is a pair (X, a) consisting of a set X and a map $a\colon X \times X \longrightarrow \mathcal{V}$ satisfying

$$k \leq a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \leq a(x,z),$$

   for all $x, y, z \in X$. Furthermore, a $\mathcal{V}$-category (X, a) is called ***separated*** whenever

$$(k \leq a(x,y) \quad \text{and} \quad k \leq a(y,x)) \implies x = y,$$

   for all $x, y \in X$.

2. A ***$\mathcal{V}$-functor*** $f\colon (X, a) \longrightarrow (Y, b)$ between $\mathcal{V}$-categories is a map $f\colon X \longrightarrow Y$ such that

$$a(x, x') \leq b(f(x), f(x')),$$

   for all $x, x' \in X$.

3. Finally, $\mathcal{V}$-categories and $\mathcal{V}$-functors define the category $\mathcal{V}\text{-}\mathsf{Cat}$, and its full subcategory defined by separated $\mathcal{V}$-categories is denoted by $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$.

We note that there is a canonical forgetful functor $\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$ sending the $\mathcal{V}$-category (X, a) to the set X. For every $\mathcal{V}$-category $X = (X, a)$, the ***dual*** $\mathcal{V}$-category X^{op} is defined as $X^{\mathrm{op}} = (X, a^\circ)$ where

$$a^\circ(x,y) = a(y,x),$$

for all $x, y \in X$. In fact, this construction defines a functor $(-)^{\mathrm{op}}\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ commuting with the forgetful functor to Set.

Examples 2.3. Below we list some of the principal examples, for more details we refer, for instance, to [Hofmann and Reis, 2018].

1. The trivial quantale $\mathsf{1} = \{*\}$ where $k = \top$. Then $\mathsf{1}\text{-}\mathsf{Cat} = \mathsf{Set}$.

2. The two element chain $\mathsf{2} = \{0 \leq 1\}$ with $\otimes = \&$ and $k = 1$. Then $\mathsf{2}\text{-}\mathsf{Cat} \sim \mathsf{Ord}$.

3. The extended real half line $\overleftarrow{[0,\infty]}$ ordered by the "greater or equal" relation $\geqslant$ and
   - the tensor product given by addition $+$, denoted by $\overleftarrow{[0,\infty]}_+$;
   - or with $\otimes = \max$, denoted as $\overleftarrow{[0,\infty]}_\wedge$.

   Then $\overleftarrow{[0,\infty]}_+\text{-}\mathsf{Cat} \sim \mathsf{Met}$ is the category of (generalised) metric spaces and non-expansive maps and $\overleftarrow{[0,\infty]}_\wedge\text{-}\mathsf{Cat} \sim \mathsf{UMet}$ is the category of (generalised) ultrametric spaces and non-expansive maps.

4. The unit interval $[0,1]$ with the "greater or equal" relation $\geqslant$ and the tensor $u \oplus v = \min\{1, u + v\}$, denoted as $\overleftarrow{[0,1]}_\oplus$. Then $\overleftarrow{[0,1]}_\oplus\text{-}\mathsf{Cat} \sim \mathsf{BMet}$ is the category of (generalised) bounded-by-one metric spaces and non-expansive maps.

5. The unit interval $[0,1]$ with the usual order $\leqslant$ and $\otimes = \wedge$ the minimum, or $\otimes = *$ the usual multiplication, or $\otimes = \odot$ the Łukasiewicz sum defined by $u \odot v = \max\{0, u + v - 1\}$. Then $[0,1]_\wedge\text{-}\mathsf{Cat} \sim \mathsf{UMet}$, $[0,1]_*\text{-}\mathsf{Cat} \sim \mathsf{Met}$, and $[0,1]_\odot\text{-}\mathsf{Cat} \sim \mathsf{BMet}$.

Example 2.4. The notion of probabilistic metric space goes back to [Menger, 1942]. Here a ***probabilistic metric*** on a set X is a map $d\colon X \times X \times [0,\infty] \longrightarrow [0,1]$, where $d(x,y,t) = u$ means that u *is the probability that the distance from* x *to* y *is less then* t. Similar to a classic metric, such a map is required to satisfy the following conditions:

0. $d(x,y,-)\colon [0,\infty] \longrightarrow [0,1]$ is left continuous,
1. $d(x,x,t) = 1$ for $t > 0$,
2. $d(x,y,r) * d(y,z,s) \leq d(x,z,r+s)$,
3. $d(x,y,t) = 1 = d(y,x,t)$ for all $t > 0$ implies $x = y$,
4. $d(x,y,t) = d(y,x,t)$ for all t,
5. $d(x,y,\infty) = 1$.

The complete lattice

$$\mathcal{D} = \{f\colon [0,\infty] \longrightarrow [0,1] \mid f(t) = \bigvee_{s<t} f(s) \text{ for all } t \in [0,\infty]\}$$

becomes a quantale with multiplication

$$(f \otimes g)(t) = \bigvee_{r+s\leqslant t} f(r) * g(s),$$

for $f, g \in \mathcal{D}$, and unit the map $\kappa\colon [0,\infty] \longrightarrow [0,1]$ with $\kappa(0) = 0$ and $\kappa(t) = 1$ for $t > 0$. In the formula above, one may substitute the multiplication $*$ by any other tensor $\otimes\colon [0,1] \times [0,1] \longrightarrow [0,1]$.

Then a probabilistic metric can be seen as a map $d\colon X \times X \longrightarrow \mathcal{D}$, and conditions (1) and (2) read as

$$\kappa \leq d(x,x) \qquad \text{and} \qquad d(x,y) \otimes d(y,z) \leq d(x,z).$$

Hence $\mathcal{D}$-$\mathsf{Cat} \sim \mathsf{ProbMet}$ is the category of (generalised) probabilistic metric spaces and non-expansive maps.

Before adding a topological component to the theory of $\mathcal{V}$-categories, we collect some well-known properties of $\mathcal{V}$-categories and $\mathcal{V}$-functors. For the relevant notions of categorical topology we refer to [Adámek *et al.*, 1990].

Theorem 2.5. *The canonical forgetful functor* $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Set}$ *is topological. Here a cone* $(f_i\colon (X,a) \longrightarrow (X_i,a_i))_{i\in I}$ *in* $\mathcal{V}$-Cat *is initial with respect to* $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Set}$ *if and only if, for all* $x, y \in X$,

$$a(x,y) = \bigwedge_{i\in I} a_i(f_i(x), f_i(y)).$$

Therefore $\mathcal{V}$-Cat *has concrete limits and colimits and a (surjective, initial monocone)-factorisation system; moreover,* $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Set}$ *has a right adjoint* $\mathsf{Set} \longrightarrow \mathcal{V}$-$\mathsf{Cat}$ *(indiscrete structures) and a left adjoint* $\mathsf{D}\colon \mathsf{Set} \longrightarrow \mathcal{V}$-$\mathsf{Cat}$ *(discrete structures). Furthermore, a morphism* $f\colon (X,a) \longrightarrow (Y,b)$ *in* $\mathcal{V}$-Cat *is*

1. *a monomorphism if and only if* f *is injective,*

2. *a regular monomorphism if and only if f is an embedding with respect to $\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$, that is, f is injective and $a(x,y) = b(f(x), f(y))$ for all $x, y \in X$,*

3. *an epimorphism if and only if f is surjective.*

Proposition 2.6. *The $\mathcal{V}$-category $\mathcal{V} = (\mathcal{V}, \hom)$ is injective with respect to embeddings and, for every $\mathcal{V}$-category X, the cone $(f : X \longrightarrow \mathcal{V})_f$ is initial with respect to the forgetful functor $\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$.*

Remark 2.7. Since $(-)^{\mathrm{op}} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ is a concrete isomorphism, Proposition 2.6 applies also to the $\mathcal{V}$-category $\mathcal{V}^{\mathrm{op}}$ *in lieu* of $\mathcal{V}$.

In the remainder of this section we assume that *the lattice $\mathcal{V}$ is completely distributive*, we refer to [Wood, 2004] for the definition and an extensive discussion of properties of this notion. In particular, under this assumption it is useful to consider the ***totally below*** relation $\lll$ on the lattice $\mathcal{V}$, which is defined by $v \lll u$ whenever

$$u \le \bigvee A \implies v \in {\downarrow} A,$$

for every subset A of $\mathcal{V}$.

Assumption 2.8. The underlying lattice of the quantale $\mathcal{V}$ is completely distributive.

Remark 2.9. Regarding the various topologies on $\mathcal{V}$ we have the following facts, for more information see [Gierz *et al.*, 2003].

1. The Lawson topology on the completely distributive lattice $\mathcal{V}$ is compact Hausdorff. With respect to this topology, as shown in [Gierz *et al.*, 2003, Proposition VII-3.10], an ultrafilter $\mathfrak{v}$ in $\mathcal{V}$ converges to
$$\xi(\mathfrak{v}) = \bigwedge_{A \in \mathfrak{v}} \bigvee A \in \mathcal{V}.$$
Moreover, the Scott topology respectively its dual topology have the following convergences:

Scott topology:	$\mathfrak{v} \to x \iff \xi(\mathfrak{v}) \ge x,$
Dual of Scott topology:	$\mathfrak{v} \to x \iff \xi(\mathfrak{v}) \le x.$

2. By [Gierz *et al.*, 2003, Lemma VII-2.7] and [Gierz *et al.*, 2003, Proposition VII-2.10], the Lawson topology of $\mathcal{V}$ coincides with the Lawson topology of $\mathcal{V}^{\mathrm{op}}$, and the set
$$\{{\uparrow} u \mid u \in \mathcal{V}\} \cup \{{\downarrow} u \mid u \in \mathcal{V}\}$$
is a subbasis for the closed sets of this topology which is known as the interval topology.

3. The sets
$$\uparrow v = \{u \in \mathcal{V} \mid v \le u\} \qquad (v \in \mathcal{V})$$
form a subbase for the closed sets of the dual of the Scott topology of $\mathcal{V}$ (see [Gierz *et al.*, 2003, Proposition VI-6.24]). We denote (the convergence of) this topology by $\xi_{\le}$.

4. The convergence $\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}$ together with the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ and the quantale $\mathcal{V}$ defines a topological theory in the sense of [Hofmann, 2007], and therefore allows for an extension of the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ on Set to a monad on $\mathcal{V}$-Cat (see [Tholen, 2009]).

Definition 2.10. We denote the corresponding Eilenberg–Moore category $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$ by $\mathcal{V}$-CatCH, and refer to its objects as ***$\mathcal{V}$-categorical compact Hausdorff spaces*** (see also [Hofmann and Reis, 2018]). In more detail, a $\mathcal{V}$-categorical compact Hausdorff space is a triple (X, a, α) where

- (X, a) is a $\mathcal{V}$-category and
- $\alpha\colon \mathsf{U}X \longrightarrow X$ is the convergence of a compact Hausdorff topology on X such that $\alpha\colon (\mathsf{U}X, \mathsf{U}a) \longrightarrow (X, a)$ is a $\mathcal{V}$-functor.

Example 2.11. The triple $\mathcal{V} = (\mathcal{V}, \hom, \xi)$ is a $\mathcal{V}$-categorical compact Hausdorff space. Moreover, for a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, also $X^{\mathrm{op}} = (X, a^\circ, \alpha)$ is a $\mathcal{V}$-categorical compact Hausdorff space.

Example 2.12. As it is pointed out in [Tholen, 2009], 2-categorical compact Hausdorff spaces are precisely Nachbin's ordered compact Hausdorff spaces.

Example 2.13. For $\mathcal{V} = \overleftarrow{[0,\infty]}_+$, we designate $\mathcal{V}$-categorical compact Hausdorff spaces as ***metric compact Hausdorff spaces***. As already pointed out in [Hofmann and Reis, 2018], these spaces can be considered as generalisations of compact metric spaces, *i.e.* metric spaces where the induced topology is compact. For instance, [Hofmann and Reis, 2018, Corollary 4.21] implies that the underlying metric of a metric compact Hausdorff space is Cauchy-complete, generalising the classic result that every compact metric space is Cauchy-complete. In other words, one can deduce Cauchy-completeness of a metric space (X, a) by exhibiting a *compatible* compact Hausdorff topology, one does not need to consider the *induced* topology.

Example 2.14. For the trivial quantale $V = 1$, 1-$\mathsf{CatCH} \simeq \mathsf{CompHaus}$.

Proposition 2.15. *For a quantale $\mathcal{V}$, the sets*

$$\{u \in \mathcal{V} \mid v \lll u\} \qquad (v \in \mathcal{V})$$

form a subbase for its Scott topology.

Proof. We start by proving that for every $v \in \mathcal{V}$ the set $\{u \in \mathcal{V} \mid v \lll u\}$ is open. Let $\mathfrak{v}$ be an ultrafilter in $\mathcal{V}$ that converges to $u \in \mathcal{V}$ such that $v \lll u$. The properties of the totally below relation guarantee that there exists $w \in \mathcal{V}$ such that $v \lll w \lll u$. Then, by Remark 2.9 (1), for every $A \in \mathfrak{v}$, $u \le \bigvee A$. Hence, for every $A \in \mathfrak{v}$ there exists $a \in A$ such that $w \le a$. Therefore, for every $A \in \mathfrak{v}$,

$$A \cap \{u \in \mathcal{V} \mid v \lll u\} \neq \varnothing.$$

We show now that the sets $\{u \in \mathcal{V} \mid v \lll u\}$ $(v \in \mathcal{V})$ induce the convergence of the Scott topology. Let w be an element of $\mathcal{V}$ and $\mathfrak{v}$ and ultrafilter on $\mathcal{V}$ such that, for every $v \lll w$ in $\mathcal{V}$, the set $\{u \in \mathcal{V} \mid v \lll u\}$ belongs to $\mathfrak{v}$. Then, since $\mathcal{V}$ is completely distributive, we have

$$w = \bigvee_{v \lll w} v \le \bigvee_{A \in \mathfrak{v}} \bigwedge A = \xi(\mathfrak{v}). \qquad \square$$

Remark 2.16. For a point-separating cone $(f_i\colon (X,a,\alpha) \longrightarrow (X_i,a_i,\alpha_i))_{i\in I}$ in $\mathcal{V}$-CatCH, the following assertions are equivalent, for details see [Tholen, 2009].

(i) For all $x,y \in X$, $a(x,y) = \bigwedge_{i\in I} a_i(f_i(x), f_i(y))$.

(ii) $(f\colon (X,a,\alpha) \longrightarrow (X_i,a_i,\alpha_i))_{i\in I}$ is initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathsf{CompHaus}$.

(iii) $(f\colon (X,a,\alpha) \longrightarrow (X_i,a_i,\alpha_i))_{i\in I}$ is initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathsf{Set}$.

In the sequel we will simply say "initial" when referring to either of these forgetful functors. We also note that a cone $(f_i\colon (X,a,\alpha) \longrightarrow (X_i,a_i,\alpha_i))_{i\in I}$ is point-separating if and only if it is a monocone in $\mathcal{V}$-CatCH.

Theorem 2.17. *The category $\mathcal{V}$-CatCH is monadic over $\mathcal{V}$-Cat and topological over $\mathsf{CompHaus}$, hence $\mathcal{V}$-CatCH is complete and cocomplete and has a (surjective, initial monocone)-factorisation system.*

Proof. See [Tholen, 2009]. □

Definition 2.18. A $\mathcal{V}$-categorical compact Hausdorff space X is called ***Priestley*** whenever the cone $\mathcal{V}$-$\mathsf{CatCH}(X,\mathcal{V}^{\mathrm{op}})$ is point-separating and initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathsf{CompHaus}$.

Example 2.19. For $\mathcal{V} = 2$, the notion of Priestley space coincides with the classical one.

Remark 2.20. By definition, the $\mathcal{V}$-categorical compact Hausdorff space $\mathcal{V}^{\mathrm{op}}$ is Priestley. Moreover, every finite separated $\mathcal{V}$-categorical compact Hausdorff space is Priestley.

We denote the full subcategory of $\mathcal{V}$-CatCH defined by all Priestley spaces by $\mathcal{V}$-Priest. Due to well-known facts about factorisation structures for cones (see [Adámek *et al.*, 1990, Section 16]), we have the following:

Proposition 2.21. *The full subcategory $\mathcal{V}$-Priest of $\mathcal{V}$-CatCH is reflective.*

We denote the left adjoint of the inclusion functor $\mathcal{V}$-$\mathsf{Priest} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$ by $\pi_0\colon \mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathcal{V}$-$\mathsf{Priest}$.

Proof. For each X in $\mathcal{V}$-CatCH, its reflection $X \longrightarrow \pi_0(X)$ into $\mathcal{V}$-Priest is given by the (surjective, initial monocone)-factorisation of the cone $(\varphi\colon X \longrightarrow \mathcal{V}^{\mathrm{op}})_\varphi$ of all morphisms from X to $\mathcal{V}^{\mathrm{op}}$ in $\mathcal{V}$-CatCH.

$$X \twoheadrightarrow \pi_0(X) \xrightarrow[\widetilde{\varphi}]{} \mathcal{V}^{\mathrm{op}}, \qquad \varphi\colon X \longrightarrow \mathcal{V}^{\mathrm{op}}$$

To show that this construction defines indeed a left adjoint to $\mathcal{V}$-$\mathsf{Priest} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$, consider $f\colon X \longrightarrow Y$ in $\mathcal{V}$-CatCH where Y is Priestley. Then, for every $\varphi\colon Y \longrightarrow \mathcal{V}^{\mathrm{op}}$, there is some arrow $\widetilde{\varphi}\colon \pi_0(X) \longrightarrow \mathcal{V}^{\mathrm{op}}$ making the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & \pi_0(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}} \\ Y & \xrightarrow[\varphi]{} & \mathcal{V}^{\mathrm{op}} \end{array} \tag{2.i}$$

commute. Since the top arrow of (2.i) is surjective and the cone $(\varphi\colon Y \longrightarrow \mathcal{V}^{\mathrm{op}})_\varphi$ is point-separating and initial, there is a diagonal arrow $\bar{f}\colon \pi_0(X) \longrightarrow Y$ in (2.i) making in particular the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & \pi_0(X) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\ Y & & \end{array}$$

commute. ☐

Corollary 2.22. *The category* $\mathcal{V}$-Priest *is complete and cocomplete.*

We already observed in [Hofmann and Nora, 2020, Remark 4.52] that a monocone in $\mathcal{V}$-Priest is initial with respect to $\mathcal{V}$-Priest $\longrightarrow$ Set if and only if it is initial with respect to $\mathcal{V}$-CatCH $\longrightarrow$ Set (the same argument as in the proof of [Hofmann and Nora, 2020, Theorem A.6] applies here). At this moment we do not know whether, for instance, every separated metric compact Hausdorff space is Priestley. However, since $[0,1]^{\mathrm{op}}$ is an initial cogenerator in PosComp (see [Nachbin, 1965, Theorem 1] and [Jung, 2004, Lemma 2.2]), we have the following fact.

Proposition 2.23. *The inclusion functor* PosComp $\longrightarrow$ $[0,1]$-CatCH *corestricts to* PosComp $\longrightarrow$ $[0,1]$-Priest*.*

3 Duality theory for enriched Priestley spaces: concretely

Based on quantale-enriched categories instead of order structures, in [Hofmann and Nora, 2018] we started to develop a theory which extends Stone-type dualities from Priestley spaces first to all partially ordered compact spaces, and eventually to Priestley spaces enriched in the complete lattice $[0,1]$ with a continuous quantale structure $\otimes\colon [0,1]\times[0,1] \longrightarrow [0,1]$ with neutral element 1. In both cases, the strategy is to

1. establish first an embedding of a category of spaces and continuous relations (in technical terms: Kleisli categories for the Hausdorff respectively Vietoris monad) into a category of finitely (co)complete $\mathcal{V}$-categories, and then

2. to identify on both sides those morphisms that correspond to functions between spaces.

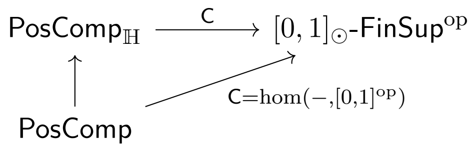
3. Finally, we would also like to identify the image of the embedding functor.

We have largely achieved these goals for the case of partially ordered compact spaces. However, for enriched Priestley spaces (and enriched relations) we were only able to provide the first step. In this section we contribute to the second step and show how to restrict the embedding result for enriched relations to categories of functions, *for the Łukasiewicz quantale* $[0,1]_\odot$. To provide the context, we recall the relevant results for partially ordered compact spaces and the embedding result for enriched relations.

In analogy with classic Halmos-type dualities for Boolean and Priestley spaces, our starting point is the category $[0,1]_\odot$-FinSup of finitely cocomplete $[0,1]_\odot$-categories and $[0,1]_\odot$-functors that preserve finite weighted colimits and the Vietoris monad $\mathbb{H} = (\mathsf{H}, w, h)$ on the category PosComp of partially ordered compact Hausdorff spaces and monotone continuous maps.

Remark 3.1. More information on power constructions in topology can be found in [Schalk, 1993a,b]. In our previous work [Hofmann and Nora, 2018; Hofmann *et al.*, 2019] we used the notation $\mathbb{V}$ instead of $\mathbb{H}$; however, in this paper we think of the classic Vietoris topology [Vietoris, 1922] as an extension of the Hausdorff metric and reserve the designation $\mathbb{V}$ for the monad based on presheafs $X \longrightarrow \mathcal{V}$ rather than subsets $A \subseteq X$.

We obtain the commutative diagram



of functors. However, unlike the functor $\mathsf{C} = \hom(-,1)\colon \mathsf{Priest}_{\mathbb{H}} \longrightarrow \mathsf{FinSup}^{\mathrm{op}}$ in the classical case, the functor $\mathsf{C}\colon \mathsf{PosComp}_{\mathbb{H}} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinSup}^{\mathrm{op}}$ is not fully faithful, as the next example shows.

Example 3.2. As observed in [Hofmann and Nora, 2018, Example 6.16], for every $u \in [0,1]$, the map $u \odot -\colon [0,1] \longrightarrow [0,1]$ is a morphism in $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ sending 1 to u. On the other hand, there are only two morphisms of type $1 \,\circ\!\!\!\longrightarrow 1$ in $\mathsf{PosComp}_{\mathbb{H}}$.

Therefore we have to consider further structure on the right-hand side. The starting point is the following observation.

Theorem 3.3. *The category* $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ *has a bimorphism representing monoidal structure.*

Proof. See [Kelly, 1982, Section 6.5]. □

This leads us to the category

$$\mathrm{Mon}([0,1]_{\odot}\text{-}\mathsf{FinSup})$$

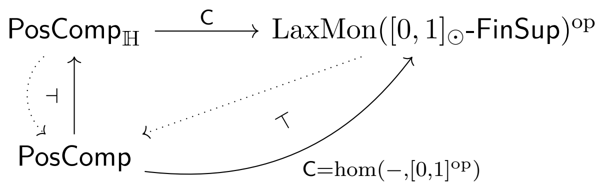
of monoids and homomorphisms in $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ with respect to the above-mentioned monoidal structure and *with neutral element the top-element*, and to the category

$$\mathrm{LaxMon}([0,1]_{\odot}\text{-}\mathsf{FinSup})$$

with the same objects as $\mathrm{Mon}([0,1]_{\odot}\text{-}\mathsf{FinSup})$, but now with morphisms those of $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ that preserve the monoid structure laxly:

$$\Phi(\psi_1 \otimes \psi_2) \le \Phi(\psi_1) \otimes \Phi(\psi_2).$$

We obtain the commutative diagram



of functors represented by solid arrows, and the monad morphism $j = (j_X)_X$ induced by C is given by the family of maps

$$j_X\colon \mathsf{H}X \longrightarrow [\mathsf{C}X,[0,1]],\; A \longmapsto \Phi_A,$$

with $\Phi_A\colon \mathsf{C}X \longrightarrow [0,1],\; \psi \longmapsto \sup_{x\in A}\psi(x)$.

Theorem 3.4. *The functor*

$$\mathsf{C}\colon \mathsf{PosComp}_{\mathbb{H}} \longrightarrow \mathrm{LaxMon}([0,1]_{\odot}\text{-}\mathsf{FinSup})^{\mathrm{op}}$$

is fully faithful.

Proof. The assertion follows from the fact that the monad morphism j is an isomorphism, see [Hofmann and Nora, 2018, Theorem 6.14]. □

To be able to restrict Theorem 3.4 to $\mathsf{PosComp}$, we have the following two results.

Proposition 3.5. *Let X be in $\mathsf{PosComp}$ and $A \subseteq X$ closed and upper. Then A is irreducible if and only if Φ_A satisfies*

$$\Phi_A(1) = 1 \quad \textit{and} \quad \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

Proof. See [Hofmann and Nora, 2018, Proposition 6.7]. □

Corollary 3.6. *Let $\varphi\colon X \longrightarrow\!\!\!\!\!\!\circ\;\; Y$ be a morphism in $\mathsf{PosComp}_{\mathbb{H}}$. Then φ is a function if and only if $\mathsf{C}\varphi$ is a morphism in $\mathrm{Mon}([0,1]_{\odot}\text{-}\mathsf{FinSup})$.*

Proof. See [Hofmann and Nora, 2018, Corollary 6.8]. □

Therefore we conclude:

Theorem 3.7. *The functor*

$$\mathsf{C}\colon \mathsf{PosComp}_{\mathbb{H}} \longrightarrow \mathrm{LaxMon}([0,1]_{\odot}\text{-}\mathsf{FinSup})^{\mathrm{op}}$$

of Theorem 3.4 restricts to a fully faithful functor

$$\mathsf{C}\colon \mathsf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_{\odot}\text{-}\mathsf{FinSup})^{\mathrm{op}}.$$

Proof. See [Hofmann and Nora, 2018, Corollary 6.15]. □

As we already observed in [Hofmann and Nora, 2018], we obtain results closer to the classical case by also enriching the topological side; that is, by considering enriched Priestley spaces and the *enriched* Vietoris monad $\mathbb{V} = (\mathsf{V}, w, h)$. The latter is introduced in [Hofmann, 2014] in the context of $\mathcal{U}$-categories and $\mathcal{U}$-functors, for a *topological theory* $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ based on the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$. For an overview of the background theory we refer to [Hofmann, 2014, Section 1], to understand the content of this section it is enough to know the following facts.

- An $\mathcal{U}$-category (X, a) is given by a set X and a map $a\colon \mathsf{U}X \times X \longrightarrow \mathcal{V}$ satisfying two axioms similar to the ones of a $\mathcal{V}$-category.
- The category of $\mathcal{U}$-categories and $\mathcal{U}$-functors is denoted as $\mathcal{U}\text{-}\mathsf{Cat}$.

- By combining the internal hom and the convergence $\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}$, the quantale $\mathcal{V}$ becomes an $\mathcal{U}$-category where $(\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v)$.
- The underlying set of $\mathsf{V}X$ is the set

$$\{\text{all } \mathcal{U}\text{-functors } \varphi\colon X \longrightarrow \mathcal{V}\}.$$

- Finally, in Remark 3.10 we provide an elementary description of the Kleisli category of the monad $\mathbb{V} = (\mathsf{V}, w, \hbar)$, restricted to $\mathcal{V}$-categorical compact Hausdorff spaces.

Examples 3.8. For $\mathcal{V} = 2$, $\mathcal{U}$-categories correspond to topological spaces and $\mathcal{U}$-functors to continuous maps (see [Barr, 1970]). The topological space 2 is the Sierpiński space with $\{1\}$ closed, and $\mathsf{V}X$ is the lower Vietoris space. On the other hand, for the multiplication $*$ on $[0,1]$, an $\mathcal{U}$-category is essentially an approach space (see [Lowen, 1997]), thanks to the isomorphism of quantales $[0,1]_* \simeq \overleftarrow{[0,\infty]}_+$.

Theorem 3.9. *The monad* $\mathbb{V} = (\mathsf{V}, w, \hbar)$ *on* $\mathcal{U}_\odot$-Cat *restricts to* $[0,1]_\odot$-CatCH *and* $[0,1]_\odot$-Priest.

Proof. See [Hofmann, 2014, Theorem 4.20] and [Hofmann and Nora, 2018, Corollary 9.7]. ☐

Remark 3.10. By [Hofmann, 2014, Section 8], the Kleisli category of the monad $\mathbb{V} = (\mathsf{V}, w, \hbar)$ on $[0,1]_\odot$-CatCH is equivalent to the category with

- objects all $[0,1]_\odot$-categorical compact Hausdorff spaces,
- and a morphism $\varphi\colon (X, a_0, \alpha) \longrightarrow\!\!\!\!\!\!\circ\;\; (Y, b_0, \beta)$ is a $[0,1]_\odot$-distributor $\varphi\colon (X, a_0) \longrightarrow (Y, b_0)$ so that the diagram

$$\begin{array}{ccc} \mathsf{U}X & \xrightarrow{\mathsf{U}\varphi} & \mathsf{U}Y \\ {\scriptstyle a_0\cdot\alpha}\downarrow & & \downarrow{\scriptstyle b_0\cdot\beta} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

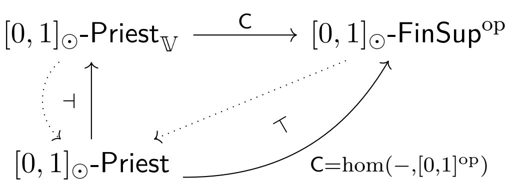
of $[0,1]_\odot$-distributors commutes.

In the sequel we will freely use this perspective, in particular, the functor

$$[0,1]_\odot\text{-}\mathsf{CatCH} \longrightarrow ([0,1]_\odot\text{-}\mathsf{CatCH})_{\mathbb{V}}$$

sends $f\colon (X, a_0, \alpha) \longrightarrow (Y, b_0, \beta)$ to $f_* = b_0 \cdot f\colon (X, a_0, \alpha) \longrightarrow\!\!\!\!\!\!\circ\;\; (Y, b_0, \beta)$. More general, a map $f\colon X \longrightarrow Y$ between $[0,1]_\odot$-categorical compact Hausdorff spaces (X, a_0, α) and (Y, b_0, β) is a morphism $f\colon (X, a_0, \alpha) \longrightarrow (Y, b_0, \beta)$ in $[0,1]_\odot$-CatCH if and only if $f_* = b_0 \cdot f$ is a morphism $f_*\colon (X, a_0, \alpha) \longrightarrow\!\!\!\!\!\!\circ\;\; (Y, b_0, \beta)$ in $([0,1]_\odot$-$\mathsf{CatCH})_{\mathbb{V}}$.

Now we come back to our study of dualities. In this setting, we obtain the commutative diagram



of functors represented by solid arrows. We stress that here the functor

$$\mathsf{C}\colon [0,1]_{\odot}\text{-}\mathsf{Priest}_{\mathbb{V}} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinSup}^{\mathrm{op}}$$

is a lifting of the hom-functor $\hom(-,1)$. The X-component of the induced monad morphism j is given by

$$j_X\colon \mathsf{V}X \longrightarrow [\mathsf{C}X,[0,1]], \quad (\varphi\colon 1 \nrightarrow X) \longmapsto \left(\psi \mapsto \psi\cdot\varphi = \bigvee_{x\in X}(\psi(x)\otimes\varphi(x))\right).$$

Similarly to Theorem 3.4, we have:

Theorem 3.11. *The functor*

$$\mathsf{C}\colon [0,1]_{\odot}\text{-}\mathsf{Priest}_{\mathbb{V}} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinSup}^{\mathrm{op}}$$

is fully faithful.

Proof. The assertion follows from the fact that the monad morphism j is an isomorphism, see [Hofmann and Nora, 2018, Theorem 9.10]. □

The principal goal of this section is to obtain a restriction of the functor of Theorem 3.11 to a fully faithful functor defined on $[0,1]_{\odot}$-Priest — a question left open in [Hofmann and Nora, 2018]. To do so, we identify those $[0,1]_{\odot}$-functors $\Phi\colon \mathsf{C}X \longrightarrow [0,1]$ which correspond to "the points of X inside $\mathsf{V}X$"; that is, to the $\mathfrak{U}_{\odot}$-functors of the form $a(\dot{x},-)\colon X \longrightarrow [0,1]$. We shall employ the fact that the quantale $[0,1]_{\odot}$ is a ***Girard quantale***: for every $u\in[0,1]$, $u = \hom(\hom(u,\bot),\bot)$. We recall that $\hom(u,\bot) = 1-u$ and put $u^{\bot} = 1-u$. Also note that $(-)^{\bot}\colon [0,1] \longrightarrow [0,1]^{\mathrm{op}}$ is an isomorphism in $[0,1]_{\odot}$-Priest.

In a nutshell, our strategy is the same as in the ordered case: we show that an additional property on Φ translates to "$\varphi\colon X \longrightarrow [0,1]$ is irreducible", and "soberness" of X guarantees $\varphi = a(\dot{x},-)$, for some $x\in X$. Therefore we need to introduce these notions for $\mathfrak{U}_{\odot}$-categories, which fortunately was already done in [Clementino and Hofmann, 2009]. In our context, "sober" means *Cauchy-complete* (called *Lawvere complete* in [Clementino and Hofmann, 2009]) and "irreducible" means *left adjoint $\mathfrak{U}_{\odot}$-distributor*. We do not introduce these notions here but rather refer to the before-mentioned literature; for our purpose it is enough to recall the following two results.

Theorem 3.12. *An $\mathfrak{U}_{\odot}$-functor $\varphi\colon X \longrightarrow [0,1]$ (viewed as an $\mathfrak{U}_{\odot}$-distributor from 1 to X) is left adjoint if and only if the representable $[0,1]_{\odot}$-functor*

$$[\varphi,-]\colon \mathfrak{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1]) \longrightarrow [0,1], \quad \varphi' \longmapsto \bigwedge_{x\in X}\hom(\varphi(x),\varphi'(x))$$

preserves copowers and finite suprema.

Proof. See [Hofmann and Stubbe, 2011, Proposition 3.5]. □

Theorem 3.13. *Every $[0,1]_{\odot}$-categorical compact Hausdorff space X is Cauchy-complete (viewed as an $\mathfrak{U}_{\odot}$-category); that is, every left adjoint $\mathfrak{U}_{\odot}$-distributor φ from 1 to X is of the form $\varphi = a(\dot{x},-)$, for some $x\in X$.*

Proof. See [Hofmann and Reis, 2018, Corollary 4.18]. $\square$

To link Theorem 3.12 with our situation, we view an $\mathcal{U}_{\odot}$-functor $\varphi\colon X \longrightarrow [0,1]$ as a $[0,1]_{\odot}$-distributor $\varphi\colon 1 \longrightarrow\!\!\!\!\!\circ\; X$ and note that the diagram

$$\begin{array}{ccc}
[0,1]_{\odot}\text{-}\mathsf{Dist}(X,1) & \xrightarrow{(-)^{\perp}} & [0,1]_{\odot}\text{-}\mathsf{Dist}(1,X)^{\mathrm{op}} \\
{\scriptstyle (-\cdot\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle [\varphi,-]^{\mathrm{op}}} \\
[0,1] & \xrightarrow[(-)^{\perp}]{} & [0,1]^{\mathrm{op}}
\end{array}$$

commutes in $[0,1]_{\odot}$-Cat (see [Hofmann and Reis, 2018, Proposition 4.35]). Furthermore, we can restrict the top line of the diagram above to the $[0,1]_{\odot}$-functor

$$(-)^{\perp}\colon \mathcal{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow \mathcal{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1])^{\mathrm{op}}.$$

Note that we consider $\mathcal{U}_{\odot}$-$\mathsf{Cat}(X,[0,1]^{\mathrm{op}})$ as $[0,1]_{\odot}$-subcategory of $[0,1]_{\odot}$-$\mathsf{Dist}(X,1)$. We obtain immediately:

Proposition 3.14. *An $\mathcal{U}_{\odot}$-functor $\varphi\colon X \longrightarrow [0,1]$ is a left adjoint $\mathcal{U}_{\odot}$-distributor φ from 1 to X if and only if the $[0,1]_{\odot}$-functor $(-\cdot\varphi)\colon \mathcal{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow [0,1]$ preserves powers and finite infima.*

Finally, for an object X in $[0,1]_{\odot}$-Priest, we will show that the inclusion $[0,1]_{\odot}$-functor

$$\mathsf{C}X \hookrightarrow \mathcal{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}})$$

is $\bigvee$-dense. This property guarantees that $-\cdot\varphi\colon \mathcal{U}_{\odot}\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow [0,1]$ preserves powers and finite infima if and only if $-\cdot\varphi\colon \mathsf{C}X \longrightarrow [0,1]$ does so.

For every $\mathcal{U}_{\odot}$-category (X,a), the $[0,1]_{\odot}$-subcategory

$$\{\text{all } \mathcal{U}_{\odot}\text{-functors } \varphi\colon X \longrightarrow [0,1]\} \subseteq [0,1]^X \tag{3.i}$$

is closed under weighted limits and finite weighted colimits (see [Hofmann, 2007, Corollary 5.3 e Proposition 6.11]); we shall show now that this property characterises the collection of all $\mathcal{U}_{\odot}$-functors $\varphi\colon X \longrightarrow [0,1]$. This way we transport a well-known fact from approach theory (see [Lowen, 1997]) to the "Łukasiewicz setting".

In general, every $[0,1]_{\odot}$-subcategory $\mathcal{R} \subseteq [0,1]^X$ closed under weighted limits and finite weighted colimits corresponds to a monad

$$\mu\colon [0,1]^X \longrightarrow [0,1]^X \qquad (1 \le \mu,\ \mu\mu \le \mu)$$

where the $[0,1]_{\odot}$-functor μ preserves finite weighted colimits. Here, given $\mathcal{R} \subseteq [0,1]^X$,

$$\mu(\alpha) = \bigwedge\{\varphi \mid \varphi \in \mathcal{R},\quad \alpha \le \varphi\},$$

and, for a monad $\mu\colon [0,1]^X \longrightarrow [0,1]^X$,

$$\mathcal{R} = \{\alpha \in [0,1]^X \mid \mu(\alpha) = \alpha\}.$$

For a subset $A \subseteq X$, we write $\chi_A\colon X \longrightarrow [0,1]$ for the characteristic function of A. The following key result is essentially [Lowen, 1997, Proposition 1.6.5].

Proposition 3.15. *Let $\mu, \mu' \colon [0,1]^X \longrightarrow [0,1]^X$ be monads that preserve finite weighted colimits. Then $\mu = \mu'$ if and only if $\mu(\chi_A) = \mu'(\chi_A)$, for all $A \subseteq X$.*

Note that, for a $[0,1]_\odot$-subcategory $\mathcal{R} \subseteq [0,1]^X$ closed under weighted limits and finite weighted colimits and with corresponding monad μ, we have

$$\mu(\chi_A)(x) = \bigwedge\{\varphi \mid \varphi \in \mathcal{R}, \chi_A \leq \varphi\} = \bigwedge\{\varphi \mid \varphi \in \mathcal{R} \text{ and, for all } z \in A,\ \varphi(z) = 1\},$$

for all $x \in X$. For an $\mathfrak{U}_\odot$-category (X, a), the monad μ corresponding to (3.i) is given by

$$\mu(\alpha)(x) = \bigvee_{\mathfrak{x} \in UX} a(\mathfrak{x}, x) \odot \xi U\alpha(\mathfrak{x}),$$

for all $\alpha \in [0,1]^X$. In particular, for every $A \subseteq X$,

$$\mu(\chi_A)(x) = \bigvee_{\mathfrak{x} \in UX} a(\mathfrak{x}, x) \odot \xi U\chi_A(\mathfrak{x}), = \bigvee_{\mathfrak{x} \in UA} a(\mathfrak{x}, x),$$

for all $x \in X$.

Lemma 3.16. *Let $\mathcal{R} \subseteq [0,1]^X$ be a $[0,1]_\odot$-subcategory closed under weighted limits and finite weighted colimits and $a \colon \mathsf{U}X \times X \longrightarrow [0,1]$ be the initial convergence on X induced by the cone $(\varphi \colon X \longrightarrow [0,1])_{\varphi \in \mathcal{R}}$ in $\mathfrak{U}_\odot$-Cat. Then the following assertions hold.*

1. $a(\mathfrak{x}, x) = \bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R}, \quad \xi U\varphi(\mathfrak{x}) = 1\}$, *for all $\mathfrak{x} \in UX$ and $x \in X$.*
2. *For all $A \subseteq X$ and $x \in X$,*

$$\bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R} \text{ and, for all } z \in A,\ \varphi(z) = 1\} = \bigvee_{\mathfrak{x} \in UA} a(\mathfrak{x}, x).$$

Proof. To see the first statement, note that

$$a(\mathfrak{x}, x) = \bigwedge\{\hom(\xi\mathsf{U}\varphi(\mathfrak{x}), \varphi(x)) \mid \varphi \in \mathcal{R}\} \leq \bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R},\ \xi\mathsf{U}\varphi(\mathfrak{x}) = 1\}.$$

On the other hand, for every $\varphi \in \mathcal{R}$, put $u = \xi\mathsf{U}\varphi(\mathfrak{x})$. Then $\hom(u, \varphi) \in \mathcal{R}$ and, since $\hom(u, -) \colon [0,1] \longrightarrow [0,1]$ is continuous with respect to the Euclidean topology,

$$\xi\mathsf{U}(\hom(u, \varphi))(\mathfrak{x}) = \xi\mathsf{U}(\hom(u, -))(\varphi(\mathfrak{x})) = \hom(u, \xi\mathsf{U}\varphi(\mathfrak{x})) = 1,$$

which proves the assertion. Regarding the second statement, the inequality

$$\bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R} \text{ and, for all } z \in A,\ \varphi(z) = 1\} \geq \bigvee_{\mathfrak{x} \in UA} a(\mathfrak{x}, x)$$

is certainly true. To see the opposite inequality, put

$$u = \bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R} \text{ and, for all } z \in A,\ \varphi(z) = 1\}.$$

Let $v < u$ und put $\varepsilon = u - v$, then $\hom(u, v) = 1 - \varepsilon$. For every $\varphi \in \mathcal{R}$ with $\varphi(x) < v$, there exists some $z \in A$ with $\varphi(z) < 1 - \varepsilon$. In fact, if $\varphi(z) \geq 1 - \varepsilon$ for all $z \in A$, then $\hom(1 - \varepsilon, \varphi(z)) = 1$ for all $z \in A$, but $\hom(1 - \varepsilon, \varphi(x)) = \varphi(x) + \varepsilon < u$. Therefore

$$\mathfrak{f} = \{\varphi^{-1}([0, 1 - \varepsilon]) \mid \varphi \in \mathcal{R}, \quad \varphi(x) < v\} \cup \{A\}$$

is a filter base, let $\mathfrak{x}$ be an ultrafilter finer than $\mathfrak{f}$. Then, for every $\varphi \in \mathcal{R}$ with $\varphi(x) < v$, $\xi\mathsf{U}\varphi(\mathfrak{x}) \leq 1 - \varepsilon$. Therefore

$$a(\mathfrak{x}, x) = \bigwedge\{\varphi(x) \mid \varphi \in \mathcal{R}, \quad \xi\mathsf{U}\varphi(\mathfrak{x}) = 1\} \geq v.$$ □

From Proposition 3.15 and Lemma 3.16 we conclude now:

Corollary 3.17. *Let $\mathcal{R} \subseteq [0,1]^X$ be a $[0,1]_\odot$-subcategory closed under weighted limits and finite weighted colimits, and consider on X the initial convergence $a\colon \mathsf{U}X \times X \longrightarrow [0,1]$ induced by $\mathcal{R}$. Then*

$$\mathcal{R} = \{\textit{all } \mathfrak{U}_\odot\textit{-functors } \varphi\colon X \longrightarrow [0,1]\}.$$

Corollary 3.18. *Let $\mathcal{R}, \mathcal{R}' \subseteq [0,1]^X$ be $[0,1]_\odot$-subcategories closed under weighted limits and finite weighted colimits. If $\mathcal{R}$ and $\mathcal{R}'$ induce the same convergence, then $\mathcal{R} = \mathcal{R}'$.*

We return now to $[0,1]_\odot$-enriched Priestley spaces.

Proposition 3.19. *Let X be in $[0,1]_\odot$-Priest and $\mathcal{R}$ be the closure under infima of $[0,1]_\odot$-$\mathsf{Priest}(X,[0,1])$ in $[0,1]^X$. Then the $[0,1]_\odot$-subcategory $\mathcal{R} \subseteq [0,1]^X$ is closed under weighted limits and finite weighted colimits.*

Proof. Since the maps

$$\vee\colon [0,1] \times [0,1] \longrightarrow [0,1], \qquad [0,1] \longrightarrow [0,1],\ u \longmapsto 0,$$
$$\wedge\colon [0,1] \times [0,1] \longrightarrow [0,1], \qquad [0,1] \longrightarrow [0,1],\ u \longmapsto 1$$

as well as the maps

$$\hom(u,-)\colon [0,1] \longrightarrow [0,1] \quad \text{and} \quad u \odot -\colon [0,1] \longrightarrow [0,1] \qquad (u \in [0,1])$$

are morphisms in $[0,1]_\odot$-Priest, the $[0,1]_\odot$-subcategory $[0,1]_\odot$-$\mathsf{Priest}(X,[0,1])$ of $[0,1]^X$ is closed under finite weighted limits and finite weighted colimits. Clearly, $\mathcal{R} \subseteq [0,1]^X$ is closed under all weighted limits. Since

$$\left(\bigwedge_{i\in I} \varphi_i\right) \vee \left(\bigwedge_{i\in J} \varphi_j\right) = \bigwedge_{(i,j)\in I\times J} (\varphi_i \vee \varphi_j),$$

$\mathcal{R}$ is closed in $[0,1]^X$ under binary suprema, and $\mathcal{R}$ is closed in $[0,1]^X$ under tensors since $u \odot -$ preserves non-empty infima. □

Corollary 3.20. *Let X be in $[0,1]_\odot$-Priest. Then every $\mathfrak{U}_\odot$-functor $X \longrightarrow [0,1]$ is an infimum of morphisms $X \longrightarrow [0,1]$ in $[0,1]_\odot$-Priest.*

Proof. Since $[0,1] \simeq [0,1]^{\mathrm{op}}$ in $[0,1]_\odot$-$\mathsf{Cat}^{\mathbb{U}}$ and X is Priestley, the cone $[0,1]_\odot$-$\mathsf{Priest}(X,[0,1])$ is point-separating and initial with respect to $[0,1]_\odot$-$\mathsf{CatCH} \longrightarrow \mathsf{CompHaus}$. Then, since the functor $\mathsf{K}\colon [0,1]_\odot$-$\mathsf{CatCH} \longrightarrow \mathfrak{U}_\odot$-$\mathsf{Cat}$ [Hofmann *et al.*, 2014, Proposition III.5.3.3] preserves initial mono-cones, the closure of $[0,1]_\odot$-$\mathsf{Priest}(X,[0,1])$ in $[0,1]^X$ under infima coincides with $\mathfrak{U}_\odot$-$\mathsf{Cat}(X,[0,1])$. □

Using the isomorphism $(-)^\perp\colon [0,1] \longrightarrow [0,1]^{\mathrm{op}}$, we obtain the desired result.

Corollary 3.21. *For every X in $[0,1]_\odot$-Priest, the inclusion $\mathsf{C}X \hookrightarrow \mathfrak{U}_\odot$-$\mathsf{Cat}(X,[0,1]^{\mathrm{op}})$ is $\bigvee$-dense (with respect to suprema in $[0,1]$).*

Corollary 3.22. *For every $\mathcal{U}_\odot$-functor $\varphi\colon X \longrightarrow [0,1]$, the $[0,1]_\odot$-functor*

$$(- \cdot \varphi)\colon \mathcal{U}_\odot\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow [0,1]$$

preserves finite weighted limits if and only if the $[0,1]_\odot$-functor

$$(- \cdot \varphi)\colon \mathsf{C}X \longrightarrow [0,1]$$

does so. Therefore, by Proposition 3.14, an $\mathcal{U}_\odot$-distributor φ from 1 to X is left adjoint if and only if the $[0,1]_\odot$-functor $(- \cdot \varphi)\colon \mathsf{C}X \longrightarrow [0,1]$ preserves finite weighted limits.

Proof. Clearly, if $(- \cdot \varphi)\colon \mathcal{U}_\odot\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow [0,1]$ preserves finite weighted limits, then so does $(- \cdot \varphi)\colon \mathsf{C}X \longrightarrow [0,1]$. Assume now that $(- \cdot \varphi)\colon \mathsf{C}X \longrightarrow [0,1]$ preserves finite weighted limits. Then certainly $(- \cdot \varphi)\colon \mathcal{U}_\odot\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}}) \longrightarrow [0,1]$ preserves the top-element, it is left to show the preservation of powers and binary infima. Let $u \in [0,1]$ and $\psi\colon X \longrightarrow [0,1]^{\mathrm{op}}$ in $\mathcal{U}_\odot$-Cat. Then $\psi = \bigvee_{i\in I} \alpha_i$ with $\alpha_i\colon X \longrightarrow [0,1]^{\mathrm{op}}$ in $[0,1]_\odot$-Priest, we may assume that $I \neq \varnothing$. Then, since the function "taking u-powers" $u \pitchfork -$ preserves non-empty suprema in $\mathcal{U}_\odot\text{-}\mathsf{Cat}(X,[0,1]^{\mathrm{op}})$, we obtain

$$\begin{aligned}(u \pitchfork \psi)\cdot\varphi = \left(u \pitchfork \bigvee_{i\in I}\alpha_i\right)\cdot\varphi = \left(\bigvee_{i\in I} u \pitchfork \alpha_i\right)\cdot\varphi = \bigvee_{i\in I}((u \pitchfork \alpha_i)\cdot\varphi)\\ = \bigvee_{i\in I}(u \pitchfork (\alpha_i\cdot\varphi)) = u \pitchfork \bigvee_{i\in I}(\alpha_i\cdot\varphi) = u \pitchfork (\psi\cdot\varphi).\end{aligned}$$

The preservation of binary infima can be shown analogously. □

Similarly to Corollary 3.6, we deduce

Corollary 3.23. *Let $\varphi\colon X \longrightarrow\!\!\!\!\circ\; Y$ be a morphism in $[0,1]_\odot$-$\mathsf{Priest}_{\mathbb{V}}$. Then φ comes from a morphism in $[0,1]_\odot$-Priest if and only if $\mathsf{C}\varphi$ preserves finite weighted limits.*

Proof. For $\varphi\colon 1 \longrightarrow\!\!\!\!\circ\; Y$, this follows immediately from Corollary 3.22 and Cauchy-completeness of Y. For the general case, observe that

- $\varphi\colon X \longrightarrow\!\!\!\!\circ\; Y$ is of the form $\varphi = f_*$ for some $f\colon X \longrightarrow Y$ in $[0,1]_\odot$-Priest if and only if, for all $x \in X$, there is some $y \in Y$ with $y_* = \varphi \cdot x_*$ (see Remark 3.10), and
- $\mathsf{C}\varphi$ preserves finite weighted limits if and only if, for all $x \in X$, $\mathrm{ev}_x \cdot \mathsf{C}\varphi$ preserves finite weighted limits. □

We let $[0,1]_\odot$-FinLat denote the category of finitely complete and finitely cocomplete $[0,1]_\odot$-categories and $[0,1]_\odot$-functors that preserve finite weighted limits and colimits. All told, we obtain:

Theorem 3.24. *The fully faithful functor*

$$\mathsf{C}\colon\ ([0,1]_\odot\text{-}\mathsf{Priest}_{\mathbb{V}})^{\mathrm{op}} \longrightarrow [0,1]_\odot\text{-}\mathsf{FinSup}$$

restricts to a fully faithful functor

$$\mathsf{C}\colon ([0,1]_\odot\text{-}\mathsf{Priest})^{\mathrm{op}} \longrightarrow [0,1]_\odot\text{-}\mathsf{FinLat}.$$

Remark 3.25. The categories $[0,1]_\odot$-FinSup and $[0,1]_\odot$-FinLat are $\aleph_1$-ary quasivarieties. For $[0,1]_\odot$-FinSup, this is shown in [Hofmann and Nora, 2018, Remark 2.10], and for $[0,1]_\odot$-FinLat this can be shown as in [Hofmann and Nora, 2018, Remark 2.10] by adding operations and equations for powers and finite infima. In particular, both categories are locally $\aleph_1$-ary presentable and the forgetful functor $[0,1]_\odot$-$\mathsf{FinLat} \longrightarrow [0,1]_\odot$-$\mathsf{FinSup}$ preserves limits and $\aleph_1$-filtered colimits.

Unfortunately, at the moment we are not able to provide a useful description of the image of C. Nevertheless, in the next section we will use Theorem 3.24 to obtain properties of the category of coalgebras for $\mathsf{V}\colon [0,1]_\odot$-$\mathsf{Priest} \longrightarrow [0,1]_\odot$-$\mathsf{Priest}$.

4 Duality theory for enriched Priestley spaces: abstractly

In Section 3 we presented some duality results for the category $[0,1]_\odot$-Priest which in particular expose some algebraic flavour of $[0,1]_\odot$-$\mathsf{Priest}^{\mathrm{op}}$. For a general quantale $\mathcal{V}$, we are still far away from concrete duality results, and in this section we investigate properties of $\mathcal{V}$-categorical compact Hausdorff spaces which help us to recognise $(\mathcal{V}\text{-}\mathsf{Priest})^{\mathrm{op}}$ as some sort of algebraic category.

Since we will use it frequently, below we recall an intrinsic characterisation of cofiltered limits in $\mathsf{CompHaus}$ which goes back to [Bourbaki, 1942]. We refer to this result commonly as the *Bourbaki-criterion*.

Theorem 4.1. *Let $D\colon I \longrightarrow \mathsf{CompHaus}$ be a cofiltered diagram. Then a cone $(p_i\colon L \longrightarrow D(i))_{i\in I}$ for D is a limit cone if and only if*

1. *$(p_i\colon L \longrightarrow D(i))_{i\in I}$ is point-separating, and*
2. *for every $i \in I$,*
$$\bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i.$$
That is, "the image of each p_i is as large as possible".

Remark 4.2. The second condition above is automatically satisfied if $p_i\colon L \longrightarrow D(i)$ is surjective.

Remark 4.3. The Bourbaki-criterion applies also to complete categories A with a limit preserving faithful functor $|-|\colon \mathsf{A} \longrightarrow \mathsf{CompHaus}$. In this case, the first condition above reads as

> the cone $(p_i\colon L \longrightarrow D(i))_{i\in I}$ is point-separating and initial with respect to the functor $|-|\colon \mathsf{A} \longrightarrow \mathsf{CompHaus}$.

Example 4.4. From the Bourbaki-criterion it follows at once that, for instance, every Priestley space X is a cofiltered limit of finite Priestley spaces. In fact, let $(p_i\colon X \longrightarrow X_i)_{i\in I}$ be the canonical cone for the canonical diagram of X with respect to all finite spaces. Clearly, the cone $(p_i\colon X \longrightarrow X_i)_{i\in I}$ is point-separating and initial since 2 is finite. For every index i, consider the image factorisation of p_i.

$$\begin{array}{lcc} & X & \\ & \big\downarrow{\scriptstyle p_i} & \searrow \\ \text{finite spaces:} & X_i & \hookleftarrow \operatorname{im}(p_i) \end{array}$$

Since $\mathrm{im}(p_i) \hookrightarrow X_i$ belongs to the diagram, the second condition is satisfied.

We can deduce in a similar fashion the well-known facts that every Boolean space X is a cofiltered limit of finite spaces, every compact Hausdorff space is a cofiltered limit of metrizable compact Hausdorff spaces, and so on.

Remark 4.5. The classic Stone/Priestley duality $\mathsf{Priest}^{\mathrm{op}} \sim \mathsf{DL}$ implies in particular that $\mathsf{Priest}^{\mathrm{op}}$ is a finitary variety, a fact which can also be seen abstractly using Theorems 1.1 and 1.2. Below we explain the argument in some detail as it serves as a motivation for the investigation in the remainder of this section.

1. Priest has all limits and colimits. This is well-known, but we stress that it is a special case of Corollary 2.22.

2. Every embedding in Priest is a regular monomorphism; therefore the class of embeddings coincides with the class of regular monomorphisms. We use the argument of [Hofmann, 2002b, Lemma 4.8]: for an embedding $m\colon X \longrightarrow Y$ in Priest, consider a presentation $(q_i\colon Y \longrightarrow Y_i)_{i\in I}$ as a cofiltered limit of finite Priestley spaces (= finite partially ordered sets). For every $i \in I$, take the (surjective, embedding)-factorisation

$$X \xrightarrow{\;p_i\;} X_i \xrightarrow{\;m_i\;} Y_i$$

of $q_i \cdot m$. Then also $(p_i\colon X \longrightarrow X_i)_{i\in I}$ is a limit cone (by the Bourbaki-criterion); moreover, m is the limit of the family $(m_i)_{i\in I}$.

(4.i)
$$\begin{array}{ccc} X & \overset{m}{\rightarrowtail} & Y \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle q_i} \\ X_i & \underset{m_i}{\rightarrowtail} & Y_i \end{array}$$

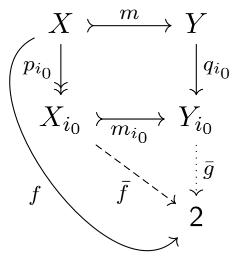
Having finite and hence discrete domain and codomain, each $m_i\colon X_i \longrightarrow Y_i$ is a regular monomorphism in $\mathsf{Pos}_{\mathrm{fin}} = \mathsf{Priest}_{\mathrm{fin}}$ (this is a special case of Theorem 2.5) and therefore also in Priest. Consequently, also $m = \lim_i m_i$ is a regular monomorphism in Priest.

3. By definition and by the above, the two-element space is a regular cogenerator in Priest.

4. The two-element space is finitely copresentable in Priest. This is very well-known; for our purpose we mention here that it is a consequence of [Hofmann, 2002a, Lemma 2.2]. In this section we observe that this result generalises beyond the finitary case (see Lemma 4.37).

5. The two-element space is regular injective in Priest. This follows immediately from finite copresentability: Consider a regular monomorphism $m\colon X \longrightarrow Y$ in Priest together with (4.i), and let $f\colon X \longrightarrow 2$ be a morphism in Priest. Since 2 is finitely copresentable, there is some $i_0 \in I$ and a morphism $\bar{f}\colon X_{i_0} \longrightarrow 2$ with $\bar{f} \cdot p_{i_0} = f$. Since 2 is injective in Pos (we stress that this is a special case of Proposition 2.6), there is some $\bar{g}\colon X_{i_0} \longrightarrow Y_{i_0}$ with

$\bar{g} \cdot m_{i_0} = \bar{f}$. Hence, $\bar{g} \cdot q_{i_0}$ is an extension of f along m.



6. Priest has effective equivalence corelations. A direct proof, even for partially ordered compact Hausdorff spaces in general, can be found in [Abbadini and Reggio, 2020].

Note that our treatment of properties of Priest rests on results about Ord and Pos, therefore we have first a look at $\mathcal{V}$-categories.

Theorem 4.6. *$\mathcal{V}$-Cat$^{\mathrm{op}}$ is a quasivariety.*

Proof. First recall from Theorem 2.5 that the regular monomorphisms in $\mathcal{V}$-Cat are precisely the embeddings, and from Proposition 2.6 that $\mathcal{V}$ is injective and $(f\colon X \longrightarrow \mathcal{V})_f$ is initial, for every $\mathcal{V}$-category X. Moreover, $\mathcal{V}_I$ (indiscrete structure) is a cogenerator in $\mathcal{V}$-Cat and therefore $\mathcal{V} \times \mathcal{V}_I$ is a regular injective regular cogenerator. Since $\mathcal{V}$-Cat is also complete, the assertion follows. □

Remark 4.7. The observation above should be compared to the fact that "Top$^{\mathrm{op}}$ is a quasivariety", for details see [Barr and Pedicchio, 1995, 1996] and [Adámek and Pedicchio, 1997; Pedicchio and Wood, 1999].

On the other hand, for every $\mathcal{V}$, the quasivariety $\mathcal{V}$-Cat$^{\mathrm{op}}$ does not have any rank. To see this, we recall first the following result from [Gabriel and Ulmer, 1971, Page 64] (see also [Ulmer, 1971]).

Proposition 4.8. *A set is copresentable in* Set *if and only if it is a singleton.*

The corresponding result for $\mathcal{V}$-Cat is now an immediate consequence of the following observation.

Proposition 4.9. *The "discrete" functor* $\mathsf{D}\colon \mathsf{Set} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ *preserves non-empty limits, in particular cofiltered limits. If $k = \top$ is the top-element of $\mathcal{V}$, then* D *preserves also the terminal object.*

Corollary 4.10. *If X is copresentable in $\mathcal{V}$-Cat, then $|X| = 1$.*

Proof. By Proposition 4.9, the forgetful functor $|-|\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$ preserves copresentable objects since, for every $\mathcal{V}$-category X, $\hom(-, |X|) \simeq \hom(D-, X)$. □

We turn now our attention to separated $\mathcal{V}$-categories.

Theorem 4.11. *The full subcategory $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ of $\mathcal{V}\text{-}\mathsf{Cat}$ is closed under initial monocones. Therefore the inclusion functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ has a left adjoint; moreover, the canonical forgetful functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ is mono-topological with left adjoint $\mathsf{D}\colon \mathsf{Set} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ (discrete structures). Consequently, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ is complete and cocomplete, with concrete limits. A morphism $f\colon X \longrightarrow Y$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ is a monomorphism if and only if the map f is injective.*

Proof. See [Hofmann and Tholen, 2010], for instance. □

Remark 4.12. We do not know if $\mathsf{Top}_0^{\mathrm{op}}$ or $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}^{\mathrm{op}}$ are quasivarieties. Note that in both cases the class of regular monomorphisms does not coincide with the class of embeddings, as we explain below (see also [Baron, 1968]).

The description of further classes of morphisms in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ is facilitated by the notion of *L-closure* introduced in [Hofmann and Tholen, 2010].

Lemma 4.13. *Let X be a $\mathcal{V}$-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent.*

(i) $x \in \overline{M}$.

(ii) *For all $f, g\colon X \longrightarrow Y$ in $\mathcal{V}\text{-}\mathsf{Cat}$, if $f|_M = g|_M$, then $f(x) \simeq g(x)$.*

(iii) *For all $f, g\colon X \longrightarrow Y$ in $\mathcal{V}\text{-}\mathsf{Cat}$ with Y separated, if $f|_M = g|_M$, then $f(x) = g(x)$.*

(iv) *For all $f, g\colon X \longrightarrow \mathcal{V}$ in $\mathcal{V}\text{-}\mathsf{Cat}$, if $f|_M = g|_M$, then $f(x) = g(x)$.*

Corollary 4.14. *The epimorphisms in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ are precisely the L-dense $\mathcal{V}$-functors, and the regular monomorphisms the closed embeddings.*

Proof. The assertion regarding epimorphisms is in [Hofmann and Tholen, 2010, Theorem 3.8]. However, both claims follow immediately from Lemma 4.13. □

We denote by $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$ the full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ formed by all Cauchy-complete separated $\mathcal{V}$-categories. The following two results follow immediately from Corollary 4.14.

Corollary 4.15. *A separated $\mathcal{V}$-category X is Cauchy-complete if and only if X is a regular subobject of a power of $\mathcal{V}$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. Moreover, the regular monomorphisms in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$ are precisely the embeddings of $\mathcal{V}$-categories.*

Corollary 4.16. *The $\mathcal{V}$-category $\mathcal{V}$ is a regular injective regular cogenerator in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$. Hence, $(\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}})^{\mathrm{op}}$ is a quasivariety.*

Remark 4.17. Clearly, the "discrete" functor $\mathsf{D}\colon \mathsf{Set} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ preserves non-empty limits. Under some conditions (see [Clementino and Hofmann, 2009, Proposition 2.2]), every discrete $\mathcal{V}$-category is Cauchy-complete and the discrete functor $\mathsf{D}\colon \mathsf{Set} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$ is left adjoint to the forgetful functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}} \longrightarrow \mathsf{Set}$ and preserves codirected limits. Hence, in this case at most a one-element $\mathcal{V}$-category can be copresentable in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$.

Remark 4.18. In general, the category $(\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,cc}})^{\mathrm{op}}$ is not a variety, *i.e.* does not have effective equivalence corelations. A counterexample is already given by the case $\mathcal{V} = 2$ since the dual of $\mathsf{Pos} \sim 2\text{-}\mathsf{Cat}_{\mathrm{sep,cc}}$ is not a variety. This fact is well-known and follows immediately from the following facts:

- $\mathsf{Pos}^{\mathrm{op}}$ is equivalent to the category TAL of totally algebraic lattices and maps preserving all suprema and all infima (see [Rosebrugh and Wood, 1994], for instance),
- TAL is a full subcategory of the category CCD of (constructively) completely distributive lattices and maps preserving all suprema and all infima,
- the unit interval $[0,1]$ is completely distributive but not totally algebraic,
- the category CCD is monadic over Set (see [Pedicchio and Wood, 1999], and [Pu and Zhang, 2015] for a generalisation to quantaloid-enriched categories). Here the free algebra over a set X is given by the complete lattice of upsets of the powerset of X, and this lattice is totally algebraic and therefore also the free totally algebraic lattice over X.

Another important property of $\mathcal{V}$-categories and $\mathcal{V}$-functors is established in [Kelly and Lack, 2001]: $\mathcal{V}\text{-}\mathsf{Cat}$ is locally presentable, for every quantale $\mathcal{V}$. Under Assumption 4.19 below, and based on [Seal, 2005, 2009], we show that $\mathcal{V}\text{-}\mathsf{Cat}$ is locally $\aleph_1$-copresentable by describing a corresponding countable limit sketch. This will help us later to identify $\mathcal{V}\text{-}\mathsf{CatCH}$ as the model category of a $\aleph_1$-ary limit sketch in $\mathsf{CompHaus}$. To do so, *in the remainder of this section we impose the following conditions on the quantale* $\mathcal{V}$.

Assumption 4.19. We assume that the underlying lattice of $\mathcal{V}$ is completely distributive, and that there is a countable subset $D \subseteq \mathcal{V}$ so that, for all $v \in \mathcal{V}$,

$$v = \bigvee \{u \in D \mid u \lll v\}.$$

Examples 4.20. The quantales of Examples 2.3 and Example 2.4 satisfy Assumption 4.19.

Remark 4.21. Under Assumption 4.19, for each $v \in \mathcal{V}$,

$$\uparrow v = \bigcap \{\uparrow u \mid u \in D, u \lll v\}.$$

Hence, by Remark 2.9 (3), the sets $\uparrow u$ $(u \in D)$ form a subbasis for the closed sets of the dual of the Scott topology of $\mathcal{V}$.

We start with the following well-known fact.

Lemma 4.22. *The assignments*

$$(\varphi\colon X \to \mathcal{V}) \quad \longmapsto \quad (\varphi^{-1}(\uparrow u)_{u \in D})$$

and

$$(B_u)_{u\in D} \quad \longmapsto \quad (\varphi\colon X \to \mathcal{V},\, x \mapsto \bigvee \{u \in D \mid x \in B_u\})$$

define a bijection between the sets

$$\mathcal{V}^X \qquad \text{and} \qquad \{(B_u)_{u\in D} \mid \text{for all } u \in D,\ B_u \subseteq X \;\&\; B_u = \bigcap_{v \lll u} B_v\}.$$

Remark 4.23. Under the bijection above, a map $a\colon X \times X \longrightarrow \mathcal{V}$ corresponds to a family $(R_u)_{u\in D}$ of binary relations R_u on X.

Proposition 4.24. *A $\mathcal{V}$-relation $a\colon X \times X \longrightarrow \mathcal{V}$ is reflexive if and only if $\Delta_X \subseteq R_k$. Moreover, $a\colon X \times X \longrightarrow \mathcal{V}$ is transitive if and only if, for all $u, v \in D$, $R_u \cdot R_v \subseteq R_{u\otimes v}$.*

Proof. See [Seal, 2009]. □

Remark 4.25. A $\mathcal{V}$-category (X, a) is separated if and only if the relation R_k on X is anti-symmetric.

Therefore the structure of a $\mathcal{V}$-category can be equivalently described by a family of binary relations, suitably interconnected. Since a map $f\colon X \longrightarrow Y$ between $\mathcal{V}$-categories is a $\mathcal{V}$-functor if and only if f preserves the corresponding relations, we obtain at once:

Corollary 4.26. *The categories $\mathcal{V}$-Cat and $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$ are model categories in Set of an $\aleph_1$-ary countable limit sketch.*

Remark 4.27. We do not know yet wether $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep,cc}}$ is locally presentable. However, we note that in [Adámek *et al.*, 2015] this property is proven for $\mathcal{V} = [0, 1]_{\odot}$, that is, for the case of bounded metric spaces.

We turn now our attention to $\mathcal{V}$-categorical compact Hausdorff spaces. First we observe that Proposition 4.9 as well as some of its consequences generalise directly to the topological case.

Proposition 4.28. *The "discrete" functors $\mathsf{D}\colon \mathsf{CompHaus} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$ and $\mathsf{D}\colon \mathsf{CompHaus} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}}$ preserve non-empty limits. If $k = \top$ is the top-element of $\mathcal{V}$, then D preserves also the terminal object.*

Regarding copresentable compact Hausdorff spaces, we recall the following result from [Gabriel and Ulmer, 1971, 6.5(c)] (see also [Ulmer, 1971]).

Theorem 4.29.
1. *The finitely copresentable compact Hausdorff spaces are precisely the finite ones.*
2. *The $\aleph_1$-copresentable compact Hausdorff spaces are precisely the metrisable ones. In particular, the unit interval $[0, 1]$ is $\aleph_1$-copresentable in $\mathsf{CompHaus}$.*

Corollary 4.30. *For every regular cardinal λ, the forgetful functors $|-|\colon \mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathsf{CompHaus}$ and $|-|\colon \mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}} \longrightarrow \mathsf{CompHaus}$ preserve λ-copresentable objects. In particular, every finitely copresentable (separated) $\mathcal{V}$-categorical compact Hausdorff space is finite and every $\aleph_1$-copresentable (separated) $\mathcal{V}$-categorical compact Hausdorff space has a metrizable topology.*

We are particularly interested in properties of the space $\mathcal{V}$. We start with the following observation.

Proposition 4.31. *A subbase for the Lawson topology on $\mathcal{V}$ is given by the sets*

$$\{u \in \mathcal{V} \mid v \lll u\} \quad \text{and} \quad \{u \in \mathcal{V} \mid v \not\leq u\} \qquad (v \in D).$$

Proof. By definition, the Lawson topology is the join of the Scott topology and the lower topology of $\mathcal{V}$ (see Remark 2.9); we recall that the latter is generated by the sets $(\uparrow v)^{\complement}$, with $v \in \mathcal{V}$. Since the lattice $\mathcal{V}$ is completely distributive, the Scott topology of $\mathcal{V}$ has as subbase the sets (see Proposition 2.15)

$$\{u \in \mathcal{V} \mid v \lll u\},$$

with $v \in \mathcal{V}$. Since "generated topology" defines a left adjoint, the sets

$$\{u \in \mathcal{V} \mid v \lll u\} \quad \text{and} \quad \{u \in \mathcal{V} \mid v \not\leq u\} \qquad (v \in \mathcal{V})$$

form a subbase for the Lawson topology of $\mathcal{V}$. Let now $v \in \mathcal{V}$. For each $v \lll u \in \mathcal{V}$, there is some $w \in D$ with $v \lll w \lll u$, therefore

$$\{u \in \mathcal{V} \mid v \lll u\} = \bigcup_{w \in D, v \lll w} \{u \in \mathcal{V} \mid w \lll u\}.$$

Finally, since $v \in \bigvee\{w \in D \mid w \lll v\}$, we obtain $\uparrow v = \bigcap\{\uparrow w \mid w \in D, w \lll v\}$ and therefore $(\uparrow v)^{\complement} = \bigcup\{(\uparrow w)^{\complement} \mid w \in D, w \lll v\}$. □

Corollary 4.32. *The Lawson topology makes $\mathcal{V}$ a $\aleph_1$-copresentable object in* $\mathsf{CompHaus}$.

Proof. By Proposition 4.31, the Lawson topology on $\mathcal{V}$ has a countable subbase and therefore also a countable base. Hence, $\mathcal{V}$ with the Lawson topology is a metrizable compact Hausdorff space and therefore, by Theorem 4.29, $\aleph_1$-copresentable in $\mathsf{CompHaus}$. □

We shall now extend Corollary 4.26 to the topological context and show that $\mathcal{V}$-CatCH is a model category of a limit sketch in $\mathsf{CompHaus}$. To prepare this, we recall an alternative way of expressing the compatibility between topology and $\mathcal{V}$-categories which is closer to Nachbin's original definition.

Proposition 4.33. *For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α) with the same underlying set X, the following assertions are equivalent.*

(i) $\alpha\colon \mathsf{U}(X, a) \longrightarrow (X, a)$ *is a $\mathcal{V}$-functor.*

(ii) $a\colon (X, \alpha) \times (X, \alpha) \longrightarrow (\mathcal{V}, \xi_{\leq})$ *is continuous.*

Proof. See [Hofmann and Reis, 2018, Proposition 3.22]. □

Lemma 4.34. *Consider $\mathcal{V}$ with the dual of the Scott topology. Then, under the correspondence of Lemma 4.22, $\varphi\colon X \longrightarrow \mathcal{V}$ is continuous if and only if, for each $u \in D$, B_u is closed in X.*

Proof. Recall from Remark 4.21 that the sets $\uparrow u$ $(u \in D)$ form a subbase for the closed sets of the dual of the Scott topology of $\mathcal{V}$. □

Applying Lemma 4.34 to the map $a\colon (X, \alpha) \times (X, \alpha) \longrightarrow (\mathcal{V}, \xi_{\leq})$ of Proposition 4.33 gives immediately:

Theorem 4.35. *Both $\mathcal{V}$-CatCH and $\mathcal{V}$-$\mathsf{CatCH}_{\mathsf{sep}}$ are model categories in $\mathsf{CompHaus}$ of a countable $\aleph_1$-ary limit sketch. Hence, both categories are locally copresentable.*

Proof. For the second affirmation, use [Adámek and Rosický, 1994, Remark 2.63]. □

Remark 4.36. At this moment we do not have any information about the rank of the locally presentable category $\mathcal{V}$-$\mathsf{CatCH}^{\mathrm{op}}$; in particular, we do not know if $\mathcal{V}$-$\mathsf{CatCH}^{\mathrm{op}}$ is $\aleph_1$-ary locally copresentable.

In order to obtain more information on copresentable objects in $\mathcal{V}$-CatCH, we adapt now [Hofmann, 2002a, Lemma 2.2] to the case of a general regular cardinal. Here we call a λ-ary limit sketch $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ ***λ-small*** whenever there is a set M of morphisms in C of cardinality less than λ so that every morphism of C is a finite composite of morphisms in M. Hence, for $\lambda > \aleph_0$, we require the category C to be λ-small.

Lemma 4.37. *Let λ be a regular cardinal and let $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ be a λ-small limit sketch. Then a model of $\mathcal{S}$ in a category X is λ-copresentable in $\mathrm{Mod}(\mathcal{S}, \mathsf{X})$ provided that each component is λ-copresentable in X.*

Proof. See [Hofmann, 2002a, Lemma 2.2]. □

By Assumption 4.19, the limit sketch for $\mathcal{V}$-CatCH is countable which allows us to derive the following properties.

Corollary 4.38. *A $\mathcal{V}$-categorical compact Hausdorff space is $\aleph_1$-ary copresentable in $\mathcal{V}$-CatCH (respectively $\mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}}$) if and only if its underlying compact Hausdorff space is metrizable. In particular, $\mathcal{V}^{\mathrm{op}}$ is $\aleph_1$-ary copresentable in $\mathcal{V}$-CatCH and in $\mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}}$.*

Corollary 4.39. *If the quantale $\mathcal{V}$ is finite, then the finitely copresentable objects of $\mathcal{V}$-CatCH (respectively $\mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}}$) are precisely the finite ones.*

Remark 4.40. The conclusion of Lemma 4.37 is not necessarily optimal. For instance, the circle line $\mathbb{T} = {}^{\mathbb{R}}/_{\mathbb{Z}}$ is $\aleph_1$-copresentable but not finitely copresentable in $\mathsf{CompHaus}$ (see [Gabriel and Ulmer, 1971, 6.5]); hence, Lemma 4.37 implies that $\mathbb{T}$ is $\aleph_1$-copresentable in the category $\mathsf{CompHausAb}$ of compact Hausdorff Abelian groups and continuous homomorphisms. However, by the famous Pontryagin duality theorem (see [Morris, 1977], for instance), $\mathbb{T}$ is even finitely copresentable in $\mathsf{CompHausAb}$ which cannot be concluded from Lemma 4.37.

Remark 4.41. In particular, the finitely copresentable partially ordered compact spaces are precisely the finite ones. Moreover, a partially ordered compact space is $\aleph_1$-copresentable in $\mathsf{PosComp}$ if and only if its underlying compact Hausdorff topology is metrisable. This characterisation is slightly different from our result in [Hofmann *et al.*, 2018] where the $\aleph_1$-copresentable objects in $\mathsf{PosComp}$ are characterised as those spaces where both – the order and the topology – are induced by the same (not necessarily symmetric) metric.

As we show next, the results above also imply that the reflector $\pi_0 \colon \mathcal{V}\text{-}\mathsf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathsf{Priest}$ preserves $\aleph_1$-cofiltered limits. In the classical case, the corresponding property is shown in [Gabriel and Ulmer, 1971, Page 67] using Stone duality; however, our proof here is based on the Bourbaki-criterion.

Proposition 4.42. *The reflection functor $\pi_0 \colon \mathcal{V}\text{-}\mathsf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathsf{Priest}$ preserves $\aleph_1$-cofiltered limits (and even cofiltered limits if $\mathcal{V}$ is finite).*

Proof. Let $(p_i \colon X \longrightarrow D(i))_{i \in I}$ be a $\aleph_1$-cofiltered limit in $\mathcal{V}$-CatCH ($\aleph_0$-cofiltered if $\mathcal{V}$ is finite). Since $\mathcal{V}^{\mathrm{op}}$ is $\aleph_1$-ary copresentable ($\aleph_0$-ary copresentable if $\mathcal{V}$ is finite) in $\mathcal{V}$-CatCH, the cone of all morphisms of type $X \longrightarrow \mathcal{V}^{\mathrm{op}}$ is given by the cone of all morphism

$$X \xrightarrow{\ p_i\ } D(i) \xrightarrow{\ \varphi\ } \mathcal{V}^{\mathrm{op}}$$

where $i \in I$ and $\varphi\colon D(i) \longrightarrow \mathcal{V}^{\mathrm{op}}$ in $[0,1]$-CatCH. Hence, for every $i \in I$ and every $\varphi\colon X \longrightarrow \mathcal{V}^{\mathrm{op}}$, we obtain the commutative diagram

$$\begin{array}{ccccc} X & \twoheadrightarrow & \pi_0(X) & & \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle \pi_0(p_i)} & \searrow & \\ D(i) & \twoheadrightarrow & \pi_0(D(i)) & \xrightarrow[\widetilde{\varphi}]{} & \mathcal{V}^{\mathrm{op}}. \end{array}$$

Therefore the cone $(\pi_0(p_i)\colon \pi_0(X) \longrightarrow \pi_0(D(i)))_{i\in I}$ is initial with respect to the forgetful functor $\mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathsf{CompHaus}$.

Let now $i \in I$ and $x \in D(i)$ with $x \in \bigcap\{\mathrm{im}(\pi_0(D(k))) \mid k\colon j \to i \text{ in } I\}$. Let $A \subseteq X$ be the inverse image of x under the reflection map $D(i) \longrightarrow \pi_0(D(i))$. Then, for every $k\colon j \to i$ in I, $\varnothing \neq A \cap \mathrm{im}(k)$. Since the set $\{\mathrm{im}(k) \mid k\colon j \to i\}$ is codirected and A is compact, we obtain

$$\varnothing \neq \bigcap_{k\colon j \to i} A \cap \mathrm{im}\, D(k) = A \cap \bigcap_{k\colon j\to i} \mathrm{im}\, D(k) = A \cap \mathrm{im}(p_i).$$

Therefore $x \in \mathrm{im}(\pi_0(p_i))$. □

Remark 4.43. Corollary 4.38 allows for an alternative proof of Corollary 4.42 for $\otimes = \odot$ in the spirit of [Gabriel and Ulmer, 1971, Page 67]. Firstly, the dualising object $[0,1]$ induces a natural dual adjunction (see [Porst and Tholen, 1991])

$$[0,1]_{\odot}\text{-}\mathsf{CatCH} \underset{\hom(-,[0,1])}{\overset{\mathsf{C}=\hom(-,[0,1])}{\rightleftarrows}}\!\!\!\!\!\!{\scriptstyle\perp}\;\; [0,1]_{\odot}\text{-}\mathsf{FinLat}^{\mathrm{op}}$$

where the fixed subcategory on the left-hand side is precisely $[0,1]_{\odot}$-Priest. Then the functor $\pi_0\colon [0,1]_{\odot}$-$\mathsf{CatCH} \longrightarrow [0,1]_{\odot}$-$\mathsf{Priest}$ is the composite of the functor $\mathsf{C}\colon [0,1]_{\odot}$-$\mathsf{CatCH} \longrightarrow [0,1]_{\odot}$-$\mathsf{FinLat}^{\mathrm{op}}$ and the right adjoint functor $[0,1]_{\odot}$-$\mathsf{FinLat}^{\mathrm{op}} \longrightarrow [0,1]_{\odot}$-$\mathsf{Priest}$ above (see [Lambek and Rattray, 1979, Theorem 2.0], and note that, for every L in $[0,1]_{\odot}$-FinLat, the space $\hom(L,[0,1])$ is Priestley by construction).

Combining Corollaries 4.42 and 4.30 we obtain:

Corollary 4.44. *1. An object is $\aleph_1$-ary copresentable in $\mathcal{V}$-Priest if and only if its underlying compact Hausdorff space is metrizable. In particular, $\mathcal{V}^{\mathrm{op}}$ is $\aleph_1$-ary copresentable in $\mathcal{V}$-Priest.*

2. Assume that $\mathcal{V}$ is finite. Then an object is finitely copresentable in $\mathcal{V}$-Priest if and only if it is finite. In particular, $\mathcal{V}^{\mathrm{op}}$ is finitely copresentable in $\mathcal{V}$-Priest.

Proof. Since the left adjoint $\pi_0\colon \mathcal{V}$-$\mathsf{CatCH} \longrightarrow \mathcal{V}$-$\mathsf{Priest}$ of $\mathcal{V}$-$\mathsf{Priest} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$ preserves $\aleph_1$-codirected limits, the inclusion functor $\mathcal{V}$-$\mathsf{Priest} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$ preserves $\aleph_1$-copresentable objects. Furthermore, since $\mathcal{V}$-Priest is closed in $\mathcal{V}$-CatCH under limits, $\mathcal{V}$-$\mathsf{Priest} \longrightarrow \mathcal{V}$-$\mathsf{CatCH}$ reflects $\aleph_1$-copresentable objects. The second affirmation follows similarly. □

Corollary 4.45. *The fully faithful right adjoint functor*

$$\mathsf{C}\colon [0,1]_{\odot}\text{-}\mathsf{Priest}^{\mathrm{op}} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinLat}$$

of Theorem 3.24 preserves $\aleph_1$-filtered colimits.

Theorem 4.46. *The category $\mathcal{V}$-Priest is locally $\aleph_1$-ary copresentable. If $\mathcal{V}$ is finite, then $\mathcal{V}$-Priest is even locally $\aleph_0$-ary copresentable.*

Proof. By the Bourbaki-criterion, every X in $\mathcal{V}$-Priest is a limit of the canonical diagram of X with respect to the full subcategory of $\mathcal{V}$-Priest defined by all $\aleph_1$-copresentable objects. Since $\mathcal{V}$-Priest is complete, we conclude that $\mathcal{V}$-Priest is locally $\aleph_1$-ary copresentable. If $\mathcal{V}$ is finite, the same argument works with $\aleph_0$ *in lieu* of $\aleph_1$. □

Remark 4.47. The results of this section together with Theorem 3.24 and Remark 3.25 allow us to deduce some nice properties of the category $\mathrm{CoAlg}(\mathsf{V})$ of coalgebras and homomorphisms for the functor $\mathsf{V}\colon [0,1]_{\odot}\text{-}\mathsf{Priest} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{Priest}$. We write A for the isomorphism closure of the image of

$$\mathsf{C}\colon ([0,1]_{\odot}\text{-}\mathsf{Priest})^{\mathrm{op}} \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinLat};$$

hence, A is locally $\aleph_1$-presentable and $\mathsf{A} \longrightarrow [0,1]\text{-}\mathsf{FinLat}_{\odot}$ is a reflective full subcategory closed under $\aleph_1$-filtered colimits. Also note that the category $\mathrm{Un}([0,1]_{\odot}\text{-}\mathsf{FinSup})$ of unary algebras and homomorphisms in $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ is locally $\aleph_1$-ary presentable and the forgetful functor $\mathrm{Un}([0,1]_{\odot}\text{-}\mathsf{FinSup}) \longrightarrow [0,1]_{\odot}\text{-}\mathsf{FinSup}$ preserves $\aleph_1$-filtered colimites (see [Adámek and Rosický, 1994, Proposition 1.53]). Finally, $\mathrm{CoAlg}(\mathsf{V})^{\mathrm{op}}$ is equivalent to the category B of A-objects equipped with a unary operation in $[0,1]_{\odot}\text{-}\mathsf{FinSup}$ and homomorphisms, which can be obtained as the pullback

$$\begin{array}{ccc} \mathrm{Un}([0,1]_{\odot}\text{-}\mathsf{FinSup}) & \longleftarrow & \mathsf{B} \\ \downarrow & & \downarrow \\ [0,1]_{\odot}\text{-}\mathsf{FinSup} & \longleftarrow & \mathsf{A} \end{array}$$

of $\aleph_1$-accessible functors. Consequently, B is locally $\aleph_1$-ary presentable (see [Bird, 1984; Bird *et al.*, 1989; Makkai and Paré, 1989]) and therefore $\mathrm{CoAlg}(\mathsf{V})$ is locally $\aleph_1$-ary copresentable. In particular, $\mathrm{CoAlg}(\mathsf{V})$ is complete.

Next, we link $\mathcal{V}$-categorical compact Hausdorff spaces with compact $\mathcal{V}$-categories. To do so, we also *impose now the following condition.*

Assumption 4.48. For the neutral element k of $\mathcal{V}$, the set

$$\{u \in \mathcal{V} \mid u \lll k\}$$

is directed.

Then $\bot < k$ and, for all $u, v \in \mathcal{V}$,

$$k \le u \vee v \implies (k \le u \quad \text{or} \quad k \le v);$$

which guarantees that the L-closure is topological (see [Hofmann and Tholen, 2010, Proposition 3.3]). Moreover, under this condition, a separated $\mathcal{V}$-category X induces a Hausdorff topology; if this topology is compact, X becomes a $\mathcal{V}$-categorical compact Hausdorff space (see [Hofmann and Reis, 2018, Theorem 3.28 and Propositions 3.26 and 3.29]). We let $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}}$ denote the full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ defined by those $\mathcal{V}$-categories with compact topology, then this construction defines a fully faithful functor

$$\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}} \longrightarrow \mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}.$$

From Lemma 4.13 and Corollary 4.14 we obtain immediately:

Corollary 4.49. *Let* $f\colon X \longrightarrow Y$ *be in* $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}}$*. Then*

1. f *is a regular monomorphism in* $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ *if and only if* f *is an embedding, and*
2. f *is an epimorphism in* $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ *if and only if* f *is surjective.*

Lemma 4.50. *If the* $\mathcal{V}$*-category* $\mathcal{V}$ *is compact, then the L-topology on* $\mathcal{V}$ *coincides with the Lawson topology.*

Proof. By [Hofmann and Nora, 2020, Remark 4.27], for every $u \in \mathcal{V}$, the sets $\uparrow u$ and $\downarrow u$ are closed in $\mathcal{V}$ with respect to the L-closure. □

Example 4.51. In particular, the L-closure on the $[0,1]_{\odot}$-category $[0,1]$ induces the Euclidean topology with convergence ξ.

Corollary 4.52. *Assume that the* $\mathcal{V}$*-category* $\mathcal{V}$ *is compact. Then we have a fully faithful functor*

$$\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}} \longrightarrow \mathcal{V}\text{-}\mathsf{Priest},$$

and every $\mathcal{V}$*-enriched Priestley space is a cofiltered limit of compact separated* $\mathcal{V}$*-categories. Moreover:*

- *every embedding* $f\colon X \longrightarrow Y$ *in* $\mathcal{V}\text{-}\mathsf{Priest}$ *is a regular monomorphism, and*
- *therefore the epimorphisms in* $\mathcal{V}\text{-}\mathsf{Priest}$ *are precisely the surjective morphisms.*

Consequently, $\mathcal{V}^{\mathrm{op}}$ *is a regular cogenerator in* $\mathcal{V}\text{-}\mathsf{Priest}$*.*

Proof. Regarding embeddings, we use the same argument as in Remark 4.5 (2). Every epimorphism e in $\mathcal{V}\text{-}\mathsf{Priest}$ factorises as $e = m\cdot g$ where g is surjective and m is a regular monomorphism, hence m is an isomorphism and therefore e is surjective. □

Remark 4.53. If $\mathcal{V}$ is finite, then $\mathcal{V}^{\mathrm{op}}$ is finitely copresentable in $\mathcal{V}\text{-}\mathsf{Priest}$ and, with the same argument as in Remark 4.5 (5), we deduce that $\mathcal{V}^{\mathrm{op}}$ is regular injective in $\mathcal{V}\text{-}\mathsf{Priest}$. Unfortunately, the same argument does not seem to work if $\mathcal{V}$ is infinite since in this case

- $\mathcal{V}^{\mathrm{op}}$ is countably but in general not finitely copresentable in $\mathcal{V}\text{-}\mathsf{Priest}$, but
- we are not able to prove that every $\mathcal{V}$-enriched Priestley space is a $\aleph_1$-cofiltered limit of compact separated $\mathcal{V}$-categories.

We finish this paper by bringing another well-known result from order theory into the enriched realm: every $\mathcal{V}$-categorical compact Hausdorff space is a quotient of a Priestley one. We shall make use of the free $\mathcal{V}$-categorical compact Hausdorff space, for $\mathcal{U}$-category (X,a), and therefore assume that our *topological theory* $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ *is strict* in the sense of [Hofmann, 2007]:

Assumption 4.54. The complete lattice $\mathcal{V}$ is completely distributive, and we consider the Lawson topology $\xi\colon U\mathcal{V} \longrightarrow \mathcal{V}$ (see Remark 2.9). Furthermore, the tensor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is continuous with respect to the Lawson topology.

We consider the free $\mathcal{V}$-categorical compact Hausdorff space

$$(\mathsf{U}X, \widehat{a}, m_X)$$

of a $\mathcal{U}$-category (X, a) where $\widehat{a} = \mathsf{U}a \cdot m_X^\circ$ (see [Hofmann *et al.*, 2014, Theorem III.5.3.5]). Moreover, by [Hofmann, 2007, Lemma 6.7 and Proposition 6.11], the map

$$\delta_A \colon X \longrightarrow \mathcal{V}, \quad x \longmapsto \bigvee\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in \mathsf{U}A\}$$

is an $\mathcal{U}$-functor, for every $A \subseteq X$, since it can be written as the composite

$$\delta_A = \Big(X \xrightarrow{\ulcorner a \urcorner} \mathcal{V}^{\mathsf{U}A} \xrightarrow{\bigvee} \mathcal{V} \Big).$$

For our next result we need to consider a stronger version of Assumption 4.48 which *we assume from now on*:

Assumption 4.55. The set $\{u \in \mathcal{V} \mid u \lll v\}$ is directed, for every $v \neq \bot$ in $\mathcal{V}$.

Lemma 4.56. *For every $\mathcal{U}$-category (X, a) and all $\mathfrak{x}, \mathfrak{y} \in \mathsf{U}X$,*

$$\widehat{a}(\mathfrak{x}, \mathfrak{y}) = \bigvee\{u \in \mathcal{V} \mid \forall A \in \mathfrak{x} \,.\, \delta_A^{-1}(\uparrow u) \in \mathfrak{y}\}.$$

Proof. Same as in [Hofmann, 2013, page 83], which in turn relies on [Hofmann, 2006, Corollary 1.5]. □

Lemma 4.57. *For every $\mathcal{U}$-category (X, a), the cone*

$$(\mathsf{U}X \xrightarrow{\xi \cdot \mathsf{U}\delta_A} \mathcal{V})_{A \subseteq X}$$

is initial in $\mathcal{V}$-CatCH.

Proof. For all $\mathfrak{x}, \mathfrak{y} \in \mathsf{U}X$, we show that

$$\widehat{a}(\mathfrak{x}, \mathfrak{y}) \geq \bigwedge\{\xi \cdot \mathsf{U}\delta_A(\mathfrak{y}) \mid A \in \mathfrak{x}\},$$

and observe that $\delta_A(\mathfrak{x}) \geq k$, for every $A \in \mathfrak{x}$. Let

$$u \lll \bigwedge\{\xi \cdot \mathsf{U}\delta_A(\mathfrak{y}) \mid A \in \mathfrak{x}\}.$$

Then, for every $A \in \mathfrak{x}$, $u \lll \xi \cdot \delta_A(\mathfrak{y})$, and therefore $\uparrow u \in \mathsf{U}\delta_A(\mathfrak{y})$, which is equivalent to $\delta_A^{-1}(\uparrow u) \in \mathfrak{y}$. Therefore $u \leq \widehat{a}(\mathfrak{x}, \mathfrak{y})$, by Lemma 4.56. □

Corollary 4.58. *For every $\mathcal{U}$-category (X, a), the $\mathcal{V}$-categorical compact Hausdorff space $(\mathsf{U}X)^{\mathrm{op}}$ is Priestley.*

Corollary 4.59. *Every $\mathcal{V}$-categorical compact Hausdorff space is a regular quotient of a Priestley one.*

Proof. With $\alpha \colon \mathsf{U}X \longrightarrow X$ denoting the convergence of X (and X^{op}),

$$\alpha \colon \mathsf{U}(X^{\mathrm{op}}) \longrightarrow X^{\mathrm{op}}$$

is a regular quotient in $\mathcal{V}$-CatCH, and hence also $\alpha \colon \mathsf{U}(X^{\mathrm{op}})^{\mathrm{op}} \longrightarrow X$. □

5 Conclusions

In this paper we have continued our study of "enriched Stone-type dualities" initiated in [Hofmann and Nora, 2018] where we extended the context from order structures to quantale-enriched (in particular metric) structures. Therefore, we passed from ordered compact Hausdorff spaces to quantale-enriched compact Hausdorff spaces which led naturally to the notion of *quantale-enriched Priestley space*. In Section 3 we complemented the duality results for categories of quantale-enriched Priestley space and continuous *distributors* by showing how these results can be restricted to categories of *maps* (Theorem 3.7). In Section 4 we investigated the category $\mathcal{V}$-Priest of quantale-enriched Priestley spaces and morphisms, with emphasis on those properties which identify the dual of this category as some kind of algebraic category. For certain quantales, we showed that the larger category $\mathcal{V}$-CatCH of quantale-enriched compact Hausdorff spaces and morphisms is a model category in CompHaus of a countable $\aleph_1$-ary limit sketch (Theorem 4.35), characterised the $\aleph_1$-copresentable objects of $\mathcal{V}$-CatCH as precisely the metrizable ones (Corollary 4.38), and showed that the left adjoint $\pi_0\colon \mathcal{V}\text{-CatCH} \longrightarrow \mathcal{V}\text{-Priest}$ of the inclusion functor $\mathcal{V}\text{-Priest} \longrightarrow \mathcal{V}\text{-CatCH}$ preserves $\aleph_1$-cofiltered limits (Proposition 4.42). Based on these results, we characterised the $\aleph_1$-copresentable objects (Corollary 4.44) and showed that this category is locally $\aleph_1$-copresentable (Theorem 4.46).

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References

Abbadini, M. (2019), The dual of compact ordered spaces is a variety, *Theory and Applications of Categories* **34** (44), 1401–1439.

Abbadini, M. and Reggio, L. (2020), On the axiomatisability of the dual of compact ordered spaces, *Applied Categorical Structures* **28** (6), 921–934.

Adámek, J. (2004), On Quasivarieties and Varieties as Categories, *Studia Logica* **78** (1/2), 7–33.

Adámek, J., Herrlich, H. and Strecker, G. E. (1990), *Abstract and concrete categories: The joy of cats*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507.

Adámek, J., Milius, S., Moss, L. S. and Urbat, H. (2015), On finitary functors and their presentations, *Journal of Computer and System Sciences* **81** (5), 813–833.

Adámek, J. and Pedicchio, M. C. (1997), A remark on topological spaces, grids, and topological systems, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **38** (3), 217–226.

Adámek, J. and Rosický, J. (1994), *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge.

Baron, S. (1968), Note on epi in τ_0, *Canadian Mathematical Bulletin* **11**, 503–504.

Barr, M. (1970), Relational algebras, in S. MacLane, editor, *Reports of the Midwest Category Seminar*

IV, pages 39–55, Springer Berlin Heidelberg, authors: H. Applegate, M. Barr, B. Day, E. Dubuc, Phreilambud, A. Pultr, R. Street, M. Tierney, S. Swierczkowski.

Barr, M. and Pedicchio, M. C. (1995), $\mathsf{Top}^{\mathrm{op}}$ is a quasi-variety, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **36** (1), 3–10.

Barr, M. and Pedicchio, M. C. (1996), Topological Spaces and Quasi-Varieties, *Applied Categorical Structures* **4** (1).

Bird, G. J. (1984), *Limits in 2-categories of locally-presentable categories*, Ph.D. thesis, University of Sydney.

Bird, G. J., Kelly, G. M., Power, A. J. and Street, R. H. (1989), Flexible limits for 2-categories, *Journal of Pure and Applied Algebra* **61** (1), 1–27.

Bonsangue, M. M., Kurz, A. and Rewitzky, I. M. (2007), Coalgebraic representations of distributive lattices with operators, *Topology and its Applications* **154** (4), 778–791.

Borceux, F. (1994), *Handbook of categorical algebra 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press (CUP), Cambridge, xviii + 443 pages, categories and structures.

Bourbaki, N. (1942), *Éléments de mathématique. 3. Pt. 1: Les structures fondamentales de l'analyse. Livre 3: Topologie générale*, Paris: Hermann & Cie.

Brümmer, G. C. L., Giuli, E. and Herrlich, H. (1992), Epireflections which are completions, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **33** (1), 71–93.

Cignoli, R., Lafalce, S. and Petrovich, A. (1991), Remarks on Priestley duality for distributive lattices, *Order* **8** (3), 299–315.

Clementino, M. M. and Hofmann, D. (2009), Lawvere completeness in topology, *Applied Categorical Structures* **17** (2), 175–210.

Dimov, G. D. and Tholen, W. (1989), A characterization of representable dualities, in J. Adámek and S. MacLane, editors, *Categorical topology and its relation to analysis, algebra and combinatorics: Prague, Czechoslovakia, 22-27 August 1988*, pages 336–357, World Scientific.

Duskin, J. (1969), Variations on Beck's tripleability criterion, in S. MacLane, editor, *Reports of the Midwest Category Seminar III*, pages 74–129, Springer Berlin Heidelberg.

Esakia, L. (1974), Topological Kripke models, *Doklady Akademii Nauk SSSR* **214**, 298–301.

Gabriel, P. and Ulmer, F. (1971), *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

Gelfand, I. (1941), Normierte Ringe, *Recueil Mathématique. Nouvelle Série* **9** (1), 3–24.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W. and Scott, D. S. (2003), *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge.

Halmos, P. R. (1956), Algebraic logic I. Monadic Boolean algebras, *Compositio Mathematica* **12**, 217–249.

Hochster, M. (1969), Prime ideal structure in commutative rings, *Transactions of the American Mathematical Society* **142**, 43–60.

Hofmann, D. (2002a), A generalization of the duality compactness theorem, *Journal of Pure and Applied Algebra* **171** (2-3), 205–217.

Hofmann, D. (2002b), On a generalization of the Stone-Weierstrass theorem, *Applied Categorical Struc-*

tures **10** (6), 569–592.

Hofmann, D. (2006), Exponentiation for unitary structures, *Topology and its Applications* **153** (16), 3180–3202.

Hofmann, D. (2007), Topological theories and closed objects, *Advances in Mathematics* **215** (2), 789–824.

Hofmann, D. (2013), Duality for distributive spaces, *Theory and Applications of Categories* **28** (3), 66–122.

Hofmann, D. (2014), The enriched Vietoris monad on representable spaces, *Journal of Pure and Applied Algebra* **218** (12), 2274–2318.

Hofmann, D., Neves, R. and Nora, P. (2018), Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces, *Theory and Applications of Categories* **33** (12), 276–295.

Hofmann, D., Neves, R. and Nora, P. (2019), Limits in categories of Vietoris coalgebras, *Mathematical Structures in Computer Science* **29** (4), 552–587.

Hofmann, D. and Nora, P. (2014), Some notes on Esakia spaces, *Textos de Matemática (Série B)* **46**, 201–220.

Hofmann, D. and Nora, P. (2018), Enriched Stone-type dualities, *Advances in Mathematics* **330**, 307–360.

Hofmann, D. and Nora, P. (2020), Hausdorff Coalgebras, *Applied Categorical Structures* **28** (5), 773–806.

Hofmann, D. and Reis, C. D. (2018), Convergence and quantale-enriched categories, *Categories and General Algebraic Structures with Applications* **9** (1), 77–138.

Hofmann, D., Seal, G. J. and Tholen, W., editors (2014), *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, volume 153 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.

Hofmann, D. and Stubbe, I. (2011), Towards Stone duality for topological theories, *Topology and its Applications* **158** (7), 913–925.

Hofmann, D. and Tholen, W. (2010), Lawvere completion and separation via closure, *Applied Categorical Structures* **18** (3), 259–287.

Johnstone, P. T. (1986), *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, reprint of the 1982 edition.

Jung, A. (2004), Stably compact spaces and the probabilistic powerspace construction, in J. Desharnais and P. Panangaden, editors, *Proceedings of the Workshop on Domain Theoretic Methods for Probabilistic Processes*, volume 87 of *Electronic Notes in Theoretical Computer Science*, pages 5–20.

Kelly, G. M. (1982), *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, Republished in: Reprints in Theory and Applications of Categories. No. 10 (2005), 1–136.

Kelly, G. M. and Lack, S. (2001), V-Cat is locally presentable or locally bounded if V is so, *Theory and Applications of Categories* **8** (23), 555–575.

Lambek, J. and Rattray, B. A. (1979), A general Stone–Gelfand duality, *Transactions of the American Mathematical Society* **248** (1), 1–35.

Lawvere, F. W. (1973), Metric spaces, generalized logic, and closed categories, *Rendiconti del Seminario*

Matemàtico e Fisico di Milano **43** (1), 135–166, Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

Lowen, R. (1997), *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press.

Makkai, M. and Paré, R. (1989), *Accessible categories: the foundations of categorical model theory*, volume 104, American Mathematical Society.

McKinsey, J. C. C. and Tarski, A. (1946), On closed elements in closure algebras, *Annals of Mathematics. Second Series* **47** (1), 122–162.

Menger, K. (1942), Statistical metrics, *Proceedings of the National Academy of Sciences of the United States of America* **28**, 535–537.

Morris, S. A. (1977), *Pontryagin duality and the structure of locally compact abelian groups*, volume 29 of *London Mathematical Society Lecture Note Series*, Cambridge University Press.

Nachbin, L. (1965), *Topology and Order*, volume 4 of *Van Nostrand Mathematical Studies*, D. Van Nostrand, Princeton, N.J.-Toronto, Ont.-London, translated from the Portuguese by Lulu Bechtolsheim.

Pedicchio, M. C. and Wood, R. J. (1999), Groupoidal completely distributive lattices, *Journal of Pure and Applied Algebra* **143** (1-3), 339–350.

Petrovich, A. (1996), Distributive lattices with an operator, *Studia Logica* **56** (1-2), 205–224, special issue on Priestley duality.

Porst, H.-E. and Tholen, W. (1991), Concrete dualities, in H. Herrlich and H.-E. Porst, editors, *Category theory at work*, volume 18 of *Research and Exposition in Mathematics*, pages 111–136, Heldermann Verlag, Berlin, with Cartoons by Marcel Erné.

Priestley, H. A. (1970), Representation of distributive lattices by means of ordered Stone spaces, *Bulletin of the London Mathematical Society* **2** (2), 186–190.

Priestley, H. A. (1972), Ordered topological spaces and the representation of distributive lattices, *Proceedings of the London Mathematical Society. Third Series* **24** (3), 507–530.

Pu, Q. and Zhang, D. (2015), Categories enriched over a quantaloid: Algebras, *Theory and Applications of Categories* **30** (21), 751–774.

Rosebrugh, R. and Wood, R. J. (1994), Constructive complete distributivity IV, *Applied Categorical Structures* **2** (2), 119–144.

Schalk, A. (1993a), *Algebras for Generalized Power Constructions*, Ph.D. thesis, Technische Hochschule Darmstadt.

Schalk, A. (1993b), Domains arising as algebras for powerspace constructions, *Journal of Pure and Applied Algebra* **89** (3), 305–328.

Seal, G. J. (2005), Canonical and op-canonical lax algebras, *Theory and Applications of Categories* **14** (10), 221–243.

Seal, G. J. (2009), A Kleisli-based approach to lax algebras, *Applied Categorical Structures* **17** (1), 75–89.

Stone, M. H. (1936), The theory of representations for Boolean algebras, *Transactions of the American Mathematical Society* **40** (1), 37–111.

Stone, M. H. (1938), Topological representations of distributive lattices and Brouwerian logics, *Časopis pro pěstování matematiky a fysiky* **67** (1), 1–25.

Stubbe, I. (2014), An introduction to quantaloid-enriched categories, *Fuzzy Sets and Systems* **256**, 95–

116, Special Issue on Enriched Category Theory and Related Topics (Selected papers from the 33^{rd} Linz Seminar on Fuzzy Set Theory, 2012).

Tholen, W. (2009), Ordered topological structures, *Topology and its Applications* **156** (12), 2148–2157.

Ulmer, F. (1971), Locally α-presentable and locally α-generated categories, in J. W. Gray, editor, *Reports of the Midwest Category Seminar V*, pages 230–247, Springer Berlin Heidelberg, authors: M. André and M. Barr and M. Bunge and A. Frei and J. W. Gray and P. A. Grillet and P. Leroux and F. E. J. Linton and J. MacDonald and P. Palmquist and P. B. Shay and F. Ulmer.

Vietoris, L. (1922), Bereiche zweiter Ordnung, *Monatshefte für Mathematik und Physik* **32** (1), 258–280.

Wood, R. J. (2004), Ordered Sets via Adjunction, in M. C. Pedicchio and W. Tholen, editors, *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*, volume 97 of *Encyclopedia of Mathematics and its Applications*, pages 5–47, Cambridge University Press, Cambridge.