Hausdorff coalgebras

Dirk Hofmann and Pedro Nora*

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal.

HASLab INESC TEC - Institute for Systems and Computer Engineering, Technology and Science, University of Minho, Portugal.

dirk@ua.pt, pedro.top.nora@gmail.com

As composites of constant, finite (co)product, identity, and powerset functors, Kripke polynomial functors form a relevant class of Set-functors in the theory of coalgebras. The main goal of this paper is to expand the theory of limits in categories of coalgebras of Kripke polynomial functors to the context of quantale-enriched categories. To assume the role of the powerset functor we consider "powerset-like" functors based on the Hausdorff $\mathcal{V}$-category structure. As a starting point, we show that for a lifting of a Set-functor to a topological category X over Set that commutes with the forgetful functor, the corresponding category of coalgebras over X is topological over the category of coalgebras over Set and, therefore, it is "as complete" but cannot be "more complete". Secondly, based on a Cantor-like argument, we observe that Hausdorff functors on categories of quantale-enriched categories do not admit a terminal coalgebra. Finally, in order to overcome these "negative" results, we combine quantale-enriched categories and topology *à la* Nachbin. Besides studying some basic properties of these categories, we investigate "powerset-like" functors which simultaneously encode the classical Hausdorff metric and Vietoris topology and show that the corresponding categories of coalgebras of "Kripke polynomial" functors are (co)complete.

Contents

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1. Introduction

Starting with early studies in the nineties to the introduction of uniform notions of *behavioural metric* in the last decade, the study of coalgebras over metric-like spaces has focused on four specific areas:

1. liftings of functors from the category Set of sets and functions to categories of metric spaces (see [8, 6, 7]), as a way of lifting state-based transition systems into transitions systems over categories of metric spaces;

2. results on the existence of terminal coalgebras and their computation (see [56, 8]), as a way of calculating the behavioural distance of two given states of a transition system;

3. the introduction of behavioural metrics with corresponding "Up-To techniques" (see [12, 8, 15]), as a way of easing the calculation of behavioural distances;

4. and the development of coalgebraic logical foundations over metric spaces (see [6, 37, 58]), as a way of reasoning in a quantitive way about transition systems.

In this paper we focus on the first two topics, with particular interest in metric versions of *Kripke polynomial functors*. As composites of constant, finite (co)product, identity, and powerset functors, Kripke polynomial functors form a pertinent class of Set-functors in the theory of coalgebras (for example, see [49], [13] and [38]), which is well-behaved in regard to the existence of limits in their respective categories of coalgebras — assuming that the powerset functor is submitted to certain cardinality restrictions. The latter constraint is essential since the powerset functor $\mathsf{P}\colon \mathsf{Set} \to \mathsf{Set}$ does not admit a terminal coalgebra; a well-known fact which follows from the following:

- in [39] it is shown that the terminal coalgebra of a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}$ is a fixed point of F, and

- in [16] it is (essentially) proven that the powerset functor $\mathsf{P}\colon \mathsf{Set} \to \mathsf{Set}$ does not have fixed points.

On the other hand, being accessible, the finite powerset functor $\mathsf{P}_{\mathrm{fin}}\colon \mathsf{Set} \to \mathsf{Set}$ does admit a terminal coalgebra (see [9]); in fact, the category of coalgebras for $\mathsf{P}_{\mathrm{fin}}\colon \mathsf{Set} \to \mathsf{Set}$ is complete. Metric counterparts of the powerset functor are often based on the Hausdorff metric, informally, we call them *Hausdorff functors*. This metric was originally introduced in [26, 45] (see also [11]), and, recently, has been considered in the more general context of quantale enriched categories (see [5, 53]) in which we discuss the results presented here.

A common theme of the papers [8] and [7] is to study liftings of Set-functors to categories of metric spaces, or more generally to the category $\mathcal{V}$-Cat of $\mathcal{V}$-categories and $\mathcal{V}$-functors, for a quantale $\mathcal{V}$, in the sense that the diagram

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Cat} \arrow[r, "\bar{\mathsf{T}}"] \arrow[d] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[d] \\
\mathsf{Set} \arrow[r, "\mathsf{T}"'] & \mathsf{Set}
\end{tikzcd}$$

commutes. In Section 2 we show that, for such a lifting of a Set-functor, the corresponding category of coalgebras over $\mathcal{V}$-Cat is topological over the category of coalgebras over Set (see Theorem 2.5). This implies that it is possible to recast over $\mathcal{V}$-Cat all the theory about limits in categories of Kripke polynomial coalgebras over Set. However, this result also highlights that "adding a $\mathcal{V}$-category structure" does not improve the situation regarding limits by itself. In particular, the Hausdorff functor that considers all subsets of a metric space does not admit a terminal coalgebra.

Besides cardinal restrictions, another way to "tame" the powerset functor is to equip a set with some kind of structure and then consider only its "structure relevant" subsets. This is precisely the strategy employed in [28] where we passed from Kripke polynomial functors to *Vietoris polynomial functors* on categories of topological spaces. For instance, it is implicitly shown in [20, page 245] that the classic Vietoris functor on the category of compact Hausdorff spaces and continuous maps has a terminal coalgebra, and this result generalises to all topological spaces when considering the compact Vietoris functor on Top which sends a space to its hyperspace of compact subsets (see [28] for details). This fact might not come as a surprise for the reader thinking of compactness as "generalised finiteness"; however, it came as a surprise to us to learn that the lower Vietoris functor on Top, where one considers all closed subsets, also admits a terminal coalgebra.

Motivated by the fact that finite topological spaces correspond precisely to finite ordered sets, over the past decades several results about topological spaces have been inspired by their finite counterparts; for a sequence of results see for instance [34, 35, 17]. One therefore might wonder if the result regarding the lower Vietoris functor on Top has an order-theoretic counterpart; in other words, does the upset functor $\mathsf{Up}\colon \mathsf{Ord} \to \mathsf{Ord}$ admit a terminal coalgebra? The answer is negative, as it follows from the "generalized Cantor Theorem" of [19].[1] Based on [19], in Section 3 we generalise Cantor's Theorem further (see Theorem 3.16) and use this result to

[1]We thank Adriana Balan for calling our attention to [19].

show that the (non-symmetric) Hausdorff functor on $\mathcal{V}$-Cat – sending a $\mathcal{V}$-category to the space of all "up-closed" subsets – does not admit a terminal coalgebra.

To overcome these "negative results" regarding completeness of categories of coalgebras, in Section 4 we add a topological component to the $\mathcal{V}$-categorical setting. More specifically, we introduce the Hausdorff construction for $\mathcal{V}$-categories equipped with a compatible compact Hausdorff topology. We note that these *$\mathcal{V}$-categorical compact Hausdorff spaces* are already studied in [55, 31], being the corresponding category denoted here by $\mathcal{V}$-CatCH. Also, we find it worthwhile to notice that the notion of $\mathcal{V}$-categorical compact Hausdorff space generalises simultaneously Nachbin's ordered compact Hausdorff spaces [43] and the classic notion of compact metric space; therefore, it provides a framework to combine and even unify both theories. For example:

- It is known that the specialisation order of a sober space is directed complete (see [36, Lemma II.1.9]); in [31] we observed that this fact implies immediately that the order relation of an ordered compact Hausdorff space is directed complete. Furthermore, an appropriate version of this result in the quantale-enriched setting implies that the metric of a metric compact Hausdorff space (i.e. a metric space with a *compatible* compact Hausdorff topology) is Cauchy complete, generalising the classical fact that a compact metric space (i.e. a metric space where the *induced* topology is compact) is Cauchy complete.
- The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ introduced in Section 4 combines the Vietoris topology and the Hausdorff metric; in particular, for a metric compact Hausdorff space, the Hausdorff metric is compatible with the Vietoris topology (Proposition 4.19). This result represents a variation of the classic fact stating that, for every compact metric space X, the Hausdorff metric induces the Vietoris topology of the compact Hausdorff space X (see [42]).

By "adding topology", and under mild assumptions on the quantale $\mathcal{V}$, we are able to show that $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ preserves codirected limits (see Theorem 4.35); which eventually allows us to conclude that, for every Hausdorff polynomial functor on $\mathcal{V}$-CatCH, the corresponding category of coalgebras is complete (see Theorem 4.47).

In the last part of this paper we consider a $\mathcal{V}$-categorical counterpart of the notion of a Priestley space. In [29] we developed already "Stone-type" duality theory for these type of spaces; here we show that $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ sends Priestley spaces to Priestley spaces, generalising a well-known fact of the Vietoris functor on the category of partially ordered compact spaces. Consequently, many results regarding coalgebras for $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ are valid for its restriction to Priestley spaces as well.

Throughout this paper we assume familiarity with the basic theory of quantale enriched categories. For the readers convenience, in Appendix A we collect some notions and results and, moreover, present some useful properties of the reflector into the category of separated $\mathcal{V}$-categories.

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2. Strict functorial liftings

The main motif of this work is to expand the study of limits in categories of coalgebras of Kripke polynomial functors to the context of quantale-enriched categories. In more general terms, this means that given an endofunctor F on a category A and a faithful functor $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$, our problem consists in studying a "lifting" of F to an endofunctor $\bar{\mathsf{F}}$ on X. In a strict sense, by "lifting" we mean that the diagram

$$\begin{tikzcd} \mathsf{X} \arrow[r, "\bar{\mathsf{F}}"] \arrow[d, "\mathsf{U}"'] & \mathsf{X} \arrow[d, "\mathsf{U}"] \\ \mathsf{A} \arrow[r, "\mathsf{F}"'] & \mathsf{A} \end{tikzcd} \tag{2.i}$$

commutes.

Remark 2.1. If in (2.i) the functor $\bar{\mathsf{F}}$ has a fix-point, then so has F. Hence, if F does not have a fix-point, then neither does $\bar{\mathsf{F}}$. In particular, any strict lifting of the powerset functor $\mathsf{P}\colon \mathsf{Set} \to \mathsf{Set}$ does not admit a terminal coalgebra.

Then, we obtain a faithful functor

$$\bar{\mathsf{U}}\colon \operatorname{CoAlg}(\bar{\mathsf{F}}) \to \operatorname{CoAlg}(\mathsf{F})$$

by "applying U". In [28, Theorem 3.11] we showed under additional conditions that, if the forgetful functor $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$ is topological, then so is the functor $\bar{\mathsf{U}}\colon \operatorname{CoAlg}(\bar{\mathsf{F}}) \to \operatorname{CoAlg}(\mathsf{F})$. We start by improving upon this result.

In the remainder of this section, let $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$ be a topological functor, for more information about this notion we refer to [4]. We recall that X is fibre-complete, and for an object A of A we use the suggestive notation (A, α) to denote an element of the fibre of A. Then we write $\alpha \le \beta$ if $1_A\colon (A,\alpha) \to (A,\beta)$ is a morphism of X. Since we also assume the existence of functors $\mathsf{F}\colon \mathsf{A} \to \mathsf{A}$ and $\bar{\mathsf{F}}\colon \mathsf{X} \to \mathsf{X}$ such that the diagram (2.i) commutes, with a slight abuse of notation, we often write $(\mathsf{F}A, \mathsf{F}\alpha)$ instead of $\bar{\mathsf{F}}(A,\alpha)$.

For a U-structured arrow $f\colon A \to \mathsf{U}(B,\beta)$ in A, we denote by $(A, f^{\triangleleft}_{\beta})$ the corresponding U-initial lift. Similarly, for $f\colon \mathsf{U}(A,\alpha) \to B$ in A, we denote by $(B, f^{\triangleright}_{\alpha})$ the corresponding U-final lift. Below we collect some well-known facts.

Proposition 2.2. *Let $f\colon A \to B$ be a morphism in A and (A,α) and (B,β) be objects in the fibres of A and B, respectively. Then the following assertions are equivalent.*

(i) $f\colon (A,\alpha) \to (B,\beta)$ *is a morphism in* X.

(ii) $\alpha \le f^{\triangleleft}_{\beta}$.

(iii) $f^{\triangleright}_{\alpha} \leq \beta$.

Proposition 2.3. *Let* (A, α) *and* (A, β) *be objects in the fibre of an object* A *of* A*. If* $\alpha \leq \beta$ *then* $\mathsf{F}\alpha \leq \mathsf{F}\beta$*.*

Proposition 2.4. *Let* $c\colon A \to \mathsf{F}A$ *be a morphism in* A *and let* $\mathcal{A}$ *be a collection of objects* (A, α) *in the fibre of* A *such that* $c\colon (A, \alpha) \to (\mathsf{F}A, \mathsf{F}\alpha)$ *is in* X*. Let* (A, α_c) *be the supremum of* $\mathcal{A}$*. Then,* $c\colon (A, \alpha_c) \to (\mathsf{F}A, \mathsf{F}\alpha_c)$ *is a morphism of* X*.*

Proof. First note that

$$(A, \alpha_c) \xrightarrow{\ c\ } \bigvee\{(\mathsf{F}A, \mathsf{F}\alpha) \mid (X, \alpha) \in \mathcal{A}\}$$

is a morphism in X, and, by Proposition 2.3, so is

$$\bigvee\{(\mathsf{F}A, \mathsf{F}\alpha) \mid (X, \alpha) \in \mathcal{A}\} \xrightarrow{\ 1_{\mathsf{F}A}\ } (\mathsf{F}A, \mathsf{F}\alpha_c).$$ □

Theorem 2.5. *The functor* $\bar{\mathsf{U}}\colon \mathrm{CoAlg}(\bar{\mathsf{F}}) \to \mathrm{CoAlg}(\mathsf{F})$ *is topological.*

Proof. Let $(A_i, \alpha_i, c_i)_{i\in I}$ be a family of objects in $\mathrm{CoAlg}(\bar{\mathsf{F}})$, and $(f_i\colon (A, c) \to (A_i, c_i))_{i\in I}$ a cone in $\mathrm{CoAlg}(\mathsf{F})$. Consider

$$\alpha_c = \bigvee\{\alpha \mid c\colon (A, \alpha) \to (\mathsf{F}A, \mathsf{F}\alpha) \text{ is in } \mathsf{X} \text{ and, for all } i \in I,\ \alpha \leq f_i{}^{\triangleleft}_{\alpha_i}\}.$$

Then, by Proposition 2.4, $c\colon (A, \alpha_c) \to (\mathsf{F}A, \mathsf{F}\alpha_c)$ is a morphism of X. Moreover, by construction, $\alpha_c \leq f_i{}^{\triangleleft}_{\alpha_i}$ for all $i \in I$; hence, $(f_i\colon (A, \alpha_c) \to (A_i, \alpha_i))_{i\in I}$ is a cone in X. Therefore, $(f_i\colon (A, \alpha_c, c) \to (A_i, \alpha_i, c_i))_{i\in I}$ is a cone in $\mathrm{CoAlg}(\bar{\mathsf{F}})$. We claim that this cone is U-initial.

Let $(g_i\colon (B, \beta, b) \to (A_i, \alpha_i, c_i)$ be a cone in $\mathrm{CoAlg}(\bar{\mathsf{F}})$, and $h\colon (B, b) \to (A, c)$ a morphism in $\mathrm{CoAlg}(\mathsf{F})$ such that, for every $i \in I$,

$$f_i \cdot h = g_i \tag{2.ii}$$

We will see that $h^{\triangleright}_{\beta} \leq \alpha_c$. First observe that it follows from (2.ii) and Proposition 2.2 that $h^{\triangleright}_{\beta} \leq f_i{}^{\triangleleft}_{\alpha_i}$ for all $i \in I$. Furthermore, since $c \cdot h = Fh \cdot b$ in A it follows that $c\colon (A, h^{\triangleright}_{\beta}) \to (FA, F(h^{\triangleright}_{\beta}))$ is a morphism of X because $h\colon (B, \beta) \to (A, h^{\triangleright}_{\beta})$ is final. Therefore, by construction of α_c, we conclude that $h^{\triangleright}_{\beta} \leq \alpha_c$. □

Corollary 2.6. *The category* $\mathrm{CoAlg}(\bar{\mathsf{F}})$ *has limits of shape* I *if and only if* $\mathrm{CoAlg}(\mathsf{F})$ *has limits of shape* I*. In particular,* $\mathrm{CoAlg}(\bar{\mathsf{F}})$ *has a terminal object if and only if* $\mathrm{CoAlg}(\mathsf{F})$ *has one.*

This means that $\mathrm{CoAlg}(\bar{\mathsf{F}})$ cannot be "more complete" than $\mathrm{CoAlg}(\mathsf{F})$, one of the reasons why in Section 3.1 we will lift the powerset functor on Set to $\mathcal{V}$-Cat but only "up to natural transformation".

On the other hand, Corollary 2.6 also means that $\mathrm{CoAlg}(\bar{\mathsf{F}})$ is "at least as complete" as $\mathrm{CoAlg}(\mathsf{F})$, which allow us to recover known results about the existence of limits in $\mathrm{CoAlg}(\bar{\mathsf{F}})$. For example, in [8, Theorem 6.2] it is proven, by implicitly constructing the right adjoint of U, that every lifting to the category of symmetric metric spaces of an endofunctor on Set that admits a terminal coalgebra also admits a terminal coalgebra. A similar result was also obtained in [7, Theorem 4.15] for "$\mathcal{V}$-Catifications" — very specific liftings from Set to $\mathcal{V}$-Cat.

Note that Theorem 2.5 even tells us how to construct limits in $\mathrm{CoAlg}(\bar{\mathsf{F}})$ from limits in $\mathrm{CoAlg}(\mathsf{F})$. In particular, if U is a forgetful functor to Set then a limit in $\mathrm{CoAlg}(\bar{\mathsf{F}})$ has the same underlying set of the corresponding limit in $\mathrm{CoAlg}(\mathsf{F})$. This behaviour was already observed in [7, Theorem 4.16] for some particular liftings to $\mathcal{V}$-Cat.

Example 2.7. Given a subfunctor F of the powerset functor on Set, the corresponding class of Kripke polynomial functors is typically defined as the smallest class of Set-functors that contains the identity functor, all constant functors and it is closed under composition with F, finite sums and finite products of functors. If we are interested in strict liftings to $\mathcal{V}$-Cat, then Theorem 2.5 tells us that is possible to recast over $\mathcal{V}$-Cat all the theory about limits in categories of Kripke polynomial coalgebras over Set. For example, if we consider a strict lifting of the finite powerset functor, then every category of coalgebras of a Kripke polynomial functor is (co)complete, and every limit is obtained as the initial lift of the corresponding limit of Set-coalgebras.

In the sequel, we give an example of a generic way of lifting a functor $\mathsf{F}\colon \mathsf{A} \to \mathsf{A}$ to a category X that is topological over A. In particular, this construction is used to lift Set-functors to categories of metric spaces in [8], and to categories of $\mathcal{V}$-categories in [7].

For a functor $\mathsf{F}\colon \mathsf{A} \to \mathsf{A}$ and A-morphisms $\psi\colon A \to \widetilde{A}$ and $\sigma\colon \mathsf{F}\widetilde{A} \to \widetilde{A}$, we denote by $\psi^{\Diamond}\colon \mathsf{F}A \to \widetilde{A}$ the composite

$$\mathsf{F}A \xrightarrow{\mathsf{F}\psi} \mathsf{F}\widetilde{A} \xrightarrow{\sigma} \widetilde{A}$$

in A.

Consider now a category X equipped with a topological functor $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$ and an X-object $\widetilde{X}$ whose underlying set $\mathsf{U}\widetilde{X}$ carries the structure $\sigma\colon \mathsf{F}\mathsf{U}\widetilde{X} \to \mathsf{U}\widetilde{X}$ of a F-algebra. Then $(\psi^{\Diamond}\colon \mathsf{F}\mathsf{U}X \to \mathsf{U}\widetilde{X})_{\psi\in\mathsf{X}(X,\widetilde{X})}$ is a U-structured cone, and we define $\bar{\mathsf{F}}X$ to be the domain of the initial lift of this cone. Clearly:

Theorem 2.8. *1. The construction above defines a functor $\bar{\mathsf{F}}\colon \mathsf{X} \to \mathsf{X}$ making the diagram*

$$\begin{tikzcd} \mathsf{X} \arrow[r, "\bar{\mathsf{F}}"] \arrow[d, "\mathsf{U}"'] & \mathsf{X} \arrow[d, "\mathsf{U}"] \\ \mathsf{A} \arrow[r, "\mathsf{F}"'] & \mathsf{A} \end{tikzcd}$$

commutative.

2. For every $\psi\colon X \to \widetilde{X}$ in X, $\psi^{\Diamond}$ is an X-morphism $\psi^{\Diamond}\colon \bar{\mathsf{F}}X \to \widetilde{X}$. In particular, $\sigma = {1_{\widetilde{X}}}^{\Diamond}$ is an X-morphism $\sigma\colon \bar{\mathsf{F}}\widetilde{X} \to \widetilde{X}$.

3. *If $\widetilde{X}$ is injective with respect to initial morphisms, then $\bar{\mathsf{F}}\colon \mathsf{X} \to \mathsf{X}$ preserves initial morphism (compare with [8, Theorem 5.8]).*

4. *Let $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ be a natural transformation such that $\sigma_{\mathsf{G}} \cdot \alpha_{\widetilde{X}} = \sigma_{\mathsf{F}}$. Then α lifts to a natural transformation between the corresponding X-functors.*

5. *If $\mathsf{F} = \mathsf{T}$ is part of a monad $\mathbb{T} = (\mathsf{T}, m, e)$ on A and $\sigma\colon \mathsf{T}|\widetilde{X}| \to |\widetilde{X}|$ is a $\mathbb{T}$-algebra, then $\mathbb{T}$ lifts naturally to a monad $\bar{\mathbb{T}} = (\bar{\mathsf{T}}, m, e)$ on X.*

Proof. The first affirmation follows immediately from the commutativity of the diagram

$$\begin{tikzcd} \mathsf{F}X \arrow[r, "\mathsf{F}f"] \arrow[dr, "(\psi\cdot f)^{\lozenge}"'] & \mathsf{F}Y \arrow[d, "\psi^{\lozenge}"] \\ & \widetilde{X}, \end{tikzcd}$$

and similarly the last two ones. The second affirmation is true by definition. In Proposition 2.10 we prove a slightly more general version of (3). $\square$

Remark 2.9. We note that in Theorem 2.8 (4), the inequality $\sigma_{\mathsf{G}} \cdot \alpha_{\widetilde{X}} \leq \sigma_{\mathsf{F}}$ does not guarantee that $\alpha_X\colon \mathsf{F}X \to \mathsf{G}X$ is an X-morphism (this contradicts [8, Theorem 8.1]). For instance, consider $\mathsf{X} = \mathsf{Met}_{\mathrm{sym}}$, $\widetilde{X} = [0, \infty]$ and $\mathsf{F}, \mathsf{G}\colon \mathsf{Set} \to \mathsf{Set}$ with $\mathsf{F} = \mathsf{G}$ being the identity functor on Set, $\lambda = 1$, $\sigma_{\mathsf{G}} = 1_{[0,\infty]}$ and $\sigma_{\mathsf{F}} = \infty$ (constant). Clearly, $\sigma_{\mathsf{G}} \cdot \lambda_{[0,\infty]} \leq \sigma_{\mathsf{F}}$. However, $\mathsf{G}\colon \mathsf{Met}_{\mathrm{sym}} \to \mathsf{Met}_{\mathrm{sym}}$ is the identity functor and $\mathsf{F}\colon \mathsf{Met}_{\mathrm{sym}} \to \mathsf{Met}_{\mathrm{sym}}$ transforms every symmetric metric space into the indiscrete space on the same underlying set. Hence, for a non-indiscrete space X, $\lambda_X\colon \mathsf{F}X \to \mathsf{G}X$ is not a morphism in $\mathsf{Met}_{\mathrm{sym}}$.

In this context it is useful to note that Theorem 2.8 (3) gives a sufficient condition for the preservation of initial morphisms that can be formulated in a slightly more general way.

Proposition 2.10. *Let $\mathsf{F}\colon \mathsf{X} \to \mathsf{X}$ be a functor, $\sigma\colon \mathsf{F}\widetilde{X} \to \widetilde{X}$ a morphism in X, and $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$ a faithful functor. Assume further that $\widetilde{X}$ is injective in X with respect to initial morphisms and, for every object X in X, the cone $(\psi^{\lozenge}\colon \mathsf{F}X \to \widetilde{X})_{\psi\in\mathsf{X}(X,\widetilde{X})}$ in X is initial. Then F preserves initial morphisms.*

Proof. Let $f\colon X \to Y$ be an initial morphism in X. Since $\widetilde{X}$ is injective with respect to initial morphisms, every morphism $\psi\colon X \to \widetilde{X}$ in X factors as $h_\psi \cdot f$, for some $h_\psi\colon Y \to \widetilde{X}$ in X. Hence, $\mathsf{F}\psi = \mathsf{F}h_\psi \cdot \mathsf{F}f$. Now, suppose that $h\colon Z \to \mathsf{F}Y$ is a morphism in X, and $g\colon \mathsf{U}Z \to \mathsf{U}\mathsf{F}X$ is a function such that $\mathsf{U}\mathsf{F}f \cdot g = \mathsf{U}h$. Then, for every morphism $\psi\colon X \to \widetilde{X}$ in X, we have

$$\mathsf{U}\psi^{\lozenge} \cdot g = \mathsf{U}\sigma \cdot h_\psi \cdot \mathsf{F}f \cdot g = \mathsf{U}\sigma \cdot h_\psi \cdot h.$$

Therefore, the claim follows because the cone $(\psi^{\lozenge}\colon \mathsf{F}X \to \widetilde{X})_{\psi\in\mathsf{X}(X,\widetilde{X})}$ is initial and U is faithful. $\square$

The injectivity-condition on $\widetilde{X}$ is often fulfilled; the proposition below collects some examples.

Proposition 2.11. *1. The $\mathcal{V}$-category $(\mathcal{V}, \hom)$ is injective in $\mathcal{V}$-Cat with respect to initial morphisms. Since $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sym}} \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ preserves initial morphisms (see Theorem A.5), the symmetrisation of $(\mathcal{V}, \hom)$ is injective in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sym}}$.*

2. *The unit interval $[0, 1]$ is injective in $\mathsf{PosComp}$ with respect to initial morphisms (see [43]).*

3. *The Sierpiński space is injective with respect to initial morphisms in the category Top of topological spaces and continuous maps.*

The next proposition shows that the Hausdorff distance between subsets of metric spaces (see [26]) emerges naturally in the context of $\mathcal{V}$-categories from the construction discussed earlier.

Proposition 2.12. *The lifting of the powerset functor P on Set to $\mathcal{V}$-Cat with respect to $\bigwedge\colon \mathsf{P}\mathcal{V} \to \mathcal{V}$ sends a $\mathcal{V}$-category (X, a) to $(\mathsf{P}X, \mathsf{H}a)$, where for all $A, B \subseteq X$,*

$$\mathsf{H}a(A, B) = \bigwedge_{y\in B} \bigvee_{x\in A} a(x, y).$$

Proof. Let (X, a) be a $\mathcal{V}$-category and $\mathsf{P}a$ the $\mathcal{V}$-category structure corresponding to the lifting aforementioned. That is, for every $A, B \in \mathsf{P}X$,

$$\mathsf{P}a(A, B) = \bigwedge_{\psi\in\mathcal{V}\text{-}\mathsf{Cat}(X,\mathcal{V})} \hom(\bigwedge_{x\in A} \psi(x), \bigwedge_{y\in B} \psi(y)).$$

First, observe that for every $u \in \mathcal{V}$ the function $\hom(u, -)\colon \mathcal{V} \to \mathcal{V}$ preserves infima and the map $\hom(-, u)\colon \mathcal{V} \to \mathcal{V}$ is antimonotone.

Hence, for every $\mathcal{V}$-functor $\psi\colon (X, a) \to (\mathcal{V}, \hom)$,

$$\mathsf{H}a(A, B) \le \bigwedge_{y\in B} \bigvee_{x\in A} \hom(\psi(x), \psi(y)) \le \bigwedge_{y\in B} \hom(\bigwedge_{x\in A} \psi(x), \psi(y)) = \hom(\bigwedge_{x\in A} \psi(x), \bigwedge_{y\in B} \psi(y)).$$

Therefore, $\mathsf{H}a(A, B) \le \mathsf{P}a(A, B)$.

To see that the reverse inequality holds, consider the $\mathcal{V}$-functor $f\colon (X, a) \to (\mathcal{V}, \hom)$ below that is obtained by combining Propositions A.3 and A.4.

$$\begin{tikzcd} X \arrow[r, "\ulcorner a \urcorner"] \arrow[rr, bend left, "f"] & \mathcal{V}^A \arrow[r, "\bigvee"] & \mathcal{V} \end{tikzcd}$$

Therefore, as $\hom(-, u)$ is antimonotone,

$$\mathsf{P}a(A, B) \le \hom(\bigwedge_{y'\in A} f(y'), \bigwedge_{y\in B} f(y)) \le \hom(k, \bigwedge_{y\in B} \bigvee_{x\in A} a(x, y)) = \bigwedge_{y\in B} \bigvee_{x\in A} a(x, y) = \mathsf{H}a(A, B).$$

□

Remark 2.13. It is well-known that the formula of Proposition 2.12 defines a $\mathcal{V}$-category structure on the powerset (for instance, see [5]).

Remark 2.14. The notion of a (symmetric) distance between subsets of a metric space goes back to [45] and was made popular by its use in [26]. For more information on the history of this idea we refer to [11].

Corollary 2.15. *The lifting of the powerset functor to* $\mathcal{V}$-Cat *of Proposition 2.12 preserves initial morphisms.*

Another idea to tackle the problem of lifting an endofunctor F on Set to $\mathcal{V}$-Cat is to consider first a lax extension $\widehat{\mathsf{F}}\colon \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$ of the functor F in the sense of [52]; that is, to require

1. $r \le r' \implies \widehat{\mathsf{F}}r \le \widehat{\mathsf{F}}r'$,
2. $\widehat{\mathsf{F}}s \cdot \widehat{\mathsf{F}}r \le \widehat{\mathsf{F}}(s \cdot r)$,
3. $\mathsf{F}f \le \widehat{\mathsf{F}}(f)$ and $(\mathsf{F}f)^\circ \le \widehat{\mathsf{F}}(f^\circ)$.

It follows immediately (see [52]) that

$$\widehat{\mathsf{F}}(s \cdot f) = \widehat{\mathsf{F}}s \cdot \mathsf{F}f \quad \text{and} \quad \widehat{\mathsf{F}}(g^\circ \cdot r) = \mathsf{F}g^\circ \cdot \widehat{\mathsf{F}}r.$$

Then, based on this lax extension, the functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ admits a natural lifting to $\mathcal{V}$-Cat (see [55]): the functor $\bar{\mathsf{F}}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ sends a $\mathcal{V}$-category (X, a) to $(\mathsf{F}X, \widehat{\mathsf{F}}a)$. One advantage of this type of lifting is that allows us to use the calculus of $\mathcal{V}$-relations. The following is a simple example.

Proposition 2.16. $\bar{\mathsf{F}}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ *preserves initial* $\mathcal{V}$*-functors.*

Proof. Let $f\colon (X, a) \to (Y, b)$ be a $\mathcal{V}$-functor with $a = f^\circ \cdot b \cdot f$. Then $\widehat{\mathsf{F}}a = \mathsf{F}f^\circ \cdot \widehat{\mathsf{F}}b \cdot \mathsf{F}f$. □

The result above generalises [8, Theorem 5.8].

Example 2.17. For a $\mathcal{V}$-relation $r\colon X \nrightarrow Y$, and subsets $A \subseteq X$, $B \subseteq Y$, the formula

$$\bigwedge_{y \in B} \bigvee_{x \in A} r(x, y)$$

defines a lax extension of the powerset functor on Set to $\mathcal{V}$-Rel (see [52]). The corresponding lifting to $\mathcal{V}$-Cat coincides with the one described in Proposition 2.12. In particular, by Proposition 2.16, we obtain another proof for the fact that this lifting preserves initial morphisms.

If we start with a monad $\mathbb{T} = (\mathsf{T}, m, e)$ on Set, a *lax extension* of $\mathbb{T} = (\mathsf{T}, m, e)$ to $\mathcal{V}$-Rel is a lax extension $\widehat{\mathsf{T}}$ of the functor T to $\mathcal{V}$-Rel such that $m\colon \widehat{\mathsf{T}}\widehat{\mathsf{T}} \to \widehat{\mathsf{T}}$ and $e\colon \mathsf{Id} \to \widehat{\mathsf{T}}$ become op-lax:

$$m_Y \cdot \widehat{\mathsf{T}}\widehat{\mathsf{T}}r \le \widehat{\mathsf{T}}r \cdot m_X, \qquad\qquad e_Y \cdot r \le \widehat{\mathsf{T}}r \cdot e_X$$

for all $\mathcal{V}$-relations $r\colon X \nrightarrow Y$.

For a lax extension of a Set-monad $\mathbb{T} = (\mathsf{T}, m, e)$ to $\mathcal{V}$-Rel, the functions $e_X \colon X \to \mathsf{T}X$ and $m_X \colon \mathsf{TT}X \to \mathsf{T}X$ become $\mathcal{V}$-functors for each $\mathcal{V}$-category X, so that we obtain a monad on $\mathcal{V}$-Cat. The Eilenberg–Moore algebras for this monad are triples (X, a, α) where (X, a) is a $\mathcal{V}$-category and (X, α) is an algebra for the Set-monad $\mathbb{T}$ such that $\alpha \colon \mathsf{T}(X, a_0) \to (X, a_0)$ is a $\mathcal{V}$-functor. A map $f \colon X \to Y$ is a homomorphism $f \colon (X, a, \alpha) \to (Y, b, \beta)$ of algebras precisely if f preserves both structures, that is, whenever $f \colon (X, a) \to (Y, b)$ is a $\mathcal{V}$-functor and $f \colon (X, \alpha) \to (Y, \beta)$ is a $\mathbb{T}$-homomorphism. For more information we refer to [55, 32].

One possible way to construct lax extensions based on a (lax) $\mathbb{T}$-algebra structure $\xi \colon \mathsf{T}\mathcal{V} \to \mathcal{V}$ is devised in [27]: for every $\mathcal{V}$-relation $r \colon X \times Y \to \mathcal{V}$ and for all $\mathfrak{x} \in \mathsf{T}X$ and $\mathfrak{y} \in \mathsf{T}Y$,

$$\widehat{\mathsf{T}}r(\mathfrak{x}, \mathfrak{y}) = \bigvee \left\{ \xi \cdot \mathsf{T}r(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in \mathsf{T}(X \times Y), \mathsf{T}\pi_1(\mathfrak{w}) = \mathfrak{x}, \mathsf{T}\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}.$$

We note that $\widehat{\mathsf{T}}$ preserves the involution on $\mathcal{V}$-Rel, that is, $\widehat{\mathsf{T}}(r^\circ) = (\widehat{\mathsf{T}}r)^\circ$ for all $\mathcal{V}$-relations $r \colon X \nrightarrow Y$ (and we write simply $\widehat{\mathsf{T}}r^\circ$).

Example 2.18. Consider the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ on Set, the quantale $\mathsf{2}$ and the $\mathbb{U}$-algebra

$$\xi \colon \mathsf{U2} \longrightarrow \mathsf{2}$$

sending every ultrafilter to its generating point. The category of algebras of the induced monad on $\mathcal{V}$-Cat is the category OrdCH of (pre)ordered compact Hausdorff spaces introduced in [43] (see also [55]).

3. Hausdorff polynomial functors on $\mathcal{V}$-Cat

In this section we study a class of endofunctors on $\mathcal{V}$-Cat that intuitively is an analogue of the class of Kripke polynomial functors on Set. We begin by describing a $\mathcal{V}$-Cat-counterpart of the powerset functor on Set that is based on the upset functor on Ord.

3.1. The Hausdorff functor on $\mathcal{V}$-Cat

We introduce now some $\mathcal{V}$-categorical versions of classical notions from order theory. We start with the "up-closure" and "down-closure" of a subset.

Definition 3.1. Let (X, a) be a $\mathcal{V}$-category. For every $A \subseteq X$, put

$$\Uparrow^a A = \{y \in X \mid k \le \bigvee_{x \in A} a(x, y)\} \qquad \text{and} \qquad \Downarrow^a A = \{y \in X \mid k \le \bigvee_{x \in A} a(y, x)\}.$$

As usual, we write $\Uparrow^a x$ and $\Downarrow^a x$ if $A = \{x\}$. We also observe that $\Uparrow^a A = \Downarrow^{a^\circ} A$ which allows us to translate results about $\Uparrow^a$ to results about $\Downarrow^a$, and *vice versa*. Considering the underlying ordered set $(X, \le)$ of (X, a), we note that

$$\Uparrow^{\le} A \subseteq \Uparrow^a A \qquad \text{and} \qquad \Downarrow^{\le} A \subseteq \Downarrow^a A$$

for every $A \subseteq X$, with equality if A is a singleton. To simplify notation, we often write $\uparrow A$ and $\downarrow A$ whenever the corresponding structure can be derived from the context.

Remark 3.2. For an ordered set X, with a denoting the $\mathcal{V}$-category structure induced by the order relation $\leq$ of X, $\uparrow^{\leq} A = \uparrow^{a} A$ and $\downarrow^{\leq} A = \downarrow^{a} A$.

Lemma 3.3. *For every $\mathcal{V}$-category (X, a) and every $A \subseteq X$,*

$$A \subseteq \uparrow A, \quad \uparrow\uparrow A \subseteq \uparrow A, \quad A \subseteq \downarrow A, \quad \downarrow\downarrow A \subseteq \downarrow A.$$

Proof. It follows immediately from the two defining properties of a $\mathcal{V}$-category. □

We call a subset $A \subseteq X$ of a $\mathcal{V}$-category (X, a) ***increasing*** whenever $A = \uparrow A$; likewise, A is called ***decreasing*** whenever $A = \downarrow A$. Clearly, $\uparrow A$ is the smallest increasing subset of X which includes A, and similarly for $\downarrow A$. For later use we record some simple facts about increasing and decreasing subsets of a $\mathcal{V}$-category.

Lemma 3.4. *The intersection of increasing (decreasing) subsets of a $\mathcal{V}$-category is increasing (decreasing).*

Lemma 3.5. *Let $f \colon X \to Y$ be a $\mathcal{V}$-functor. Then the following assertions hold.*

1. *For every increasing (decreasing) subset $B \subseteq Y$, $f^{-1}(B)$ is increasing (decreasing) in X.*
2. *For every $A \subseteq X$, $f(\uparrow A) \subseteq \uparrow f(A)$ and $f(\downarrow A) \subseteq \downarrow f(A)$.*

In contrast to the situation for ordered sets, the complement of an increasing set is not necessarily decreasing. This motivates the following notation.

Definition 3.6. Let (X, a) be a $\mathcal{V}$-category and $A \subseteq X$. Then A is called ***co-increasing*** whenever $A^{\complement}$ is increasing, and A is called ***co-decreasing*** whenever $A^{\complement}$ is decreasing.

For a $\mathcal{V}$-category (X, a), we consider the $\mathcal{V}$-category

$$\mathsf{H}X = \{A \subseteq X \mid A \text{ is increasing}\},$$

equipped with

$$\mathsf{H}a(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y),$$

for all $A, B \in \mathsf{H}X$ (see Remark 2.13).

Moreover, we have the following formulas.

Lemma 3.7. *Let (X, a) be a $\mathcal{V}$-category. Then, for all $A, B \subseteq X$, the following assertions hold.*

1. *$k \leq \mathsf{H}a(A, B) \iff B \subseteq \uparrow A$.*

2. $\mathsf{H}a(A, \uparrow B) = \mathsf{H}a(A, B)$ *and* $\mathsf{H}a(\uparrow A, B) = \mathsf{H}a(A, B)$.

Proof. The first assertion is clear, and so are the inequalities $\mathsf{H}a(A, \uparrow B) \leq \mathsf{H}a(A, B)$ and $\mathsf{H}a(\uparrow A, B) \geq \mathsf{H}a(A, B)$. Furthermore, $\mathsf{H}a(A, B) \leq \mathsf{H}a(A, B) \otimes \mathsf{H}a(B, \uparrow B) \leq \mathsf{H}a(A, \uparrow B)$ and $\mathsf{H}a(\uparrow A, B) \leq \mathsf{H}a(A, \uparrow A) \otimes \mathsf{H}a(\uparrow A, B) \leq \mathsf{H}a(A, B)$. □

Corollary 3.8. *For every $\mathcal{V}$-category (X, a), the $\mathcal{V}$-category $\mathsf{H}(X, a)$ is separated. Moreover, the underlying order is containment $\supseteq$.*

For a $\mathcal{V}$-functor $f\colon (X, a) \to (X, a')$, the map

$$\mathsf{H}f\colon \mathsf{H}(X, a) \longrightarrow \mathsf{H}(Y, a')$$

sends an increasing subset $A \subseteq X$ to $\uparrow f(A)$. Then, by Lemma 3.7,

$$\mathsf{H}a(A, B) \leq \mathsf{H}a'(f(A), f(B)) = \mathsf{H}a'(\uparrow f(A), \uparrow f(B))$$

for all $A, B \in \mathsf{H}X$. Clearly, for the identity morphism $1_X\colon X \to X$ in $\mathcal{V}$-Cat, $\mathsf{H}(1_X)$ is the identity morphism on $\mathsf{H}X$. Moreover, for all $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{V}$-Cat and $A \subseteq X$, by Lemma 3.5,

$$\uparrow g(f(A)) \subseteq \uparrow g(\uparrow f(A)) \subseteq \uparrow\uparrow g(f(A)) \subseteq \uparrow g(f(A));$$

which proves that the construction above defines a functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$.

We note that this functor is naturally isomorphic to the dual construction $\mathcal{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ of the "Hausdorff functor" of [53], witnessed by the family $(d_X\colon \mathsf{H}X \to \mathcal{H}X)_X$ where $A \in \mathsf{H}X$ is sent to the presheaf $\mathsf{H}a(A, \{-\})$ on X. By [53, Section 5.2], each d_X is fully faithful and surjective; since $\mathsf{H}X$ is separated, d_X is an isomorphism in $\mathcal{V}$-Cat. For $f\colon (X, a) \to (Y, a')$ in $\mathcal{V}$-Cat, $A \subseteq X$ increasing and $y \in Y$, we calculate

$$\begin{aligned}
\bigvee_{z \in A} a'(f(z), y) &\leq \bigvee_{x \in X} \bigvee_{z \in A} (a(z, x) \otimes a'(f(x), y)) \\
&\leq \bigvee_{x \in X} \bigvee_{z \in A} (a'(f(z), f(x)) \otimes a'(f(x), y)) \leq \bigvee_{z \in A} a'(f(z), y)
\end{aligned}$$

which proves that $(d_X)_X$ is indeed a natural transformation. Consequently, the functor H is part of a Kock–Zöberlein monad $\mathbb{H} = (\mathsf{H}, w, h)$ on $\mathcal{V}$-Cat where

$$\begin{aligned}
h_X\colon X &\longrightarrow \mathsf{H}X, & \qquad w_X\colon \mathsf{H}\mathsf{H}X &\longrightarrow \mathsf{H}X, \\
x &\longmapsto \uparrow x & \mathcal{A} &\longmapsto \bigcup \mathcal{A}
\end{aligned}$$

for all $\mathcal{V}$-categories X. Clearly, h corresponds to the unit of the "Hausdorff monad" of [53]; the following remark justifies the corresponding claim regarding the multiplication.

Remark 3.9. For all $\mathcal{A} \in \mathsf{H}\mathsf{H}X$,

$$\bigcup \mathcal{A} = \{x \in X \mid \exists A \in \mathcal{A} \,.\, x \in A\} = \{x \in X \mid \uparrow x \in \mathcal{A}\} = h_X^{-1}(\mathcal{A}),$$

therefore $\bigcup \mathcal{A}$ is indeed increasing. Furthermore, we conclude that $w_X \dashv \mathsf{H}\, h_X$ in $\mathcal{V}$-Cat.

3.2. Coalgebras of Hausdorff polynomial functors on $\mathcal{V}$-Cat

The notion of Kripke polynomial functor is typically formulated in the context of sets and functions. In this section we study an intuitive $\mathcal{V}$-Cat-counterpart, where the Hausdorff functor on $\mathcal{V}$-Cat takes the role of the powerset functor on Set. For previous studies of Kripke polynomial functors see [49, 13, 38].

Definition 3.10. Let X be a subcategory of $\mathcal{V}$-Cat closed under finite limits and finite colimits such that the Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ restricts to X. We call a functor ***Hausdorff polynomial*** on X if it belongs to the smallest class of endofunctors on X that contains the identity functor, all constant functors and is closed under composition with H, finite products and finite sums of functors.

In the sequel, we will see that the category of coalgebras of a Hausdorff polynomial functor on $\mathcal{V}$-Cat is not necessarily complete. Nevertheless, thanks to the next theorem, we are some small steps away from proving that equalisers always exist.

Theorem 3.11 ([44, Theorem 2.5.24])**.** *Let F be an endofunctor over a cocomplete category X that has an (E, M)-factorisation structure such that E is contained in the class of X-epimorphisms and X is M-wellpowered. If F sends morphisms in M to morphisms in M, then $\mathrm{CoAlg}(\mathsf{F})$ has equalisers.*

Corollary 3.12. *The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves initial morphisms.*

Proof. Let $f\colon (X, a) \to (Y, b)$ be an initial morphism in $\mathcal{V}$-Cat. Consider the map $\Uparrow(-)\colon \mathsf{P}(Y, b) \to \mathsf{H}(Y, b)$ defined by $A \mapsto \uparrow A$. By Lemma 3.7, $\Uparrow(-)$ is an initial morphism in $\mathcal{V}$-Cat. Therefore, by Corollary 2.15, we can express $\mathsf{H}f$ as the following composition of initial morphisms

$$\begin{tikzcd} \mathsf{H}(X,a) \arrow[r, "\mathsf{H}f"] \arrow[d] & \mathsf{H}(Y,b) \\ \mathsf{P}(X,a) \arrow[r, "\mathsf{P}f"'] & \mathsf{P}(Y,b) \arrow[u, "\Uparrow(-)"'] \end{tikzcd} .$$

∎

Proposition 3.13. *The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves initial monomorphisms.*

Proof. We already know from Corollary 3.12 that H preserves initial morphisms, and from Corollary 3.8 that the image by H of every $\mathcal{V}$-category is separated. Therefore, H preserves initial monomorphisms. ∎

Proposition 3.14. *The category of coalgebras of a Hausdorff polynomial functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ has equalisers.*

Proof. Being a topological category over Set, the category $\mathcal{V}$-Cat is (surjective, initial mono)-structured and satisfies all conditions necessary to apply Theorem 3.11. By Proposition 3.13, the Hausdorff functor preserves initial monomorphisms and the remaining cases follow from standard arguments. Therefore, we can apply Theorem 3.11. $\square$

In the remainder of the section, we show that the Hausdorff functor does not admit a terminal coalgebra. This part is inspired by [19].

Given elements x, y of a $\mathcal{V}$-category (X, a), we write $x \prec y$ if $k \leq a(x, y)$ and $a(y, x) = \bot$, and we denote by Υx the set $\{y \in X \mid x \prec y\}$.

Proposition 3.15. *Let (X, a) be a $\mathcal{V}$-category. Then, for every $x, y \in X$, the following assertions hold.*

1. *The set Υx is increasing.*
2. *$\uparrow x \prec \Upsilon x$ in $\mathsf{H}(X, a)$.*
3. *For every initial $\mathcal{V}$-functor $(X, a) \to (Y, b)$, if $x \prec y$ then $fx \prec fy$.*

Proof. The set Υx is the intersection of the increasing sets $\uparrow x$ and $a(-, x)^{-1}\{\bot\}$. Regarding the second affirmation, observe that $\mathsf{H}a(\Upsilon x, \uparrow x) \leq \bigvee_{y \in \Upsilon x} a(y, x) = \bot$. The third affirmation is trivial. $\square$

Theorem 3.16. *Let $\mathcal{V}$ be a non-trivial quantale, and (X, a) a $\mathcal{V}$-category. A morphism of type $\mathsf{H}(X, a) \to (X, a)$ cannot be an embedding.*

Proof. Suppose that there exists an embedding $\phi \colon \mathsf{H}(X, a) \to (X, a)$. We will see that this implies that there exists $x \in X$ such that $\uparrow x = \Upsilon x$, which is a contradiction as $\mathcal{V}$ is non-trivial.

Since $\mathsf{H}X$ is a complete lattice the map $\hbar_X \cdot \phi \colon \mathsf{H}X \to \mathsf{H}X$ has a greatest fixed point A that is given by

$$\bigvee \{I \in \mathsf{H}X \mid I \leq \uparrow\phi(I)\}.$$

We claim that $x = \phi(A)$ has the desired property. The morphism ϕ is initial and $\uparrow x \prec \Upsilon x$, hence, by Proposition 3.15, $x = \phi(\uparrow x) \prec \phi(\Upsilon x)$ and, consequently, $\Upsilon x \leq \uparrow\phi(\Upsilon x)$. Therefore, $\Upsilon x \leq \uparrow x$ because $\uparrow x$ is the greatest fixed point. $\square$

Corollary 3.17. *Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $\mathsf{H} \colon \mathcal{V}$-$\mathsf{Cat} \to \mathcal{V}$-$\mathsf{Cat}$ does not admit fixed points.*

Remark 3.18. If $\mathcal{V}$ is trivial, that is $\mathcal{V} = \{k\}$, then $\mathsf{H} \colon \mathsf{Set} \to \mathsf{Set}$ is the functor that sends every set X to the set $\{X\}$. Therefore, the fixed points of $\mathsf{H} \colon \mathsf{Set} \to \mathsf{Set}$ are the terminal objects.

Corollary 3.19. *Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}$-Cat.*

Example 3.20. In particular, the (non-symmetric) Hausdorff functor on Met does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces. Passing to the symmetric version does not remedy the situation. Here, for a symmetric compact metric space (X,a), we consider now the metric $\mathsf{H}a$ defined by

$$\mathsf{H}a(A,A') = \max\left\{\sup_{x\in A}\inf_{x'\in A'} a(x,x'), \sup_{x\in A'}\inf_{x'\in A} a(x',x)\right\} \tag{3.i}$$

on the set $\mathsf{H}X$ of all closed subsets. Note that $\mathsf{H}a(\varnothing, A) = \infty$, for every non-empty subset $A \subseteq X$. Then, if $s\colon (\mathsf{H}X, \mathsf{H}a) \to (X,a)$ is an isomorphism, we construct recursively a sequence $(x_n)_{n\in\mathbb{N}}$ in X as follows:

$$x_0 = s(\varnothing) \quad \text{and} \quad x_{n+1} = s(\{x_n\}).$$

Then $a(x_m, x_l) = \infty$, for all $m,k \in \mathbb{N}$ with $m \neq k$; which contradicts compactness of (X,a).

Remark 3.21. If we disallow the empty set in the definition of H, then determining if H admits a terminal coalgebra becomes a trivial problem since, in this variation, $\mathsf{H}1 \simeq 1$. In fact, if a category X has a terminal object 1, and an endofunctor $\mathsf{F}\colon \mathsf{X} \to \mathsf{X}$ preserves it, then the unique arrow $1 \to \mathsf{F}1$ defines a terminal coalgebra.

Variants of the Hausdorff functor on categories of metric spaces are studied by various authors. For instance, in [15] it is shown that the category CMet of 1-bounded complete metric spaces and non-expansive maps is accessible, and that the Hausdorff functor $\mathcal{H}\colon \mathsf{CMet} \to \mathsf{CMet}$ sending a complete metric space to the space of all non-empty and compact subsets with distance defined as in (3.i) is accessible. Since the constant functor $1\colon \mathsf{CMet} \to \mathsf{CMet}$ is accessible, so is the functor $\mathsf{CMet} \to \mathsf{CMet}$ sending X to $\mathcal{H}X + 1$ (see [15, Propositions 2–4]), which is isomorphic to the functor $\mathsf{H}\colon \mathsf{CMet} \to \mathsf{CMet}$ sending a complete metric space to the space of all compact subsets with distance defined as in (3.i). Therefore also $\mathsf{H}\colon \mathsf{CMet} \to \mathsf{CMet}$ admits a terminal coalgebra (see [15, Theorem 1]). Trading compact with finite, in [8, Example 5.31 and Theorem 6.2] it is shown that the "finite Hausdorff functor" on $\mathsf{Met}_{\mathsf{sym}}$ admits a terminal coalgebra. Also, we point the reader to [1] where the terminal coalgebra for the Hausdorff functor on the category CUMet of compact ultrametric spaces and *continuous* maps is studied. Among other results, it is shown in [1] that the category CUMet is equivalent to the category of second countable Stone spaces and continuous maps, and the Hausdorff functor corresponds to the Vietoris functor on this category. Finally, an extensive study of categories of coalgebras for Vietoris functors on categories of (compact) spaces was conducted in [28].

4. Hausdorff polynomial functors on $\mathcal{V}$-CatCH

In Section 3.2 we saw that the image of a $\mathcal{V}$-category under the Hausdorff functor H on $\mathcal{V}$-Cat has "too many" elements for H to admit a terminal coalgebra. To filter them, in this section we add a topological component to our study of $\mathcal{V}$-Cat.

4.1. Adding topology

To "add topology", we use the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ on Set. Furthermore:

Assumption 4.1. Throughout this section we assume that $\mathcal{V}$ is completely distributive quantale (see [48, 21]).

Then

$$\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}, \quad \mathfrak{v} \longmapsto \bigwedge_{A\in\mathfrak{v}} \bigvee A$$

is the structure of an $\mathbb{U}$-algebra on $\mathcal{V}$, and represents the convergence of a compact Hausdorff topology. Therefore, as discussed at the end of Section 2, we obtain a lax extension of the ultrafilter monad to $\mathcal{V}$-Rel that induces a monad on $\mathcal{V}$-Cat. Its algebras are $\mathcal{V}$-categories equipped with a *compatible* compact Hausdorff topology (see [55, 31]); we call them ***$\mathcal{V}$-categorical compact Hausdorff spaces***, and denote the corresponding Eilenberg–Moore category by

$$\mathcal{V}\text{-}\mathsf{CatCH}.$$

Then we have a natural forgetful functor

$$\mathcal{V}\text{-}\mathsf{CatCH} \longrightarrow \mathsf{OrdCH}$$

sending (X, a, α) to $(X, \le, \alpha)$ where

$$x \le y \quad \text{whenever} \quad k \le a(x, y).$$

Moreover, $(\mathcal{V}, \hom, \xi)$ is a $\mathcal{V}$-categorical compact Hausdorff space with underlying ordered compact Hausdorff space $(\mathcal{V}, \le, \xi)$, where $\le$ is the order of $\mathcal{V}$. We denote by $\xi_{\le}$ the induced stably compact topology. We provide now some information on the topologies of $\mathcal{V}$.

Remark 4.2. Since $\mathcal{V}$ is in particular a continuous lattice, the convergence ξ is the convergence of the Lawson topology of $\mathcal{V}$ (see [25, Proposition VII-3.10]). A subbasis for this topology is given by the sets

$$\{u \in \mathcal{V} \mid v \ll u\} \quad \text{and} \quad \{u \in \mathcal{V} \mid v \not\le u\} \qquad (v \in \mathcal{V}),$$

where $\ll$ denotes the way-below relation of $\mathcal{V}$. Furthermore, by [2, Proposition 2.3.6], the sets

$$\{u \in \mathcal{V} \mid v \ll u\} \qquad (v \in \mathcal{V})$$

form a basis for the Scott topology of $\mathcal{V}$. By the proof of [25, Lemma V-5.15], the sets

$$\{u \in \mathcal{V} \mid v \not\le u\} = (\uparrow v)^{\complement} \qquad (v \in \mathcal{V})$$

form a subbasis of the dual of Scott topology of $\mathcal{V}$, which is precisely $\xi_{\le}$.

Since, moreover, $\mathcal{V}$ is (ccd), we have the following.

- By [25, Lemma VII-2.7] and [25, Proposition VII-2.10], the Lawson topology of $\mathcal{V}$ coincides with the Lawson topology of $\mathcal{V}^{\mathrm{op}}$, and the set

$$\{\uparrow u \mid u \in \mathcal{V}\} \cup \{\downarrow u \mid u \in \mathcal{V}\}$$

is a subbasis for the closed sets of this topology which is known as the interval topology.

- Therefore the Scott topology of $\mathcal{V}$ coincides with the dual of the Scott topology of $\mathcal{V}^{\mathrm{op}}$; in particular, the sets $\downarrow v$ $(v \in \mathcal{V})$ form a subbasis for the closed sets of the Scott topology of $\mathcal{V}$.

- Finally, with $\lll$ denoting the totally below relation of $\mathcal{V}$, also the sets

$$\{u \in \mathcal{V} \mid v \lll u\} \qquad (v \in \mathcal{V})$$

form a subbasis of the Scott topology of $\mathcal{V}$.

We aim now at $\mathcal{V}$-categorical generalisations of some results of [43] regarding ordered compact Hausdorff spaces. Firstly, we recall [31, Proposition 3.22]:

Proposition 4.3. *For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α) with the same underlying set X, the following assertions are equivalent.*

(i) $\alpha\colon \mathsf{U}(X, a) \to (X, a)$ *is a $\mathcal{V}$-functor.*

(ii) $a\colon (X, \alpha) \times (X, \alpha) \to (\mathcal{V}, \xi_{\leq})$ *is continuous.*

For the quantale $\mathcal{V} = 2$, the result above reveals that Nachbin's ordered compact spaces are precisely the separated $\mathcal{V}$-categorical compact Hausdorff spaces: the topological space $(\mathcal{V}, \xi_{\leq})$ is the Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed, and, therefore, the assertion (ii) translates to "the order relation a is closed in $X \times X$" (see also [55, Proposition 4])).

Corollary 4.4. *For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α) with the same underlying set X, (X, a, α) is a $\mathcal{V}$-categorical compact Hausdorff space if and only if, for all $x, y \in X$ and $u \in \mathcal{V}$ with $u \not\leq a(x, y)$, there exist neighbourhoods V of x and W of y so that, for all $x' \in V$ and $y' \in W$, $u \not\leq a(x', y')$.*

Proof. It follows from the fact that the sets $(\uparrow u)^{\complement}$ $(u \in \mathcal{V})$ form a subbasis for the topology $\xi_{\leq}$ on $\mathcal{V}$ (see Remark 4.2). $\square$

We consider now the full subcategory $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ of $\mathcal{V}\text{-}\mathsf{CatCH}$ defined by the separated $\mathcal{V}$-categorical compact Hausdorff spaces; i.e. those spaces where the underlying $\mathcal{V}$-category is separated. The results above imply that the separated reflector $\mathsf{R}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ lifts to a functor $\mathsf{S}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ which is left adjoint to the inclusion functor $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}} \to \mathcal{V}\text{-}\mathsf{CatCH}$. In fact, for a $\mathcal{V}$-categorical compact Hausdorff space (X, a, α), the equivalence relation $\sim$ on X

is closed in $X \times X$ with respect to the product topology, therefore the quotient topology on $X/{\sim}$ is compact Hausdorff and, with $p\colon X \to X/{\sim}$ denoting the projection map, the diagram

$$\begin{tikzcd} X \times X \arrow[r, "p\times p"] \arrow[dr, "a"'] & X/{\sim} \times X/{\sim} \arrow[d, "\widetilde{a}"] \\ & \mathcal{V} \end{tikzcd}$$

commutes. Consequently, the $\mathcal{V}$-category $(X/{\sim}, \widetilde{a})$ together with the quotient topology on $X/{\sim}$ is a $\mathcal{V}$-categorical compact Hausdorff space. In contrast to Remark A.9, now we have the following result.

Proposition 4.5. *The functor* $\mathsf{S}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ *preserves codirected limits.*

Proof. Let $D\colon I \to \mathcal{V}\text{-}\mathsf{CatCH}$ be a codirected diagram with limit cone $(\pi_i\colon X \to X_i)_{i\in I}$. Let $(\rho_i\colon L \to \mathsf{S}X_i)_{i\in I}$ be a limit cone of $\mathsf{S}D$ in $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$ and $q\colon \mathsf{S}X \to L$ be the canonical comparison map. By Corollary A.8, q is initial with respect to the forgetful functor $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}} \to \mathsf{CompHaus}$. Since the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\pi_i"'] & \mathsf{S}X \arrow[r, "q"] & L \arrow[d, "\rho_i"] \\ X_i \arrow[rr, "p_i"'] & & \mathsf{S}X_i \end{tikzcd}$$

commutes, $q \cdot p$ is surjective by [14, I.9.6, Corollary 2] hence q is surjective and therefore an isomorphism in $\mathcal{V}\text{-}\mathsf{CatCH}_{\mathrm{sep}}$. $\square$

Besides the compact Hausdorff space (X, α), we also consider the stably compact topology $a_{\leq}$ induced by α and the underlying order of a, as well as the dual space $(X, \alpha_{\leq})^{\mathrm{op}}$ of $(X, \alpha_{\leq})$. We remark that the identity map $1_X\colon X \to X$ is continuous of types

$$(X, \alpha) \longrightarrow (X, \alpha_{\leq}) \qquad \text{and} \qquad (X, \alpha) \longrightarrow (X, \alpha_{\leq})^{\mathrm{op}}.$$

Therefore a subset $A \subseteq X$ of X is open (closed) in (X, α) if it is open (closed) in $(X, \alpha_{\leq})$ or in $(X, \alpha_{\leq})^{\mathrm{op}}$. Moreover, every closed subset of (X, α) is compact in $(X, \alpha_{\leq})$ and in $(X, \alpha_{\leq})^{\mathrm{op}}$.

Corollary 4.6. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space. Then also*

$$a\colon (X, \alpha_{\leq})^{\mathrm{op}} \times (X, \alpha_{\leq}) \longrightarrow (\mathcal{V}, \xi_{\leq})$$

is continuous. Hence, for all $x, y \in X$ and $u \in \mathcal{V}$ with $u \not\leq a(x, y)$, there exist a neighbourhood V of x in $(X, \alpha_{\leq})^{\mathrm{op}}$ and a neighbourhood W of y in $(X, \alpha_{\leq})$ so that, for all $x' \in V$ and $y' \in W$, $u \not\leq a(x', y')$.

Proof. Follows from the facts that $a\colon X \times X \to \mathcal{V}$ is continuous of type $(X, \alpha) \times (X, \alpha) \to (\mathcal{V}, \xi_{\leq})$ and monotone of type $(X, \leq)^{\mathrm{op}} \times (X, \leq) \to (\mathcal{V}, \leq)$. $\square$

Remark 4.7. The result above allows us to construct some useful continuous maps. For instance, for $A \subseteq X$ compact in $(X, \alpha_{\leq})^{\mathrm{op}}$, the map $a\colon A \times X \to \mathcal{V}$ is continuous where we consider on A the subspace topology. Therefore the composite arrow

$$\begin{tikzcd} X \arrow[r, "\ulcorner a \urcorner"] \arrow[rr, bend left, "\Uparrow_A^a"] & \mathcal{V}^A \arrow[r, "\bigvee"] & \mathcal{V} \end{tikzcd}$$

is continuous of type $(X, \alpha_{\leq}) \to (\mathcal{V}, \xi_{\leq})$. Note that

$$\Uparrow_A^a(x) = \bigvee\{a(z, x) \mid z \in A\},$$

for every $x \in X$. Similarly, for $A \subseteq X$ compact in $(X, \alpha_{\leq})$, we obtain a continuous map $\Downarrow_A^a\colon (X, \alpha_{\leq})^{\mathrm{op}} \to (\mathcal{V}, \xi_{\leq})$ sending $x \in X$ to

$$\Downarrow_A^a(x) = \bigvee\{a(x, z) \mid z \in A\}.$$

Lemma 4.8. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space and $A \subseteq X$. Then the following assertions hold.*

1. *If A is compact in $(X, \alpha_{\leq})^{\mathrm{op}}$, then $\uparrow^a A$ is closed in $(X, \alpha_{\leq})$ and therefore also in (X, α).*
2. *If A is compact in $(X, \alpha_{\leq})$, then $\downarrow^a A$ is closed in $(X, \alpha_{\leq})^{\mathrm{op}}$ and therefore also in (X, α).*

In particular, if A is closed in (X, α), then $\uparrow^a A$ and $\downarrow^a A$ are closed in $(X, \alpha_{\leq})$ and hence also in (X, α).

Proof. Use the maps to Remark 4.7 and observe that

$$\uparrow^a A = (\Uparrow_A^a)^{-1}(\uparrow k) \qquad \text{and} \qquad \downarrow^a A = (\Downarrow_A^a)^{-1}(\uparrow k).$$ □

From now on we assume the following condition.

Assumption 4.9. The subset

$$\Downarrow k = \{u \in \mathcal{V} \mid u \lll k\}$$

of $\mathcal{V}$ is directed; which implies in particular that $k \neq \perp$. A quantale satisfying this condition is called *value quantale* in [23], whereby in [31] the designation *k is approximated* is used.

Example 4.10. Consider the quantale $\mathcal{V} = \mathsf{P}\mathbb{N}$ the powerset of $\mathbb{N}$ with order given by subset inclusion and with $\otimes = \cap$ intersection. In this case, the neutral element of $\mathcal{V}$ is given by $\mathbb{N}$ and, for $A, B \in \mathsf{P}\mathbb{N}$,

$$A \lll B \iff A = \{x\} \text{ for some } x \in B.$$

Hence, $\Downarrow \mathbb{N} = \{\{n\} \mid n \in \mathbb{N}\}$ is not directed, that is, $\mathcal{V} = \mathsf{P}\mathbb{N}$ does not satisfy Assumption 4.9.

Assumption 4.9 implies some further pleasant properties of $\mathcal{V}$, as we recall next.

Lemma 4.11. *The $\otimes$-neutral element k satisfies the conditions*

$$(k \leq u \vee v) \implies ((k \leq u) \text{ or } (k \leq v)),$$

for all $u, v \in \mathcal{V}$, and

$$k \leq \bigvee_{u \lll k} u \otimes u.$$

Proof. See [22, Theorem 1.12] and [30, Remark 4.21]. $\square$

Lemma 4.12. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \varnothing$, A is increasing and compact in $(X, \alpha_{\leq})^{\mathrm{op}}$ and B is compact in $(X, \alpha_{\leq})$. Then there exists some $u \lll k$ so that, for all $x \in A$ and $y \in B$, $u \not\leq a(x, y)$.*

Proof. Let $y \in B$. Since A is increasing and $y \notin A$, there is some $v_y \lll k$ so that

$$v_y \not\leq \bigvee_{x \in A} a(x, y).$$

Hence, by Corollary 4.6, for every $x \in A$ there exists U_{xy} open in $(X, \alpha_{\leq})^{\mathrm{op}}$ and W_{xy} open in $(X, \alpha_{\leq})$ such that $y \in W_{xy}$ and

$$\forall x' \in U_{xy}, y' \in W_{xy} \,.\, v_y \not\leq a(x', y').$$

Therefore, by compactness of A, there exists an open subset U_y in $(X, \alpha_{\leq})^{\mathrm{op}}$ and an open subset W_y in $(X, \alpha_{\leq})$ such that $A \subseteq U_y$, $y \in W_y$ and

$$\forall x' \in U_y, y' \in W_y \,.\, v_y \not\leq a(x', y').$$

Then $B \subseteq \bigcup \{W_y \mid y \in X, y \notin V\}$ and, since B is compact, there are finitely many elements $y_1, \dots, y_n \in B$ with

$$B \subseteq W_{y_1} \cup \dots \cup W_{y_n}.$$

Put $u = v_{y_1} \vee \dots \vee v_{y_n}$. Then $u \lll k$ since $\Downarrow k$ is directed; moreover, $u \not\leq a(x, y)$, for all $x \in A$ and $y \in B$. $\square$

Lemma 4.13. *Let $A \subseteq \mathcal{V}$ be compact subset in $(\mathcal{V}, \xi_{\leq})$. If $k \leq \bigvee A$, then there is some $u \in A$ with $k \leq u$.*

Proof. Assume that $\uparrow k \cap \downarrow A = \varnothing$. Since $\uparrow k$ is increasing and compact in $(\mathcal{V}, \xi_{\leq})^{\mathrm{op}}$ and $\downarrow A$ is compact in $(\mathcal{V}, \xi_{\leq})$, by Lemma 4.12, there is some $u \lll k$ so that, for all $v \in A$, $u \not\leq \hom(k, v) = v$. Therefore $k \not\leq \bigvee A$. $\square$

Combining Remark 4.7 with Lemma 4.13, we obtain:

Lemma 4.14. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space with underlying order $\leq$. Then, for every compact subset $A \subseteq X$ of $(X, \alpha_{\leq})^{\mathrm{op}}$, $\Uparrow^{a} A = \Uparrow^{\leq} A$; likewise, for every compact subset $A \subseteq X$ of $(X, \alpha_{\leq})$, $\Downarrow^{a} A = \Downarrow^{\leq} A$. In particular, for every closed subset $A \subseteq X$ of (X, α), $\Downarrow^{a} A = \Downarrow^{\leq} A$ and $\Uparrow^{a} A = \Uparrow^{\leq} A$.*

Thanks to Lemma 4.14 we can transport several well-known result for ordered compact Hausdorff spaces to metric compact Hausdorff spaces.

Lemma 4.15. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space, $A \subseteq X$ closed and $W \subseteq X$ open and co-increasing with $A \subseteq W$. Then $\downarrow A \subseteq W$.*

Proof. Apply Lemma 4.12 to $W^{\complement} \subseteq A^{\complement}$. $\square$

The following result is [43, Proposition 5].

Proposition 4.16. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space, $A \subseteq X$ closed and increasing and $V \subseteq X$ open with $A \subseteq V$. Then there exists $W \subseteq X$ open and co-decreasing with $A \subseteq W \subseteq V$.*

Theorem 4.17. *Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space, $A \subseteq X$ closed and decreasing and $B \subseteq X$ closed and increasing with $A \cap B = \varnothing$. Then there exist $V \subseteq X$ open and co-increasing and $W \subseteq X$ open and co-decreasing with*

$$A \subseteq V, \quad B \subseteq W, \quad V \cap W = \varnothing.$$

Proof. See [43, Theorem 4]. $\square$

For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$\mathsf{H}X = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

and consider on $\mathsf{H}X$ the restriction of the Hausdorff structure $\mathsf{H}a$ to $\mathsf{H}X$ and the ***hit-and-miss topology***, that is, the topology generated by the sets

$$V^{\Diamond} = \{A \in \mathsf{H}X \mid A \cap V \neq \varnothing\} \qquad (V \text{ open, co-increasing})$$

and

$$W^{\Box} = \{A \in \mathsf{H}X \mid A \subseteq W\} \qquad (W \text{ open, co-decreasing}).$$

Note that, by Lemma 4.14, the topological part of $\mathsf{H}X$ coincides with the Vietoris topology for the underlying ordered compact Hausdorff space. In particular:

Proposition 4.18. *For every $\mathcal{V}$-categorical compact Hausdorff space X, the hit-and-miss topology on $\mathsf{H}X$ is compact and Hausdorff.*

Proposition 4.19. *For every $\mathcal{V}$-categorical compact Hausdorff space X, $\mathsf{H}X$ equipped with the hit-and-miss topology and the Hausdorff structure is a $\mathcal{V}$-categorical compact Hausdorff space.*

Proof. Consider a $\mathcal{V}$-categorical compact Hausdorff space (X, a, α). To establish the compatibility between the topology and the Hausdorff $\mathcal{V}$-category structure, we use Corollary 4.4. Let $A, B \in \mathsf{H}X$ and $u \in \mathcal{V}$. Assume $u \not\leq \mathsf{H}a(A, B)$. Since $\mathcal{V}$ is (ccd), there is some $v \lll u$ with $v \not\leq \mathsf{H}a(A, B)$. Hence, there is some $y \in B$ with $v \not\leq \bigvee_{x\in A} a(x, y)$. Therefore $v \not\leq a(x, y)$ for all $x \in A$. By Corollary 4.4 and compactness of A, there exist open subsets $U, V \subseteq X$ with $A \subseteq U$, $y \in V$, and $v \not\leq a(x', y')$ for all $x' \in U$ and $y' \in V$; by Proposition 4.16, we may assume that U is co-decreasing and V is co-increasing. We conclude that

$$A \in U^{\Box}, \quad B \in V^{\Diamond}, \quad u \not\leq \mathsf{H}a(A', B')$$

for all $A' \in U^{\Box}$ and $B' \in V^{\Diamond}$. $\square$

Lemma 4.20. *Let $f \colon X \to Y$ be in $\mathcal{V}$-CatCH. Then the map*

$$\mathsf{H}f \colon \mathsf{H}X \longrightarrow \mathsf{H}Y,\ A \longmapsto \uparrow f(A)$$

is continuous and a $\mathcal{V}$-functor.

Clearly, the construction of Lemma 4.20 defines a functor

$$\mathsf{H} \colon \mathcal{V}\text{-}\mathsf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathsf{CatCH}.$$

Moreover:

Proposition 4.21. *The diagrams*

$$\begin{tikzcd} \mathsf{OrdCH} \arrow[r, "\mathsf{H}"] \arrow[d] & \mathsf{OrdCH} \arrow[d] \\ \mathcal{V}\text{-}\mathsf{CatCH} \arrow[r, "\mathsf{H}"'] & \mathcal{V}\text{-}\mathsf{CatCH} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V}\text{-}\mathsf{CatCH} \arrow[r, "\mathsf{H}"] \arrow[d] & \mathcal{V}\text{-}\mathsf{CatCH} \arrow[d] \\ \mathsf{OrdCH} \arrow[r, "\mathsf{H}"'] & \mathsf{OrdCH} \end{tikzcd}$$

of functors commutes.

Remark 4.22. Despite the commutative diagrams of Proposition 4.21 above, we cannot apply Theorem 2.5 because, in general, the functors are not topological. In fact, even the functor $\mathsf{Met} \to \mathsf{Ord}$ fails to be fibre-complete since a metric d in the fibre of $\{0 \leq 1\}$ is completely determined by the value $d(1, 0) \in (0, \infty]$. On the other hand, the functor $\mathcal{V}\text{-}\mathsf{CatCH} \to \mathsf{CompHaus}$ is topological and it is easy to see that the Hausdorff $\mathcal{V}$-category structure is compatible with the classical Vietoris topology on $\mathsf{CompHaus}$. Therefore, Theorem 2.5 tell us that equipping the Vietoris space on $\mathsf{CompHaus}$ with the Hausdorff $\mathcal{V}$-category structure yields a "powerset kind of" functor on $\mathcal{V}$-CatCH that, in some sense, disregards the $\mathcal{V}$-category structure of the objects, but whose category of coalgebras is (co)complete.

Theorem 4.23. *The functor* H *is part of a Kock–Zöberlein monad* $\mathbb{H} = (\mathsf{H}, w, \hbar)$ *on* $\mathcal{V}$-CatCH*; for every* X *in* $\mathcal{V}$-CatCH*, the components* $\hbar_X$ *and* w_X *are given by*

$$\hbar_X \colon X \longrightarrow \mathsf{H}X, \qquad\qquad w_X \colon \mathsf{HH}X \longrightarrow \mathsf{H}X.$$
$$x \longmapsto {\uparrow} x \qquad\qquad \mathcal{A} \longmapsto \bigcup \mathcal{A}$$

We recall from [33] that to every $\mathcal{V}$-category one can associate a canonical closure operator which generalises the classic topology associated to a metric space.

Proposition 4.24. *For every* $\mathcal{V}$*-category* (X, a)*,* $A \subseteq X$ *and* $x \in X$*,*

$$x \in \overline{A} \iff k \leq \bigvee_{z \in A} a(x, z) \otimes a(z, x).$$

Moreover, the closure operator $\overline{(-)}$ *is topological for every* $\mathcal{V}$*-category and defines a functor*

$$\mathsf{L}_{\mathcal{V}} \colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Top}$$

which commutes with the forgetful functors to Set*. Moreover,* $L_{\mathcal{V}}(X) = L_{\mathcal{V}}(X^{\mathrm{op}})$ *for every* $\mathcal{V}$*-category* X*.*

Proof. See [33]. □

Recall that we assume $\Downarrow k$ to be directed.

Proposition 4.25. 1. *For every* $\mathcal{V}$*-category* (X, a)*, the topology of* $\mathsf{L}_{\mathcal{V}}(X, a)$ *is generated by the* left centered balls

$$\mathrm{L}(x, u) = \{y \in X \mid u \lll a(x, y)\} \qquad (x \in X,\, u \lll k)$$

and the right centered balls

$$\mathrm{R}(x, u) = \{y \in X \mid u \lll a(y, x)\} \qquad (x \in X,\, u \lll k).$$

2. *For every separated* $\mathcal{V}$*-category* (X, a)*, the space* $\mathsf{L}(X, a)$ *is Hausdorff.*

Proof. Regarding first statement, see [30, Remark 4.21] and [22]. The proof of the second statement is analogous to the one for classic metric spaces. In fact, assume that (X, a) is separated and let $x, y \in X$ with $x \neq y$. Without loss of generality, we may assume that $k \nleq a(x, y)$. Hence, there is some $u \lll k$ with $u \nleq a(x, y)$. Take $v, w \lll k$ with $u \leq v \otimes w$. Then

$$\mathrm{L}(x, v) \cap \mathrm{R}(y, w) = \varnothing$$

since, if $z \in \mathrm{L}(x, v) \cap \mathrm{R}(y, w)$, then

$$u \leq v \otimes w \leq a(x, z) \otimes a(z, y) \leq a(x, y),$$

a contradiction. □

Until the end of this section we require also the following condition.

Assumption 4.26. For all $u, v \in \mathcal{V}$,

$$(k \leq u \otimes v) \implies (k \leq u \quad \text{and} \quad k \leq v).$$

Remark 4.27. For every subset $A \subseteq X$ of a $\mathcal{V}$-category (X, a),

$$\overline{A} \subseteq \uparrow A \cap \downarrow A.$$

In fact, if $x \in \overline{A}$, then

$$k \leq \bigvee_{z \in A} (a(x,z) \otimes a(z,x)) \leq \left(\bigvee_{z \in A} a(x,z)\right) \otimes \left(\bigvee_{z \in A} a(z,x)\right)$$

and therefore $k \leq \bigvee_{z \in A} a(x,z)$ and $k \leq \bigvee_{z \in A} a(z,x)$). In particular, every increasing and every decreasing subset of X are closed with respect to the closure operator of (X, a).

Corollary 4.28. *The identity map on $\mathcal{V}$ is continuous of type $\mathsf{L}\mathcal{V} \to (\mathcal{V}, \xi_{\leq})$.*

Recall from [31, Proposition 3.29] that the identity map on $X \times X$ is continuous of type

$$L_{\mathcal{V}}(X,a) \times L_{\mathcal{V}}(X,a) \longrightarrow L_{\mathcal{V}}((X,a) \otimes (X,a)),$$

for every $\mathcal{V}$-category (X, a); hence, the composite map

$$L_{\mathcal{V}}(X,a)^{\mathrm{op}} \times L_{\mathcal{V}}(X,a) \longrightarrow L_{\mathcal{V}}((X,a)^{\mathrm{op}} \otimes (X,a)) \xrightarrow{\ a\ } L_{\mathcal{V}}\mathcal{V} \longrightarrow (\mathcal{V}, \xi_{\leq})$$

is continuous. Therefore, if (X, a) is separated and $L_{\mathcal{V}}(X, a)$ is compact, then these two structures define a $\mathcal{V}$-categorical compact Hausdorff space. In fact, with $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{comp,sep}}$ denoting the full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ defined by those separated $\mathcal{V}$-categories X where $\mathsf{L}_{\mathcal{V}}$ is compact, the construction above defines a functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{comp,sep}} \to \mathcal{V}\text{-}\mathsf{CatCH}$ (see [31, Theorem 3.28]). Similarly to a well-known property of metric spaces, [31, Corollary 4.21] affirms that, under suitable conditions, every compact separated $\mathcal{V}$-category is Cauchy-complete.

For classical compact metric spaces, it is well-known that the Hausdorff metric induces the hit-and-miss topology. Below we give an asymmetric version of this result in the context of $\mathcal{V}$-categories.

Lemma 4.29. *For the $\mathcal{V}$-categorical compact Hausdorff space induced by a compact separated $\mathcal{V}$-category X, the hit-and-miss topology on $\mathsf{H}X$ coincides with the topology induced by the Hausdorff structure on $\mathsf{H}X$.*

Proof. Let (X, a) be a compact separated $\mathcal{V}$-category. We show that the topology induced by $\mathsf{H}a$ is contained in the hit-and-miss topology; then, since the former is Hausdorff and the latter is compact, both topologies coincide.

Let $A \in \mathsf{H}X$ and $u \lll k$. For every $v \in \mathcal{V}$ with $u \lll v \lll k$, put

$$U_v = \bigcup_{x \in A} \mathrm{L}(x, v).$$

We show that $\mathrm{L}(A, u) = \bigcup_{u \lll v \lll k} U_v^{\Box}$. To see this, let $B \in \mathrm{L}(A, u)$, hence, $u \lll \mathsf{H}a(A, B)$. Let $v \in \mathcal{V}$ with $u \lll v \lll \mathsf{H}a(A, B)$. Then, for every $y \in B$, exists $x \in A$ with $v \lll a(x, y)$, that is, $y \in \mathrm{L}(x, v)$. Therefore $B \subseteq U_v$, which is equivalent to $B \in U_V^{\Box}$. Let now $B \in U_v^{\Box}$, for some $u \lll v \lll k$. Then, for all $y \in B$, there is some $x \in A$ with $v \lll a(x, y)$; hence

$$u \lll v \le \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y) = \mathsf{H}a(A, B).$$

Let now $B \in \mathrm{R}(A, u)$, and take $u', v \in \mathcal{V}$ with $u \lll u' \lll v \lll \mathsf{H}a(B, A)$. For every $x \in A$, there exists $y \in B$ with $v \lll a(y, x)$, that is, $y \in B \cap \mathrm{R}(x, v)$. Take $w \lll k$ with $u' \lll v \otimes w$. By compactness, there exist $x_1, \dots, x_n \in A$ with

$$A \subseteq \mathrm{L}(x_1, w) \cup \dots \cup \mathrm{L}(x_n, w).$$

Then $B \in \mathrm{R}(x_1, v)^{\Diamond} \cap \dots \cap \mathrm{R}(x_n, v)^{\Diamond}$; moreover, $\mathrm{R}(x_1, v)^{\Diamond} \cap \dots \cap \mathrm{R}(x_n, v)^{\Diamond} \subseteq \mathrm{R}(A, u)$. To see the latter, let $B' \in \mathrm{R}(x_1, v)^{\Diamond} \cap \dots \cap \mathrm{R}(x_n, v)^{\Diamond}$ and $x \in A$, then $x \in \mathrm{L}(x_i, w)$ for some $i \in \{1, \dots, n\}$. Let $y \in B' \cap \mathrm{R}(x_i, v)$, then

$$u' \lll v \otimes w \le a(y, x_i) \otimes a(x_i, x) \le a(y, x),$$

which implies $u \lll u' \le \mathsf{H}a(B', A)$. $\square$

Theorem 4.30. *The functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ restricts to the category $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{comp,sep}}$, moreover, the diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{comp,sep}} \arrow[r, "\mathsf{H}"] \arrow[d] & \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{comp,sep}} \arrow[d] \\
\mathcal{V}\text{-}\mathsf{CatCH} \arrow[r, "\mathsf{H}"'] & \mathcal{V}\text{-}\mathsf{CatCH}
\end{tikzcd}$$

commutes.

4.2. Coalgebras of Hausdorff polynomial functors on $\mathcal{V}$-CatCH

In this section we show that by "adding topology" we can improve the results of Section 3.2 about limits in categories of coalgebras of Hausdorff polynomial functors. Throughout this section we still require Assumptions 4.1 and 4.9.

We begin by showing that the category of coalgebras of the Hausdorff functor on $\mathcal{V}$-CatCH is complete. The following result summarizes our strategy.

Theorem 4.31. *Let X be a category that is complete, cocomplete and has an (E, M)-factorisation structure such that X is M-wellpowered and E is contained in the class of X-epimorphisms. If a functor $\mathsf{F}\colon \mathsf{X} \to \mathsf{X}$ sends morphisms in M to morphisms in M and preserves codirected limits, then the category of coalgebras of F is complete.*

Proof. The claim follows by combining Corollary 3.11, [10, Proposition 7 of Section 9.4], [3, Remark 4.4] and [41, Corollary 2]. $\square$

Also, the theorem bellow will help us to replace "preserves codirected limits" with "preserves codirected initial cones".

Theorem 4.32 ([4, Proposition 13.15])**.** *Let* $\mathsf{U}\colon \mathsf{X} \to \mathsf{A}$ *be a limit preserving faithful functor and* $\mathsf{D}\colon \mathsf{I} \to \mathsf{X}$ *a diagram. A cone* $\mathcal{C}$ *for* D *is a limit in* X *if and only if the cone* $\mathsf{U}\mathcal{C}$ *is a limit of* UD *in* A *and* $\mathcal{C}$ *is initial with respect to* U*.*

Proposition 4.33. *The Hausdorff functor on* $\mathcal{V}$-CatCH *preserves codirected initial cones with respect to the forgetful functor* $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$*.*

Proof. Let $(f_i\colon (X, a, \alpha) \to (X_i, a_i, \alpha_i))_{i\in\mathsf{I}}$ be a codirected initial cone with respect to the functor $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$. We will show that for every $A, B \subseteq X$ the inequality

$$u = \bigwedge_{i\in\mathsf{I}} \mathsf{H}a_i(\mathsf{H}f_i(A), \mathsf{H}f_i(B)) \le \mathsf{H}a(A, B)$$

holds. Note that since $\mathcal{V}$ is (ccd) it is sufficient to prove that $v \le \mathsf{H}a(A, B)$ for every $v \lll u$.

Let $b \in B$ and fix $v \in \mathcal{V}$ such that $v \lll u$. Then, for every $i \in \mathsf{I}$,

$$u \le \mathsf{H}a_i(f_i(A), f_i(B)) \le \bigvee_{x\in A} a_i(f_i(x), f_i(b)),$$

since $\mathsf{H}a_i(\mathsf{H}f_i(A), \mathsf{H}f_i(B)) = \mathsf{H}a_i(f_i(A), f_i(B))$ by lemma 3.7. Hence, for every $i \in \mathsf{I}$, there exists an element $x_i \in A$ such that $v \le a_i(f_i(x_i), f_i(b))$. Thus, for every $i \in \mathsf{I}$, the set

$$A_i = A \cap \{x \in X \mid v \le a_i(f_i(x), f_i(b))\}$$

is non-empty and closed because $\uparrow v \subseteq \mathcal{V}$ is closed (see Remark 4.2) and $a\colon (X, \alpha) \to (\mathcal{V}, \xi_{\le})$ is continuous (see Proposition 4.3). This way we obtain a codirected family of closed subsets of X that has the finite intersection property since the cone $(f_i)_{i\in\mathsf{I}}$ is codirected. Consequently, by compactness of X, there exists $x_b \in \bigcap_{i\in\mathsf{I}} A_i$ such that for every $i \in \mathsf{I}$, $v \le a_i(f_i(x_b), f_i(b))$. Therefore, $v \le a(x_b, b)$ since the cone $(f_i)_{i\in\mathsf{I}}$ is initial, which implies $v \le \mathsf{H}a(A, B)$. $\square$

Corollary 4.34. *The Hausdorff functor on* $\mathcal{V}$-CatCH *preserves initial monomorphisms with respect to the forgetful functor* $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$*.*

Theorem 4.35. *The functor* $\mathsf{H}\colon \mathcal{V}$-$\mathsf{CatCH} \to \mathcal{V}$-$\mathsf{CatCH}$ *preserves codirected limits.*

Proof. From Proposition 4.21, the diagram below commutes.

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{CatCH} \arrow[r, "\mathsf{H}"] \arrow[d] & \mathcal{V}\text{-}\mathsf{CatCH} \arrow[r] \arrow[d] & \mathsf{CompHaus} \\
\mathsf{OrdCH} \arrow[r, "\mathsf{H}"'] & \mathsf{OrdCH} \arrow[ur] &
\end{tikzcd}$$

Therefore, taking into account Theorem 4.32, the claim follows from Proposition 4.33. ☐

Corollary 4.36. *For* $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$*, the forgetful functor* $\mathrm{CoAlg}(\mathsf{H}) \to \mathcal{V}\text{-}\mathsf{CatCH}$ *is comonadic.*

Corollary 4.37. *The category of coalgebras of the Hausdorff functor* $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ *is complete. Moreover, the functor* $\mathrm{CoAlg}\,\mathsf{H} \to \mathcal{V}\text{-}\mathsf{CatCH}$ *preserves codirected limits.*

Proof. Being a topological category over $\mathsf{CompHaus}$, the category $\mathcal{V}\text{-}\mathsf{CatCH}$ is (surjective, initial mono)-structured. Therefore, the category $\mathcal{V}\text{-}\mathsf{CatCH}$ satisfies all conditions necessary to apply Theorem 4.31. Furthermore, the previous results show that H also satisfies the necessary requirements to apply Theorem 4.31. ☐

In the sequel we describe the terminal coalgebra of the Hausdorff functor on $\mathcal{V}\text{-}\mathsf{CatCH}$; which is the limit of the codirected diagram

$$1 \longleftarrow \mathsf{H}1 \longleftarrow \mathsf{H}\mathsf{H}1 \longleftarrow \dots, \tag{4.i}$$

where the morphisms are obtained by applying successively H to the unique morphism $f_!\colon \mathsf{H}1 \to 1$.

First, we analyse the case of $\mathcal{V} = 2$. To do so, let (X, τ^d) denote the discrete space with underlying set X, and observe that for every positive integer n,

$$\mathsf{H}(n, \geq, \tau^d) = (n+1, \geq, \tau^d) \qquad \text{and} \qquad \mathsf{H}^n f_!(k) = \min(k, n).$$

Lemma 4.38. *Consider the one-point compactification* $(\mathbb{N} + \infty, \tau^*)$ *of the space* $(\mathbb{N}, \tau^d)$*. The cone*

$$(\min(-, n)\colon (\mathbb{N} + \infty, \geq, \tau^*) \longrightarrow (n+1, \geq, \tau^d))_{n \in \mathbb{N}} \tag{4.ii}$$

is a limit in OrdCH *of the diagram* (4.i)*.*

Proof. The assertion follows immediately from the "Bourbaki" criterion described in [28, Theorem 3.29]: firstly, for every $n \in \mathbb{N}$, the map $\min(-, n)\colon (\mathbb{N} + \infty, \geq, \tau^*) \to (n+1, \geq, \tau^d)$ is surjective, monotone and continuous; secondly, the cone (4.ii) is point-separating and initial with respect to the canonical forgetful functor $\mathsf{OrdCH} \to \mathsf{CompHaus}$. ☐

Theorem 4.39. *The map* $f\colon (\mathbb{N} + \infty, \geq, \tau^*) \to \mathsf{H}(\mathbb{N} + \infty, \geq, \tau^*)$ *defined by*

$$f(n) = \begin{cases} \varnothing, & n = 0 \\ \mathbb{N} + \infty, & n = \infty \\ \Uparrow(n-1), & \textit{otherwise}, \end{cases}$$

is a terminal coalgebra for $\mathsf{H}\colon \mathsf{OrdCH} \to \mathsf{OrdCH}$*.*

Proof. Since $\mathsf{H}\colon \mathsf{OrdCH} \to \mathsf{OrdCH}$ preserves codirected limits we can compute its terminal coalgebra from the limit of the diagram of Lemma 4.38. Therefore, the assertion holds by routine calculation. $\square$

Remark 4.40. The set $\mathbb{N}$ is an upset in $(\mathbb{N}+\infty, \geq, \tau^*)$ but it is not compact.

As a consequence of the theorem above we can describe the terminal coalgebra of the lower Vietoris functor on Top.

Corollary 4.41. *Consider the lower Vietoris functor $\mathsf{V}\colon \mathsf{Top} \to \mathsf{Top}$ and the space $(\mathbb{N}+\infty, \tau)$ whose topology is generated by the sets $[n, \infty]$, for $n \in \mathbb{N}$. The map $f\colon (\mathbb{N}+\infty, \tau) \to \mathsf{V}(\mathbb{N}+\infty, \tau)$ defined by*

$$f(n) = \begin{cases} \varnothing, & n = 0 \\ \mathbb{N}+\infty, & n = \infty \\ \Uparrow(n-1), & \textit{otherwise}. \end{cases}$$

is a terminal coalgebra for $\mathsf{V}\colon \mathsf{Top} \to \mathsf{Top}$.

Proof. The lower Vietoris functor $\mathsf{V}\colon \mathsf{Top} \to \mathsf{Top}$ restricts to the category $\mathsf{StablyComp}$ of stably compact spaces and spectral maps (see [50]) which is isomorphic to the category $\mathsf{OrdCH}_{\mathrm{sep}}$ (see [24]). As observed in [28, Theorem 3.36], the terminal coalgebra of the lower Vietoris functor on Top can be obtained from the terminal coalgebra of the lower Vietoris on $\mathsf{StablyComp}$. Since $\mathsf{H}\colon \mathsf{OrdCH}_{\mathrm{sep}} \to \mathsf{OrdCH}_{\mathrm{sep}}$ preserves codirected limits (see [28, Corollary 3.33] or Theorem 4.35 and Proposition 4.5) and the limit and diagram of Lemma 4.38 actually live in $\mathsf{OrdCH}_{\mathrm{sep}}$, the claim follows by applying the functor

$$\mathsf{OrdCH}_{\mathrm{sep}} \xrightarrow{\ \simeq\ } \mathsf{StablyComp} \longrightarrow \mathsf{Top}$$

to the map of Theorem 4.39. $\square$

In the following we will see that the terminal coalgebra of $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ "coincides" with the terminal coalgebra of $\mathsf{H}\colon \mathsf{OrdCH} \to \mathsf{OrdCH}$.

Proposition 4.42. *Consider the lattice homomorphism $i\colon 2 \to \mathcal{V}$; that is $i(0) = \bot$ and $i(1) = \top$. The map i induces a limit-preserving functor $\mathsf{I}\colon \mathsf{OrdCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ that keeps morphisms unchanged and sends an ordered compact Hausdorff space (X, a, τ) to $(X, i \cdot a, \tau)$.*

Proof. Let (X, a, τ) be an ordered compact Hausdorff space. First, observe that i is a lax homomorphism of quantales, hence $(X, i \cdot a)$ is a $\mathcal{V}$-category; furthermore, it is clear that i is a continuous function from $(2, \xi_{\leq}) \to (\mathcal{V}, \xi_{\leq})$, hence by Proposition 4.3, $(X, i \cdot a, \tau)$ defines an object of $\mathcal{V}\text{-}\mathsf{CatCH}$. Now, a limit in $\mathcal{V}\text{-}\mathsf{CatCH}$ is a limit in $\mathsf{CompHaus}$ equipped with the initial structure with respect to the functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$. Therefore, since i preserves infima, it follows that $\mathsf{I}\colon \mathsf{OrdCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ preserves limits. $\square$

Corollary 4.43. *The map* $f\colon (\mathbb{N}+\infty, i\cdot \geq, \tau^*) \to \mathsf{H}(\mathbb{N}+\infty, i\cdot \geq, \tau^*)$ *defined by*

$$f(n) = \begin{cases} \varnothing, & n = 0 \\ \mathbb{N}+\infty, & n = \infty \\ \uparrow\!(n-1), & \textit{otherwise}, \end{cases}$$

is a terminal coalgebra for $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$.

Proof. Let H' denote the Hausdorff functor on OrdCH. Since $\mathsf{I}\colon \mathsf{OrdCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ preserves limits then $\mathsf{I}(1)$ is the terminal object in $\mathcal{V}\text{-}\mathsf{CatCH}$. Moreover, the lattice homomorphism $i\colon 2 \to \mathcal{V}$ preserves infima and suprema, thus we obtain $\mathsf{I}\cdot\mathsf{H}' = \mathsf{H}\cdot\mathsf{I}$. Consequently,

$$\mathsf{I}(1 \longleftarrow \mathsf{H}'1 \longleftarrow \mathsf{H}'\mathsf{H}'1 \longleftarrow \dots) = 1 \longleftarrow \mathsf{H}1 \longleftarrow \mathsf{H}\mathsf{H}1 \longleftarrow \dots.$$

Therefore, the claim follows from Theorem 4.39 and Proposition 4.42. ☐

The corollary above affirms implicitly that, in general, the terminal coalgebra of the Hausdorff functor on $\mathcal{V}$-CatCH is rather simple. After all, independently of the quantale $\mathcal{V}$, we end up with a terminal coalgebra whose carrier is an ordered set. Hausdorff polynomial functors seem far more interesting in this regard.

Definition 4.44. Let X be a subcategory of $\mathcal{V}$-CatCH closed under finite limits and finite colimits such that the Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ restricts to X. We call a functor ***Hausdorff polynomial*** on X if it belongs to the smallest class of endofunctors on X that contains the identity functor, all constant functors and is closed under composition with H, finite products and finite sums of functors.

Proposition 4.45. *Every Hausdorff polynomial functor on* $\mathcal{V}$-CatCH *preserves initial monomorphisms with respect to the functor* $\mathcal{V}\text{-}\mathsf{CatCH} \to \mathsf{CompHaus}$.

Proof. Immediate consequence of Corollary 4.34 since the remaining cases trivially preserve initial monomorphisms. ☐

Proposition 4.46. *Every Hausdorff polynomial functor on* $\mathcal{V}$-CatCH *preserves codirected limits.*

Proof. We already know from Theorem 4.35 that $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{CatCH} \to \mathcal{V}\text{-}\mathsf{CatCH}$ preserves codirected limits. Moreover, a routine calculation reveals that the sum of functors that preserve codirected initial cones with respect to the forgetful functor $\mathcal{V}\text{-}\mathsf{CatCH} \to \mathsf{CompHaus}$ also does so. Consequently, the sum preserves codirected limits by Theorem 4.32 since the sum on $\mathsf{CompHaus}$ preserves codirected limits (for instance, see [28]). The remaining cases are trivial. ☐

In light of the previous results, now we can apply Theorem 4.31 to obtain:

Theorem 4.47. *The category of coalgebras of a Hausdorff polynomial functor on* $\mathcal{V}$-CatCH *is (co)complete.*

Note that for Hausdorff polynomial functors, in general, we cannot apply the same reasoning that led us to conclude that the terminal coalgebra of the Hausdorff functor on $\mathcal{V}$-CatCH "coincides" with the terminal coalgebra of the Hausdorff functor on OrdCH. For example, if A is a $\mathcal{V}$-categorical compact Hausdorff space that does not come from an ordered set, then applying the Hausdorff polynomial functor $\mathsf{H} \cdot (A \times \mathsf{Id})$ to the terminal object of $\mathcal{V}$-CatCH does not necessarily yields a $\mathcal{V}$-category structure that comes from an ordered set.

Now, by taking advantage of the results of Appendix A, we can deduce similar results for Hausdorff polynomial functors on $\mathcal{V}$-$\mathsf{CatCH}_{\mathrm{sep}}$. However, to avoid repetion, we conclude this paper by generalising the more interesting case of Hausdorff polynomial functors on Priest discussed in [28].

Assumption 4.48. Until the end of the section we assume that $\mathcal{V}$ is a commutative and unital quantale such that for every $u \in \mathcal{V}$ the map $\hom(u, -) \colon (\mathcal{V}, \xi) \to (\mathcal{V}, \xi)$ is continuous.

Definition 4.49. We call a $\mathcal{V}$-categorical compact Hausdorff space X ***Priestley*** if the cone $\mathcal{V}$-$\mathsf{CatCH}(X, \mathcal{V}^{\mathrm{op}})$ is initial and point-separating. We denote the full subcategory of $\mathcal{V}$-CatCH defined by all Priestley spaces by $\mathcal{V}$-Priest.

Example 4.50. For $\mathcal{V} = 2$, our notion of Priestley space coincides with the usual nomenclature for ordered compact Hausdorff spaces (see [46, 47]).

Proposition 4.51. *The category* $\mathcal{V}$-Priest *is closed under finite coproducts in* $\mathcal{V}$-CatCH.

Proof. Let A and B be Priestley spaces. Note that for every morphism $f \colon A \to \mathcal{V}^{\mathrm{op}}$ and $g \colon B \to \mathcal{V}^{\mathrm{op}}$ in $\mathcal{V}$-CatCH, the maps $f + \bot$ and $\bot + g$, where $\bot$ represents the constant function $\bot$, are morphisms of type $A + B \to \mathcal{V}^{\mathrm{op}}$ in $\mathcal{V}$-CatCH. Since A and B are Priestley spaces, it follows that the cone of all these morphisms is initial and point-separating with respect to the functor $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$. □

Remark 4.52. The inclusion functor $\mathcal{V}$-$\mathsf{Priest} \hookrightarrow \mathcal{V}$-$\mathsf{CatCH}$ is right adjoint (see [4, Theorem 16.8]); in particular $\mathcal{V}$-Priest is complete and cocomplete and $\mathcal{V}$-$\mathsf{Priest} \hookrightarrow \mathcal{V}$-$\mathsf{CatCH}$ preserves and reflects limits. Moreover, a mono-cone $(f_i \colon X \to X_i)_{i \in I}$ in $\mathcal{V}$-Priest is initial with respect to $\mathcal{V}$-$\mathsf{Priest} \to \mathsf{CompHaus}$ if and only if it is initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$.

Proposition 4.53. *The* $\mathcal{V}$*-categorical compact Hausdorff space* $\mathcal{V}^{\mathrm{op}}$ *is an algebra for* H *with algebra structure* $\inf \colon \mathsf{H}\mathcal{V}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ *that sends an element* $A \in \mathsf{H}\mathcal{V}^{\mathrm{op}}$ *to* $\bigvee A$ *(taken in* $\mathcal{V}$*).*

Proof. Clearly, $\inf \colon \mathsf{H}\mathcal{V}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is a $\mathcal{V}$-functor; moreover, by [25, Proposition IV-3.9], inf is also continuous. □

We recall that, given a morphism $\psi\colon X \to \mathcal{V}^{\mathrm{op}}$ of $\mathcal{V}$-CatCH, we denote by $\psi^{\Diamond}$ the composite

$$\mathsf{H}X \xrightarrow{\mathsf{H}\psi} \mathsf{H}(\mathcal{V}^{\mathrm{op}}) \xrightarrow{\inf} \mathcal{V}^{\mathrm{op}}$$

in $\mathcal{V}$-CatCH. With respect to the algebra structure of Proposition 4.53 above, we have the following result. For the pertinent notions from enriched category theory, we refer to [54].

Proposition 4.54. *Let X be a $\mathcal{V}$-categorical compact Hausdorff space. Consider a $\mathcal{V}$-subcategory $\mathcal{R} \subseteq \mathcal{V}^X$ that is closed under finite weighted limits and such that $(\psi\colon X \to \mathcal{V}^{\mathrm{op}})_{\psi\in\mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$. Then the cone $(\psi^{\Diamond}\colon \mathsf{H}X \to \mathcal{V}^{\mathrm{op}})_{\psi\in\mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\mathsf{CatCH} \to \mathsf{CompHaus}$.*

Proof. We denote by $\gg$ the totally above relation of $\mathcal{V}$. Recall from Remark 4.2 that, for every $u \in \mathcal{V}$, the set

$$\Downarrow u = \{w \in \mathcal{V} \mid u \gg w\}$$

is open with respect to $\xi_{\leq}$. Let $A, B \in \mathsf{H}X$ and

$$u \gg \mathsf{H}a(A,B) = \bigwedge_{y\in B}\bigvee_{x\in A} a(x,y).$$

Hence, there is some $y \in B$ so that, for all $x \in A$,

$$u \gg a(x,y) = \bigwedge_{\psi\in\mathcal{R}} \hom(\psi(y), \psi(x)).$$

Let $x \in A$. There is some $\psi \in \mathcal{R}$ with $u \gg \hom(\psi(y), \psi(x))$. With $v = \psi(y)$ we put $\widehat{\psi} = \hom(v, \psi(-))$. Then $\widehat{\psi} \in \mathcal{R}$ since $\mathcal{R}$ is closed under cotensors and

$$u \gg \widehat{\psi}(x) \qquad \text{and} \qquad k \leq \widehat{\psi}(y).$$

Therefore

$$A \subseteq \bigcup\{\psi^{-1}(\Downarrow u) \mid \psi \in \mathcal{R}, k \leq \psi(y)\};$$

by compactness, there exist finitely many $\psi_1, \dots, \psi_n \in \mathcal{R}$ so that $k \leq \psi_i(y)$ and

$$A \subseteq \psi_1^{-1}(\Downarrow u) \cup \dots \cup \psi_n^{-1}(\Downarrow u).$$

Put $\widehat{\psi} = \psi_1 \wedge \dots \wedge \psi_n$. By hypothesis, $\widehat{\psi} \in \mathcal{R}$. Then $k \leq \widehat{\psi}(y)$ and $u \gg \widehat{\psi}(x)$, for all $x \in A$. Therefore

$$\hom(\widehat{\psi}^{\Diamond}(B), \widehat{\psi}^{\Diamond}(A)) \leq \hom(k,u) = u. \qquad \square$$

Proposition 4.55. *Let $(f\colon X \to X_i)_{i\in I}$ be a codirected cone in $\mathcal{V}$-CatCH. Then*

$$\{\varphi f_i \mid i \in I,\ \varphi\colon X_i \to \mathcal{V}^{\mathrm{op}} \in \mathcal{V}\text{-}\mathsf{CatCH}\} \subseteq \mathcal{V}^X$$

is closed under finite weighted limits.

By Proposition 4.54, the Hausdorff functor restricts to a functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathsf{Priest} \to \mathcal{V}\text{-}\mathsf{Priest}$, hence the Hausdorff monad $\mathbb{H}$ restricts to $\mathcal{V}$-Priest.

Theorem 4.56. *Every Hausdorff polynomial functor on* $\mathcal{V}$-Priest *preserves codirected limits.*

Proof. Every Hausdorff polynomial functor on $\mathcal{V}$-Priest corresponds to the restriction to $\mathcal{V}$-Priest of a Hausdorff polynomial functor on $\mathcal{V}$-CatCH and the inclusion functor $\mathcal{V}\text{-}\mathsf{Priest} \to \mathcal{V}\text{-}\mathsf{CatCH}$ preserves and reflects limits (see Proposition 4.51 and Remark 4.52). □

Corollary 4.57. *For every Hausdorff polynomial functor* F *on* $\mathcal{V}$-Priest*, the forgetful functor* $\mathrm{CoAlg}(\mathsf{F}) \to \mathcal{V}\text{-}\mathsf{Priest}$ *is comonadic.*

Theorem 4.58. *The category of coalgebras of a Hausdorff polynomial functor* F *on* $\mathcal{V}$-Priest *is complete. Moreover, the functor* $\mathrm{CoAlg}(\mathsf{F}) \to \mathcal{V}\text{-}\mathsf{Priest}$ *preserves codirected limits.*

Proof. The category $\mathcal{V}$-Priest inherits the (surjective, initial mono-cone)-factorisation structure from $\mathcal{V}$-CatCH. Therefore, the previous discussion shows that we can apply Theorem 4.31. □

As a consequence of Theorem 4.56 and Remark 4.52, we can describe a terminal coalgebra of the Hausdorff functor on $\mathcal{V}$-Priest.

Corollary 4.59. *The map of Corollary 4.43 is a terminal coalgebra for the Hausdorff functor on* $\mathcal{V}$-Priest.

Proof. Note that the sequence $1 \longleftarrow \mathsf{H}1 \longleftarrow \mathsf{HH}1 \longleftarrow \dots$ in $\mathcal{V}$-CatCH actually lives in $\mathcal{V}$-Priest. □

A. Appendix

In this section we collect some facts about $\mathcal{V}$-categories and $\mathcal{V}$-functors, where $\mathcal{V}$ is a quantale; for more information we refer to [40, 54]. Furthermore, we present some useful properties of the reflector into the category of separated $\mathcal{V}$-categories that follow from standard arguments, but seem to be absent from the literature.

Definition A.1. Let $\mathcal{V}$ be a commutative and unital quantale. A ***$\mathcal{V}$-category*** is a pair (X, a) consisting of a set X and a map $a\colon X \times X \to \mathcal{V}$ satisfying

$$k \leq a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \leq a(x,z),$$

for all $x, y, z \in X$. Given $\mathcal{V}$-categories (X, a) and (Y, b), a ***$\mathcal{V}$-functor*** $f\colon (X,a) \to (Y,b)$ is a map $f\colon X \to Y$ such that

$$a(x,y) \leq b(f(x), f(y)),$$

for all $x, y \in X$.

In particular, the quantale $\mathcal{V}$ becomes a $\mathcal{V}$-category with structure $\hom\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$. We refer to [54] for a list of examples of quantales $\mathcal{V}$ and the corresponding categories $\mathcal{V}$-Cat of $\mathcal{V}$-categories and $\mathcal{V}$-functors.

For every $\mathcal{V}$-category (X,a), $a^\circ(x,y)=a(y,x)$ defines another $\mathcal{V}$-category structure on X, and the $\mathcal{V}$-category $(X,a)^{\mathrm{op}} := (X,a^\circ)$ is called the ***dual*** of (X,a). A $\mathcal{V}$-category (X,a) is called ***symmetric*** whenever $(X,a)=(X,a)^{\mathrm{op}}$.

Clearly, $\mathcal{V}$-categories and $\mathcal{V}$-functors define a category, denoted as $\mathcal{V}$-Cat. The full subcategory of $\mathcal{V}$-Cat defined by all symmetric $\mathcal{V}$-categories is denoted as $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sym}}$.

Remark A.2. Given $\mathcal{V}$-categories (X,a) and (Y,b), we define the tensor product of (X,a) and (Y,b) to be the $\mathcal{V}$-category $(X,a)\otimes(Y,b)=(X\times Y, a\otimes b)$, with

$$a\otimes b((x,y),(x',y'))=a(x,x')\otimes b(y,y').$$

This operation makes $\mathcal{V}$-Cat a symmetric monoidal closed category, where the internal hom of (X,a) and (Y,b) is the $\mathcal{V}$-category $[(X,a),(Y,b)]=(\mathcal{V}\text{-}\mathsf{Cat}((X,a),(X,b)),[-,-])$, with

$$[f,g]=\bigwedge_{x\in X} b(f(x),g(x)).$$

We note that $[(X,a),(Y,b)]$ is a $\mathcal{V}$-subcategory of the X-fold product $(Y,b)^X$ of (Y,b).

The following propositions are particularly useful to construct $\mathcal{V}$-functors when combined with the fact that $\mathcal{V}$-Cat is symmetrical monoidal closed.

Proposition A.3. *For every set I, the assignments $f\mapsto\bigvee_{i\in I} f(i)$ and $f\mapsto\bigwedge_{i\in I} f(i)$ define $\mathcal{V}$-functors of type $\mathcal{V}^I\to\mathcal{V}$.*

Proposition A.4. *For every $\mathcal{V}$-category (X,a), the map $a\colon (X,a)^{\mathrm{op}}\otimes(X,a)\to(\mathcal{V},\hom)$ is a $\mathcal{V}$-functor.*

The category $\mathcal{V}$-Cat is well behaved regarding (co)limits.

Theorem A.5. *The canonical forgetful functor $\mathcal{V}$-$\mathsf{Cat}\to\mathsf{Set}$ is topological. For a structured cone $(f_i\colon X\to(X_i,a_i))$, the initial lift (X,a) is given by*

$$a(x,y)=\bigwedge_{i\in I} a_i(f_i(x),f_i(y)),$$

for all $x,y\in X$. Moreover, $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sym}}$ is closed in $\mathcal{V}$-Cat under initial cones; therefore the canonical forgetful functor $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sym}}\to\mathsf{Set}$ is topological as well, and the inclusion functor $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sym}}\hookrightarrow\mathcal{V}$-$\mathsf{Cat}$ has a left adjoint.

We also recall that $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sym}} \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ has a concrete right adjoint which sends the $\mathcal{V}$-category (X, a) to its ***symmetrisation*** (X, a_s) given by

$$a_s(x, y) = a(x, y) \wedge a(y, x),$$

for all $x, y \in X$.

Every $\mathcal{V}$-category (X, a) carries a natural order defined by

$$x \leq y \text{ whenever } k \leq a(x, y),$$

which can be extended pointwise to $\mathcal{V}$-functors making $\mathcal{V}\text{-}\mathsf{Cat}$ a *2-category*. The natural order of $\mathcal{V}$-categories defines a faithful functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$. A $\mathcal{V}$-category is called ***separated*** whenever its underlying ordered set is anti-symmetric, and we denote by $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ the full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}$ defined by all separated $\mathcal{V}$-categories. Tautologically, an ordered set is separated if and only if it is anti-symmetric.

Theorem A.6. *$\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ is closed in $\mathcal{V}\text{-}\mathsf{Cat}$ under monocones. Hence, the forgetful functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \to \mathsf{Set}$ is mono-topological and the inclusion functor $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ has a left adjoint.*

Let us describe the left adjoint $S \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat}$. To do so, consider a $\mathcal{V}$-category (X, a). Then

$$x \sim y \qquad \text{whenever} \qquad x \leq y \quad \text{and} \quad y \leq x$$

defines an equivalence relation on X, and the quotient set $X/{\sim}$ becomes a $\mathcal{V}$-category $(X/{\sim}, \widetilde{a})$ by putting

$$\widetilde{a}([x], [y]) = a(x, y); \tag{A.i}$$

this is indeed independent of the choice of representants of the equivalence classes. Then the projection map

$$q_{(X,a)} \colon X \longrightarrow X/{\sim},\; x \longmapsto [x]$$

is a $\mathcal{V}$-functor $q_{(X,a)} \colon (X, a) \to (X/{\sim}, \widetilde{a})$, it is indeed the unit of this adjunction at (X, a). Furthermore, by (A.i), $q_{(X,a)} \colon (X, a) \to (X/{\sim}, \widetilde{a})$ is a universal quotient and initial with respect to $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$.

Lemma A.7. *A cone $(f_i \colon (X, a) \to (X_i, a_i))_{i \in I}$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ is initial with respect to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \to \mathsf{Set}$ if and only if*

$$a(x, y) = \bigwedge_{i \in I} a_i(f_i(x), f_i(y)), \tag{A.ii}$$

for all $x, y \in X$.

Proof. Clearly, if (A.ii) is satisfied then $(f_i\colon (X,a) \to (X_i,a_i))_{i\in I}$ is initial with respect to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \to \mathsf{Set}$ since it is initial with respect to $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$. Suppose now that $(f_i\colon (X,a) \to (X_i,a_i))_{i\in I}$ is initial with respect to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}} \to \mathsf{Set}$. Fix $x, y \in X$. Then

$$a(x,y) \le \bigwedge_{i\in I} a_i(f_i(x), f_i(y)) = u$$

because $f_i\colon (X,a) \to (X_i,a_i)$ is a $\mathcal{V}$-functor for every $i \in I$. It is left to show that $u \le a(x,y)$. This is certainly true if $u = \bot$; assume now that $\bot < u$. Let 2_u be the separated $\mathcal{V}$-category with underlying set $\{0,1\}$ and structure a_u defined by

$$a_u(0,1) = u, \quad a_u(0,0) = a_u(1,1) = k, \quad \text{and} \quad a_u(1,0) = \bot.$$

Consider $h\colon \{0,1\} \to X$ with $h(0) = x$ and $h(1) = y$. Then $f_i \cdot h$ is a $\mathcal{V}$-functor, for every $i \in I$. Hence, since $(f_i\colon (X,a) \to (X_i,a_i))_{i\in I}$ is initial, $h\colon 2_u \to X$ is a $\mathcal{V}$-functor, which implies $u \le a(x,y)$. □

Corollary A.8. *The functor* $\mathsf{S}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *preserves initial cones with respect to the canonical forgetful functors.*

Proof. Let $(f_i\colon (X,a) \to (X_i,a_i))_{i\in I}$ be an initial cone with respect to $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$. Then, for every $[x], [y] \in \mathsf{S}(X,a) = (X/{\sim}, \widetilde{a})$, and with $\mathsf{S}(X_i,a_i) = (X/{\sim}, \widetilde{a}_i)$ for all $i \in I$,

$$\widetilde{a}([x],[y]) = a(x,y) = \bigwedge_{i\in I} a_i(f_i(x), f_i(y)) = \bigwedge_{i\in I} \widetilde{a}_i([f_i(x)],[f_i(y)]) = \bigwedge_{i\in I} \widetilde{a}_i(\mathsf{S}f_i([x]), \mathsf{S}f_i([y])).$$

Therefore, the claim follows by Lemma A.7. □

Remark A.9. In [18] it is shown that $\mathsf{S}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ preserves finite products. However, S does not preserve limits in general, in particular, S does not preserve codirected limits. For instance, consider the "empty limit" of [57] and equip every X_i ($i \in I$) with the indiscrete $\mathcal{V}$-category structure a_i where $a_i(x,y) = \top$ for all $x, y \in X_i$. Then $S(X_i,a_i)$ has exactly one element, for each $i \in I$; hence the limit of the corresponding diagram in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ has one element.

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References

[1] Abramsky, S.: A Cook's Tour of the Finitary Non-Well-Founded Sets. In: S. Artemov, H. Barringer, A.A. Garcez (eds.) We Will Show Them! Essays in Honour of Dov Gabbay, vol. 1, pp. 1–18. College Publications, London (2005) 16

[2] Abramsky, S., Jung, A.: Domain Theory. In: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (eds.) Handbook of Logic in Computer Science: Semantic Structures, vol. 3, pp. 1–168. Oxford University Press (1995) 17

[3] Adámek, J.: Introduction to coalgebra. Theory and Applications of Categories **14**(8), 157–199 (2005) 27

[4] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and concrete categories: The joy of cats. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York (1990). URL http://tac.mta.ca/tac/reprints/articles/17/tr17abs.html. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507 5, 27, 31

[5] Akhvlediani, A., Clementino, M.M., Tholen, W.: On the categorical meaning of Hausdorff and Gromov distances, I. Topology and its Applications **157**(8), 1275–1295 (2010). DOI 10.1016/j.topol.2009.06.018 3, 9

[6] Băbuş, O., Kurz, A.: On the logic of generalised metric spaces. In: I. Hasuo (ed.) Coalgebraic Methods in Computer Science, pp. 136–155. Springer (2016). DOI 10.1007/978-3-319-40370-0_9 2

[7] Balan, A., Kurz, A., Velebil, J.: Extending set functors to generalised metric spaces. Logical Methods in Computer Science **15**(1) (2019) 2, 3, 7

[8] Baldan, P., Bonchi, F., Kerstan, H., König, B.: Coalgebraic Behavioral Metrics. Logical Methods in Computer Science **14**(3), 1860–5974 (2018). DOI 10.23638/lmcs-14(3:20)2018 2, 3, 7, 8, 10, 16

[9] Barr, M.: Terminal coalgebras in well-founded set theory. Theoretical Computer Science **114**(2), 299–315 (1993). DOI 10.1016/0304-3975(93)90076-6 3

[10] Barr, M., Wells, C.: Toposes, triples and theories. Springer-Verlag New York (1985). DOI 10.1007/978-1-4899-0021-0. URL http://www.tac.mta.ca/tac/reprints/articles/12/tr12abs.html. Republished in: Reprints in Theory and Applications of Categories, No. 12, 2005, pp. 1 – 288. 27

[11] Birsan, T., Tiba, D.: One hundred years since the introduction of the set distance by Dimitrie Pompeiu. In: F. Pandolfi, A. Ceragioli, H. Dontchev, K. Furuta, L. Marti (eds.) System Modeling and Optimization, *IFIP Advances in Information and Communication Technology*, vol. 199, pp. 35–39. Springer (2006). DOI 10.1007/0-387-33006-2_4. Proceedings of the 22nd IFIP TC7 Conference, July 18-22, 2005, Turin, Italy 3, 10

[12] Bonchi, F., König, B., Petrişan, D.: Up-To Techniques for Behavioural Metrics via Fibrations. In: Schewe and Zhang [51], pp. 17:1–17:17. DOI 10.4230/LIPIcs.CONCUR.2018.17. URL http://drops.dagstuhl.de/opus/volltexte/2018/9555/. 29th International Conference on Concurrency Theory, September 4-7, 2018 - Beijing, China 2

[13] Bonsangue, M., Rutten, J., Silva, A.: An algebra for Kripke polynomial coalgebras. In: 24th Annual IEEE Symposium on Logic in Computer Science, 11-14 August 2009, Los Angeles, CA, USA, pp. 49–58. IEEE (2009). DOI 10.1109/LICS.2009.18 2, 14

[14] Bourbaki, N.: General topology, part I. Hermann, Paris and Addison-Wesley (1966). Chapters 1–4 19

[15] van Breugel, F., Hermida, C., Makkai, M., Worrell, J.: An accessible approach to behavioural pseudometrics. In: L. Caires, G.F. Italiano, L. Monteiro, C. Palamidessi, M. Yung (eds.) Automata, Languages and Programming, pp. 1018–1030. Springer Berlin Heidelberg (2005). DOI 10.1007/11523468_82. Proceedings of the 32nd International Colloquium, ICALP 2005, Lisbon, Portugal, July 11-15, 2005 2, 16

[16] Cantor, G.: Über eine elementare Frage der Mannigfaltigkeitslehre. Jahresbericht der Deutschen Mathematiker-Vereinigung **1**, 75–78 (1891). URL http://mickindex.sakura.ne.jp/cantor/cnt_uFM_gm.html 2

[17] Clementino, M.M., Hofmann, D.: Triquotient maps via ultrafilter convergence. Proceedings of the American Mathematical Society **130**(11), 3423–3431 (2002). DOI 10.1090/S0002-9939-02-06472-9 3

[18] Clementino, M.M., Hofmann, D., Ribeiro, W.: Cartesian closed exact completions in topology. Journal of Pure and Applied Algebra **224**(2), 610–629 (2020). DOI 10.1016/j.jpaa.2019.06.003 36

[19] Dilworth, R.P., Gleason, A.M.: A generalized Cantor theorem. Proceedings of the American Mathematical Society **13**(5), 704–705 (1962). DOI 10.1090/S0002-9939-1962-0144824-3 3, 15

[20] Engelking, R.: General topology, *Sigma Series in Pure Mathematics*, vol. 6, 2 edn. Heldermann Verlag, Berlin (1989). Translated from the Polish by the author 3

[21] Fawcett, B., Wood, R.J.: Constructive complete distributivity. I. Mathematical Proceedings of the Cambridge Philosophical Society **107**(1), 81–89 (1990). DOI 10.1017/S0305004100068377 17

[22] Flagg, R.C.: Completeness in continuity spaces. In: R.A.G. Seely (ed.) Category Theory 1991: Proceedings of an International Summer Category Theory Meeting, held June 23-30, 1991, *CMS Conference Proceedings*, vol. 13. American Mathematical Society (1992) 21, 24

[23] Flagg, R.C.: Quantales and continuity spaces. Algebra Universalis **37**(3), 257–276 (1997). DOI 10.1007/s000120050018 20

[24] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S.: A compendium of continuous lattices. Springer-Verlag, Berlin (1980). DOI 10.1007/978-3-642-67678-9 29

[25] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S.: Continuous lattices and domains, *Encyclopedia of Mathematics and its Applications*, vol. 93. Cambridge University Press, Cambridge (2003). DOI 10.1017/CBO9780511542725 17, 18, 31

[26] Hausdorff, F.: Grundzüge der Mengenlehre. Veit & Comp, Leipzig (1914) 3, 9, 10

[27] Hofmann, D.: Topological theories and closed objects. Advances in Mathematics **215**(2), 789–824 (2007). DOI 10.1016/j.aim.2007.04.013 11

[28] Hofmann, D., Neves, R., Nora, P.: Limits in categories of Vietoris coalgebras. Mathematical Structures in Computer Science **29**(4), 552–587 (2019). DOI 10.1017/S0960129518000269 3, 5, 16, 28, 29, 30, 31

[29] Hofmann, D., Nora, P.: Enriched Stone-type dualities. Advances in Mathematics **330**, 307–360 (2018). DOI 10.1016/j.aim.2018.03.010 4

[30] Hofmann, D., Reis, C.D.: Probabilistic metric spaces as enriched categories. Fuzzy Sets and Systems **210**, 1–21 (2013). DOI 10.1016/j.fss.2012.05.005 21, 24

[31] Hofmann, D., Reis, C.D.: Convergence and quantale-enriched categories. Categories and General Algebraic Structures with Applications **9**(1), 77–138 (2018). URL `http://cgasa.sbu.ac.ir/article_58262.html` 4, 17, 18, 20, 25

[32] Hofmann, D., Seal, G.J., Tholen, W. (eds.): Monoidal Topology. A Categorical Approach to Order, Metric, and Topology, *Encyclopedia of Mathematics and its Applications*, vol. 153. Cambridge University Press, Cambridge (2014). DOI 10.1017/cbo9781107517288. URL `http://www.cambridge.org/pt/academic/subjects/mathematics/logic-categories-and-sets/monoidal-topology-categorical-approach-order-metric-and-topology`. Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen 11

[33] Hofmann, D., Tholen, W.: Lawvere completion and separation via closure. Applied Categorical Structures **18**(3), 259–287 (2010). DOI 10.1007/s10485-008-9169-9 24

[34] Janelidze, G., Sobral, M.: Finite preorders and topological descent. I. Journal of Pure and Applied Algebra **175**(1-3), 187–205 (2002). DOI 10.1016/S0022-4049(02)00134-2. Special volume celebrating the 70$^{\text{th}}$ birthday of Professor Max Kelly 3

[35] Janelidze, G., Sobral, M.: Finite preorders and topological descent. II. étale descent. Journal of Pure and Applied Algebra **174**(3), 303–309 (2002). DOI 10.1016/S0022-4049(02)00046-4 3

[36] Johnstone, P.T.: Stone spaces, *Cambridge Studies in Advanced Mathematics*, vol. 3. Cambridge University Press, Cambridge (1986). Reprint of the 1982 edition 4

[37] König, B., Mika-Michalski, C.: (Metric) Bisimulation Games and Real-Valued Modal Logics for Coalgebras. In: Schewe and Zhang [51], pp. 37:1–37:17. DOI 10.4230/LIPIcs.CONCUR.2018.37. URL http://drops.dagstuhl.de/opus/volltexte/2018/9575/. 29$^{\text{th}}$ International Conference on Concurrency Theory, September 4-7, 2018 - Beijing, China 2

[38] Kupke, C., Kurz, A., Venema, Y.: Stone coalgebras. Theoretical Computer Science **327**(1-2), 109–134 (2004). DOI 10.1016/j.tcs.2004.07.023 2, 14

[39] Lambek, J.: A fixpoint theorem for complete categories. Mathematische Zeitschrift **103**(2), 151–161 (1968). DOI 10.1007/BF01110627 2

[40] Lawvere, F.W.: Metric spaces, generalized logic, and closed categories. Rendiconti del Seminario Matemàtico e Fisico di Milano **43**(1), 135–166 (1973). DOI 10.1007/bf02924844. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37 33

[41] Linton, F.E.J.: Coequalizers in categories of algebras. In: B. Eckmann (ed.) Seminar on Triples and Categorical Homology Theory, *Lecture Notes in Mathematics*, vol. 80, pp. 75–90. Springer Berlin Heidelberg, Berlin, Heidelberg (1969). DOI 10.1007/bfb0083082. URL http://www.tac.mta.ca/tac/reprints/articles/18/tr18abs.html 27

[42] Michael, E.: Topologies on spaces of subsets. Transactions of the American Mathematical Society **71**(1), 152–182 (1951). DOI 10.1090/S0002-9947-1951-0042109-4 4

[43] Nachbin, L.: Topology and Order. No. 4 in Van Nostrand Mathematical Studies. D. Van Nostrand, Princeton, N.J.-Toronto, Ont.-London (1965). Translated from the Portuguese by Lulu Bechtolsheim 4, 9, 11, 18, 22

[44] Nora, P.: Kleisli dualities and Vietoris coalgebras. Ph.D. thesis, University of Aveiro (2019) 14

[45] Pompeiu, D.: Sur la continuité des fonctions de variables complexes. Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques. 2ième Série **7**(3), 265–315 (1905). DOI 10.5802/afst.226 3, 10

[46] Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bulletin of the London Mathematical Society **2**(2), 186–190 (1970). DOI 10.1112/blms/2.2.186 31

[47] Priestley, H.A.: Ordered topological spaces and the representation of distributive lattices. Proceedings of the London Mathematical Society. Third Series **24**(3), 507–530 (1972). DOI 10.1112/plms/s3-24.3.507 31

[48] Raney, G.N.: Completely distributive complete lattices. Proceedings of the American Mathematical Society **3**(5), 677–680 (1952). DOI 10.1090/s0002-9939-1952-0052392-3 17

[49] Rutten, J.: Universal coalgebra: a theory of systems. Theoretical Computer Science **249**(1), 3–80 (2000). DOI 10.1016/s0304-3975(00)00056-6 2, 14

[50] Schalk, A.: Algebras for generalized power constructions. Ph.D. thesis, Technische Hochschule Darmstadt (1993). URL http://www.cs.man.ac.uk/~schalk/publ/diss.ps.gz 29

[51] Schewe, S., Zhang, L. (eds.): CONCUR 2018, *LIPICS*, vol. 118. Schloss Dagstuhl - Leibniz-Zentrum für Informatik GmbH, Wadern/Saarbruecken, Germany (2018). 29th International Conference on Concurrency Theory, September 4-7, 2018 - Beijing, China 37, 40

[52] Seal, G.J.: Canonical and op-canonical lax algebras. Theory and Applications of Categories **14**(10), 221–243 (2005). URL http://www.tac.mta.ca/tac/volumes/14/10/14-10abs.html 10

[53] Stubbe, I.: "Hausdorff distance" via conical cocompletion. Cahiers de Topologie et Géométrie Différentielle Catégoriques **51**(1), 51–76 (2010) 3, 13

[54] Stubbe, I.: An introduction to quantaloid-enriched categories. Fuzzy Sets and Systems **256**, 95–116 (2014). DOI 10.1016/j.fss.2013.08.009. Special Issue on Enriched Category Theory and Related Topics (Selected papers from the 33rd Linz Seminar on Fuzzy Set Theory, 2012) 32, 33, 34

[55] Tholen, W.: Ordered topological structures. Topology and its Applications **156**(12), 2148–2157 (2009). DOI 10.1016/j.topol.2009.03.038 4, 10, 11, 17, 18

[56] Turi, D., Rutten, J.: On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. Mathematical Structures in Computer Science **8**(5), 481–540 (1998). DOI 10.1017/S0960129598002588 2

[57] Waterhouse, W.C.: An empty inverse limit. Proceedings of the American Mathematical Society **36**(2), 618 (1972). DOI 10.1090/s0002-9939-1972-0309047-x 36

[58] Wild, P., Schröder, L., Pattinson, D., König, B.: A van Benthem Theorem for Fuzzy Modal Logic. In: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science - LICS '18. ACM Press (2018). DOI 10.1145/3209108.3209180 2