

# CARTESIAN CLOSED EXACT COMPLETIONS IN TOPOLOGY

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**ABSTRACT.** Using generalized enriched categories, in this paper we show that Rosický's proof of cartesian closedness of the exact completion of the category of topological spaces can be extended to a wide range of topological categories over **Set**, like metric spaces, approach spaces, ultrametric spaces, probabilistic metric spaces, and bitopological spaces. In order to do so we prove a sufficient criterion for exponentiability of  $(\mathbb{T}, V)$ -categories and show that, under suitable conditions, every injective  $(\mathbb{T}, V)$ -category is exponentiable in  $(\mathbb{T}, V)$ -**Cat**.

## 1. INTRODUCTION

As Lawvere has shown in his celebrated paper [Law73], when  $V$  is a closed category the category  $V$ -**Cat** of  $V$ -enriched categories and  $V$ -functors is also monoidal closed. This result extends neither to the cartesian structure nor to the more general setting of  $(\mathbb{T}, V)$ -categories. Indeed, cartesian closedness of  $V$  does not guarantee cartesian closedness of  $V$ -**Cat**: take for instance the category of (Lawvere's) metric spaces  $P_+$ -**Cat**, where  $P_+$  is the complete real half-line, ordered with the  $\geq$  relation, and equipped with the monoidal structure given by addition  $+$ ;  $P_+$  is cartesian closed but  $P_+$ -**Cat** is not (see [CH06] for details); and, even when the monoidal structure of  $V$  is the cartesian one, the category  $(\mathbb{T}, V)$ -**Cat** of  $(\mathbb{T}, V)$ -categories and  $(\mathbb{T}, V)$ -functors (see [CT03]) does not need to be cartesian closed, as it is the case of the category **Top** of topological spaces and continuous maps, that is  $(\mathbb{U}, 2)$ -**Cat** for  $\mathbb{U}$  the ultrafilter monad.

Rosický showed in [Ros99] that **Top** is weakly cartesian closed, and, consequently, that its exact completion is cartesian closed. Weak cartesian closedness of **Top** follows from the existence of enough injectives in its full subcategory **Top**<sub>0</sub> of  $T_0$ -spaces and the fact that they are exponentiable, and this feature, together with several good properties of **Top**, gives cartesian closedness of its exact completion. More precisely, Rosický has shown in [Ros99] the following theorem.

**Theorem 1.1.** *Let  $\mathbf{C}$  be a complete, infinitely extensive and well-powered category with (reg epi, mono)-factorizations such that  $f \times 1$  is an epimorphism whenever  $f$  is a regular epimorphism. Then the exact completion of  $\mathbf{C}$  is cartesian closed provided that  $\mathbf{C}$  is weakly cartesian closed.*

In this paper we use the setting of  $(\mathbb{T}, V)$ -categories, for a quantale  $V$  and a **Set**-monad  $\mathbb{T}$  laxly extended to  $V$ -**Rel** to conclude, in a unified way, that several topological categories over **Set** share with **Top** the cartesian closedness of the exact completion. This was recently used by Adámek and Rosický in the study of free completions of categories [AR18]. In fact, the category  $(\mathbb{T}, V)$ -**Cat** is topological over **Set** [CH03, CT03], hence complete and with (reg epi, mono)-factorizations such that  $f \times 1$  is an epimorphism whenever  $f$  is, and it is infinitely extensive [MST06]. To assure weak

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2010 *Mathematics Subject Classification.* 18B30, 18B35, 18D15, 18D20, 54B30, 54E35, 54E70.

*Key words and phrases.* quantale, enriched category, (probabilistic) metric space, exponentiation, (weakly) cartesian closed category, exact completion.

Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2019, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT – UID/MAT/04106/2019, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. W. Ribeiro also acknowledges the FCT PhD Grant PD/BD/128059/2016.

cartesian closedness of  $(\mathbb{T}, V)\text{-Cat}$  we consider two distinct scenarios, either restricting to the case that  $V$  is a frame – so that its monoidal structure is the cartesian one – or considering the case that the lax extension is determined by a  $\mathbb{T}$ -algebraic structure on  $V$ , as introduced in [Hof07] under the name of topological theory. In the latter case the proof generalizes Rosický’s proof for  $\mathbf{Top}_0$ , after observing that, using the Yoneda embedding of [CH09, Hof11], every separated  $(\mathbb{T}, V)$ -category can be embedded in an injective one, and, moreover, these are exponentiable in  $(\mathbb{T}, V)\text{-Cat}$ . For general  $(\mathbb{T}, V)$ -categories one proceeds again as in [Ros99], using the fact that the reflection of  $(\mathbb{T}, V)\text{-Cat}$  into its full subcategory of separated  $(\mathbb{T}, V)$ -categories preserves finite products. As observed by Rosický, the exact completion of  $\mathbf{Top}$  relates to the cartesian closed category of equilogical spaces [BBS04]. Analogously, our approach leads to the study of generalized equilogical spaces, as developed in [Rib18].

The paper is organized as follows. In Section 2 we introduce  $(\mathbb{T}, V)$ -categories and list their properties used throughout the paper. In Section 3 we revisit the exponentiability problem in  $(\mathbb{T}, V)\text{-Cat}$ , establishing a sufficient criterion for exponentiability which generalizes the results obtained in [Hof07, HS15]. In Section 4 we study the properties of injective  $(\mathbb{T}, V)$ -categories which will be used in the forthcoming section to conclude that, under suitable assumptions, injective  $(\mathbb{T}, V)$ -categories are exponentiable (Theorem 5.8). This result will allow us to conclude, in Theorem 6.3, that  $(\mathbb{T}, V)\text{-Cat}$  is weakly cartesian closed, and, finally, thanks to Theorem 1.1, that the exact completion of  $(\mathbb{T}, V)\text{-Cat}$  is cartesian closed. We conclude our paper with a section on examples, which include, among others, metric spaces, approach spaces, probabilistic metric spaces, and bitopological spaces.

## 2. THE CATEGORY OF $(\mathbb{T}, V)$ -CATEGORIES

Throughout  $V$  is a commutative and unital quantale, i.e.  $V$  is a complete lattice with a symmetric and associative tensor product  $\otimes$ , with unit  $k$  and right adjoint  $\text{hom}$ , so that  $u \otimes v \leq w$  if, and only if,  $v \leq \text{hom}(u, w)$ , for all  $u, v, w \in V$ . Further assume that  $V$  is a Heyting algebra, so that  $u \wedge -$  also has a right adjoint, for every  $u \in V$ . We denote by  $V\text{-Rel}$  the 2-category of  $V$ -relations (or  $V$ -matrices), having as objects sets, as 1-cells  $V$ -relations  $r : X \rightharpoonup Y$ , i.e. maps  $r : X \times Y \rightarrow V$ , and 2-cells  $\varphi : r \rightarrow r'$  given by componentwise order  $r(x, y) \leq r'(x, y)$ . Composition of 1-cells is given by relational composition.  $V\text{-Rel}$  has an involution, given by transposition: the transpose of  $r : X \rightharpoonup Y$  is  $r^\circ : Y \rightharpoonup X$  with  $r^\circ(y, x) = r(x, y)$ .

We fix a non-trivial monad  $\mathbb{T} = (T, m, e)$  on  $\mathbf{Set}$  satisfying  $(BC)$ , i.e.  $T$  preserves weak pullbacks and the naturality squares of the natural transformation  $m$  are weak pullbacks (see [CHJ14]). In general we do not assume that  $T$  preserves products. Later we will make use of the comparison map  $\text{can}_{X,Y} : T(X \times Y) \rightarrow TX \times TY$  defined by  $\text{can}_{X,Y}(\mathfrak{w}) = (T\pi_X(\mathfrak{w}), T\pi_Y(\mathfrak{w}))$  for all  $\mathfrak{w} \in T(X \times Y)$ , where  $\pi_X$  and  $\pi_Y$  are the product projections. Moreover, we assume that  $\mathbb{T}$  has an extension to  $V\text{-Rel}$ , which we also denote by  $\mathbb{T}$ , in the following sense:

- there is a lax functor  $T : V\text{-Rel} \rightarrow V\text{-Rel}$  which extends  $T : \mathbf{Set} \rightarrow \mathbf{Set}$ ;
- $T(r^\circ) = (Tr)^\circ$  for all  $V$ -relations  $r$ ;

- the natural transformations  $e : 1_{V\text{-}\mathbf{Rel}} \rightarrow T$  and  $m : T^2 \rightarrow T$  become op-lax; that is, for every  $r : X \rightarrowtail Y$ ,

$$e_Y \cdot r \leq Tr \cdot e_X, \quad m_Y \cdot TTr \leq Tr \cdot m_X.$$

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ r \downarrow & \leq & \downarrow Tr \\ Y & \xrightarrow{e_Y} & TY \end{array} \quad \begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ TTr \downarrow & \leq & \downarrow Tr \\ TTY & \xrightarrow{m_Y} & TY \end{array}$$

We note that our conditions are stronger than those used in [HST14].

A  $(\mathbb{T}, V)$ -category is a pair  $(X, a)$  where  $X$  is a set and  $a : TX \rightarrowtail X$  is a  $V$ -relation such that

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & \searrow \leq & \downarrow a \\ & 1_X & X \end{array} \quad \text{and} \quad \begin{array}{ccc} T^2X & \xrightarrow{m_X} & TX \\ Ta \downarrow & \leq & \downarrow a \\ TX & \xrightarrow{a} & X \end{array}$$

that is, the map  $a : TX \times X \rightarrow V$  satisfies the conditions:

- (R) for each  $x \in X$ ,  $k \leq a(e_X(x), x)$ ;
- (T) for each  $\mathfrak{X} \in T^2X$ ,  $\mathfrak{x} \in TX$ ,  $x \in X$ ,  $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$ .

Given  $(\mathbb{T}, V)$ -categories  $(X, a)$ ,  $(Y, b)$ , a  $(\mathbb{T}, V)$ -functor  $f : (X, a) \rightarrow (Y, b)$  is a map  $f : X \rightarrow Y$  such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ a \downarrow & \leq & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

that is, for each  $\mathfrak{x} \in TX$  and  $x \in X$ ,  $a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$ ;  $f$  is said to be *fully faithful* when this inequality is an equality.

$(\mathbb{T}, V)$ -categories and  $(\mathbb{T}, V)$ -functors form the category  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ . If  $(X, a : TX \rightarrowtail X)$  satisfies (R) (and not necessarily (T)), we call it a  $(\mathbb{T}, V)$ -graph. The category  $(\mathbb{T}, V)\text{-}\mathbf{Gph}$ , of  $(\mathbb{T}, V)$ -graphs and  $(\mathbb{T}, V)$ -functors, contains  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  as a full reflective subcategory.

We present the examples in detail in the last section. We mention here, however, that the leading examples are obtained when one considers the quantale  $2 = (\{0, 1\}, \leq, \&, 1)$  and Lawvere's real half-line  $P_+ = ([0, \infty], \geq, +, 0)$ , the identity monad  $\mathbb{I}$  and the ultrafilter monad  $\mathbb{U}$  on  $\mathbf{Set}$ . Thus we obtain the following examples:

- $(\mathbb{I}, V)\text{-}\mathbf{Cat}$  is the category of  $V$ -categories and  $V$ -functors; in particular,  $(\mathbb{I}, 2)\text{-}\mathbf{Cat}$  is the category  $\mathbf{Ord}$  of (pre)ordered sets and monotone maps, while  $(\mathbb{I}, P_+)\text{-}\mathbf{Cat}$  is the category  $\mathbf{Met}$  of Lawvere's metric spaces and non-expansive maps (see [Law73]).
- $(\mathbb{U}, 2)\text{-}\mathbf{Cat}$  is the category  $\mathbf{Top}$  of topological spaces and continuous maps.
- $(\mathbb{U}, P_+)\text{-}\mathbf{Cat}$  is the category  $\mathbf{App}$  of Lowen's approach spaces and non-expansive maps (see [Low97]).

We recall (see [AHS90, Definition 21.1]) that a functor  $G : \mathbf{A} \rightarrow \mathbf{B}$  is said to be *topological* if every source  $(f_i : B \rightarrow GA_i)_{i \in I}$  in  $\mathbf{B}$  has a unique  $G$ -initial lift  $(\overline{f}_i : A \rightarrow A_i)_{i \in I}$ . The following was proved in [CH03] (see also [CT03]).

**Theorem 2.1.** *The forgetful functors  $(\mathbb{T}, V)\text{-}\mathbf{Cat} \rightarrow \mathbf{Set}$  and  $(\mathbb{T}, V)\text{-}\mathbf{Gph} \rightarrow \mathbf{Set}$  are topological.*

This shows, in particular, that (see [AHS90, Chapter 21] for details):

- $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is complete and cocomplete.

- Monomorphisms in  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  are the morphisms whose underlying map is injective; therefore, since the  $(\mathbb{T}, V)$ -structures on any set form a set,  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is well-powered.
- Every topological category over  $\mathbf{Set}$  has two factorization systems,  $(\text{reg epi}, \text{mono})$  and  $(\text{epi}, \text{reg mono})$ ; in  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  the former one is in general not stable (that is, regular epimorphisms need not be stable under pullback –  $\mathbf{Top}$  is such an example), but the latter one is. Indeed, epimorphisms in  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  are the  $(\mathbb{T}, V)$ -functors which are surjective as maps, the forgetful functor  $(\mathbb{T}, V)\text{-}\mathbf{Cat} \rightarrow \mathbf{Set}$  preserves pullbacks, and surjective maps are stable under pullback in  $\mathbf{Set}$ . Therefore, as  $f \times 1_Z$  is the pullback of  $f : X \rightarrow Y$  along  $\pi_Y : Y \times Z \rightarrow Y$ , we conclude that  $f \times 1_Z$  is an epimorphism provided  $f$  is.

$(\mathbb{T}, V)\text{-}\mathbf{Cat}$  has a natural structure of 2-category: for  $(\mathbb{T}, V)$ -functors  $f, g : (X, a) \rightarrow (Y, b)$ ,  $f \leq g$  if  $g \cdot a \leq b \cdot T f$ . This condition can be equivalently written as  $k \leq b(e_Y(f(x)), g(x))$  for every  $x \in X$  (see [CT03] for details). We write  $f \simeq g$  if  $f \leq g$  and  $g \leq f$ .

Extensivity of  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  was studied in [MST06]:

**Theorem 2.2.**  *$(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is infinitely extensive.*

In general  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is not cartesian closed, while  $(\mathbb{T}, V)\text{-}\mathbf{Gph}$  is. In fact, the following was proved in [CHT03]:

**Theorem 2.3.**  *$(\mathbb{T}, V)\text{-}\mathbf{Gph}$  is a quasi-topos.*

We also note that the tensor product of  $V$  induces a canonical structure  $c$  on  $X \times Y$  defined by

$$c(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y),$$

where  $\mathfrak{w} \in T(X \times Y)$ ,  $x \in X$ ,  $y \in Y$ . We put

$$(X, a) \otimes (Y, b) = (X \times Y, c),$$

and this construction is in an obvious way part of a functor  $\otimes : (\mathbb{T}, V)\text{-}\mathbf{Gph} \times (\mathbb{T}, V)\text{-}\mathbf{Gph} \rightarrow (\mathbb{T}, V)\text{-}\mathbf{Gph}$ . However, the tensor product of two  $(\mathbb{T}, V)$ -categories is in general not a  $(\mathbb{T}, V)$ -category (see [Hof07, Lemma 6.1]).

Weak cartesian closedness of  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  needs a thorough study of injective  $(\mathbb{T}, V)$ -categories and some extra conditions. This is the subject of the following sections.

### 3. EXPONENTIABLE $(\mathbb{T}, V)$ -CATEGORIES

Recall that an object  $C$  of a category  $\mathbf{C}$  with finite products is *exponentiable* whenever the functor  $C \times - : \mathbf{C} \rightarrow \mathbf{C}$  has a right adjoint. The category  $\mathbf{C}$  is *cartesian closed* if every object  $C$  of  $\mathbf{C}$  is exponentiable. Equivalently, if for each pair of objects  $A, B$  of  $\mathbf{C}$  there exists an object  $\langle A, B \rangle$  and a morphism  $\text{ev} : \langle A, B \rangle \times A \rightarrow B$  such that, for each morphism  $f : C \times A \rightarrow B$  there exists a unique morphism  $\bar{f} : C \rightarrow \langle A, B \rangle$  with  $\text{ev} \cdot (\bar{f} \times 1_A) = f$ . Dropping uniqueness of  $\bar{f}$  gives the notion of *weakly cartesian closed category*.

In this section we present a sufficient condition for a  $(\mathbb{T}, V)$ -category  $X$  to be exponentiable in  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ , which generalises [Hof06, Theorem 4.3] and [Hof07, Theorem 6.5]. To start, we recall that  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  can be fully embedded into the cartesian closed category  $(\mathbb{T}, V)\text{-}\mathbf{Gph}$ . Here, for  $(\mathbb{T}, V)$ -graphs  $(X, a)$  and  $(Y, b)$ , the exponential  $\langle (X, a), (Y, b) \rangle$  has as underlying set

$$Z := \{h : (X, a) \times (1, e_1^\circ) \rightarrow (Y, b) \mid h \text{ is a } (\mathbb{T}, V)\text{-functor}\},$$

which becomes a  $(\mathbb{T}, V)$ -graph when equipped with the largest structure  $b^a$  making the evaluation map

$$\text{ev} : Z \times X \rightarrow Y, (h, x) \mapsto h(x)$$

a  $(\mathbb{T}, V)$ -functor: for  $\mathbf{p} \in TZ$  and  $h \in Z$ , put

$$b^a(\mathbf{p}, h) = \bigvee \{v \in V \mid \forall \mathbf{q} \in (T\pi_Z)^{-1}(\mathbf{p}), x \in X \cdot a(T\pi_X(\mathbf{q}), x) \wedge v \leq b(\text{Tev}(\mathbf{q}), h(x))\},$$

where  $\pi_X$  and  $\pi_Z$  are the product projections. Note that the supremum above is even a maximum since  $- \wedge -$  distributes over suprema.

Given  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ , we define in  $V\text{-}\mathbf{Rel}$   $r \otimes s : X \times Y \rightarrowtail X' \times Y'$  by  $(r \otimes s)((x, y), (x', y')) = r(x, x') \wedge s(y, y')$ . That is,  $r \otimes s = (\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^\circ \cdot s \cdot \pi_Y)$  in the ordered set  $V\text{-}\mathbf{Rel}(X \times Y, X' \times Y')$ .

**Theorem 3.1.** *Assume that the diagram*

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{\text{can}_{X,Y}} & TX \times TY \\ T(r \otimes s) \downarrow & & \downarrow (Tr) \otimes (Ts) \\ T(X' \times Y') & \xrightarrow{\text{can}_{X',Y'}} & TX' \times TY' \end{array} \quad (3.i)$$

*commutes, for all  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ . Then a  $(\mathbb{T}, V)$ -category  $(X, a)$  is exponentiable provided that*

$$\bigvee_{\mathbf{r} \in TX} (Ta(\mathfrak{X}, \mathbf{r}) \wedge u) \otimes (a(\mathbf{r}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v), \quad (3.ii)$$

*for all  $\mathfrak{X} \in TT X$ ,  $x \in X$  and  $u, v \in V$ .*

*Proof.* We show that the  $(\mathbb{T}, V)$ -graph structure  $b^a$  on  $Z$  is transitive, for each  $(\mathbb{T}, V)$ -category  $(Y, b)$ . To this end, let  $\mathfrak{P} \in TT Z$ ,  $\mathbf{p} \in TZ$ ,  $h \in Z$ ,  $x \in X$  and  $\mathfrak{w} \in T(Z \times X)$  with  $T\pi_Z(\mathfrak{w}) = m_Z(\mathfrak{P})$ . We have to show that

$$(T(b^a)(\mathfrak{P}, \mathbf{p}) \otimes b^a(\mathbf{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \leq b(\text{Tev}(\mathfrak{w}), h(x)).$$

Since  $m$  has (BC), there is some  $\mathfrak{Q} \in TT(Z \times X)$  with  $TT\pi_Z(\mathfrak{Q}) = \mathfrak{P}$  and  $m_{Z \times X}(\mathfrak{Q}) = \mathfrak{w}$ . Hence,  $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X(\mathfrak{w})$ , and we calculate:

$$\begin{aligned} & (T(b^a)(\mathfrak{P}, \mathbf{p}) \otimes b^a(\mathbf{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \\ & \leq \bigvee_{\mathbf{r} \in TX} ((T(b^a)(TT\pi_Z(\mathfrak{Q}), \mathbf{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}), \mathbf{r})) \otimes (b^a(\mathbf{p}, h) \wedge a(\mathbf{r}, x))) \quad (\text{by (3.ii)}) \\ & \leq \bigvee_{\mathbf{r} \in TX} \bigvee_{\mathbf{q} \in \text{can}^{-1}(\mathbf{p}, \mathbf{r})} T(b^a \otimes a)(T\text{can}_{Z,X}(\mathfrak{Q}), \mathbf{q}) \otimes (b^a \otimes a)(\text{can}_{Z,X}(\mathbf{q}), (h, x)) \quad (\text{using (3.i)}) \\ & = \bigvee_{\mathbf{q} \in (T\pi_Z)^{-1}(\mathbf{p})} T(b^a \otimes a)(T\text{can}_{Z,X}(\mathfrak{Q}), \mathbf{q}) \otimes (b^a \otimes a)(\text{can}_{Z,X}(\mathbf{q}), (h, x)) \\ & = \bigvee_{\mathbf{q} \in (T\pi_Z)^{-1}(\mathbf{p})} T(b^a \times a)(\mathfrak{Q}, \mathbf{q}) \otimes (b^a \times a)(\mathbf{q}, (h, x)) \\ & \leq \bigvee_{\mathbf{q} \in (T\pi_Z)^{-1}(\mathbf{p})} Tb(TT\text{ev}(\mathfrak{Q}), T\text{ev}(\mathbf{q})) \otimes b(T\text{ev}(\mathbf{q}), h(x)) \\ & \leq b(m_Y \cdot TT\text{ev}(\mathfrak{Q}), h(x)) = b(T\text{ev}(\mathfrak{w}), h(x)). \end{aligned}$$

□

*Remark 3.2.* We note that the inequality  $\text{can}_{X',Y'} \cdot T(r \otimes s) \leq ((Tr) \otimes (Ts)) \cdot \text{can}_{X,Y}$  is automatically true. Firstly, this inequality is equivalent to  $T(r \otimes s) \leq \text{can}_{X',Y'}^\circ \cdot ((Tr) \otimes (Ts)) \cdot \text{can}_{X,Y}$ ; secondly,

$$\begin{aligned} T(r \otimes s) &= T((\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^\circ \cdot s \cdot \pi_Y)) \\ &\leq T(\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge T(\pi_{Y'}^\circ \cdot s \cdot \pi_Y) \\ &\leq \text{can}_{X',Y'}^\circ \cdot ((Tr) \otimes (Ts)) \cdot \text{can}_{X,Y}. \end{aligned}$$

It is worthwhile to notice that, when  $V$  is a frame, that is  $\otimes = \wedge$ , the condition above is equivalent to

$$\bigvee_{\mathfrak{r} \in TX} Ta(\mathfrak{X}, \mathfrak{r}) \wedge a(\mathfrak{r}, x) \geq a(m_X(\mathfrak{X}), x),$$

for all  $\mathfrak{X} \in TTX$  and  $x \in X$ . Therefore:

**Corollary 3.3.** *When  $V$  is a frame and (3.i) commutes for all  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ , a  $(\mathbb{T}, V)$ -category  $(X, a)$  is exponentiable provided that*

$$a \cdot m_X = a \cdot Ta.$$

#### 4. INJECTIVE AND REPRESENTABLE $(\mathbb{T}, V)$ -CATEGORIES

In this section we recall an important class of  $(\mathbb{T}, V)$ -categories, the so-called *representable* ones. More information on this type of  $(\mathbb{T}, V)$ -categories can be found in [CCH15, HST14]. We also recall from [CH09, Hof07, Hof11] that every injective  $(\mathbb{T}, V)$ -category is representable.

Based on the lax extension of the **Set**-monad  $\mathbb{T} = (T, m, e)$  to  $V\text{-}\mathbf{Rel}$ ,  $\mathbb{T}$  admits a natural extension to a monad on  $V\text{-}\mathbf{Cat}$ , in the sequel also denoted by  $\mathbb{T} = (T, m, e)$  (see [Tho09]). Here the functor  $T : V\text{-}\mathbf{Cat} \rightarrow V\text{-}\mathbf{Cat}$  sends a  $V$ -category  $(X, a_0)$  to  $(TX, Ta_0)$ , and  $e_X : X \rightarrow TX$  and  $m_X : TTX \rightarrow TX$  become  $V$ -functors for each  $V$ -category  $X$ . The Eilenberg–Moore algebras for this monad can be described as triples  $(X, a_0, \alpha)$  where  $(X, a_0)$  is a  $V$ -category and  $(X, \alpha)$  is an algebra for the **Set**-monad  $\mathbb{T}$  such that  $\alpha : T(X, a_0) \rightarrow (X, a_0)$  is a  $V$ -functor. For  $\mathbb{T}$ -algebras  $(X, a_0, \alpha)$  and  $(Y, b_0, \beta)$ , a map  $f : X \rightarrow Y$  is a homomorphism  $f : (X, a_0, \alpha) \rightarrow (Y, b_0, \beta)$  precisely if  $f$  preserves both structures, that is, whenever  $f : (X, a_0) \rightarrow (Y, b_0)$  is a  $V$ -functor and  $f : (X, \alpha) \rightarrow (Y, \beta)$  is a  $\mathbb{T}$ -homomorphism.

There are canonical adjoint functors

$$(V\text{-}\mathbf{Cat})^{\mathbb{T}} \begin{array}{c} \xrightarrow{K} \\ \mathbb{T} \\ \xleftarrow{M} \end{array} (\mathbb{T}, V)\text{-}\mathbf{Cat}.$$

The functor  $K$  associates to each  $X = (X, a_0, \alpha)$  in  $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$  the  $(\mathbb{T}, V)$ -category  $KX = (X, a)$ , where  $a = a_0 \cdot \alpha$ , and keeps morphisms unchanged. Its left adjoint  $M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \rightarrow (V\text{-}\mathbf{Cat})^{\mathbb{T}}$  sends a  $(\mathbb{T}, V)$ -category  $(X, a)$  to  $(TX, Ta \cdot m_X^\circ, m_X)$  and a  $(\mathbb{T}, V)$ -functor  $f$  to  $Tf$ . Via the adjunction  $M \dashv K$  one obtains a lifting of the **Set**-monad  $\mathbb{T} = (T, m, e)$  to a monad on  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ , also denoted by  $\mathbb{T} = (T, m, e)$ .

In this setting we can define ‘duals’ in  $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$  and carry them into  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ . Indeed, since  $T : V\text{-}\mathbf{Rel} \rightarrow V\text{-}\mathbf{Rel}$  commutes with the involution  $(-)^{\circ}$ : for every  $\mathbb{T}$ -algebra  $X = (X, a_0, \alpha)$  also  $(X, a_0^\circ, \alpha)$  is a  $\mathbb{T}$ -algebra. Moreover, if  $(X, a)$  is a  $(\mathbb{T}, V)$ -category, we define  $X^{\text{op}}$  by mapping  $(X, a)$  into  $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$  via  $M$ , dualizing the image in  $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ , and then carrying it back to  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ ; that is,

$$X^{\text{op}} = K((M(X, a))^{\text{op}}) = (TX, m_X \cdot (Ta)^{\circ} \cdot m_X).$$

Since the monad  $\mathbb{T} = (T, m, e)$  on  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is lax idempotent (i.e. of Kock–Zöberlein type), an algebra structure  $\alpha : TX \rightarrow X$  on a  $(\mathbb{T}, V)$ -category  $X$  is left adjoint to the unit  $e_X : X \rightarrow TX$ .

We call a  $(\mathbb{T}, V)$ -category  $X$  *representable* whenever  $e_X : X \rightarrow TX$  has a left adjoint in  $(\mathbb{T}, V)\text{-Cat}$ ; equivalently, whenever there is some  $(\mathbb{T}, V)$ -functor  $\alpha : TX \rightarrow X$  with  $\alpha \cdot e_X \simeq 1_X$ , since then

$$e_X \cdot \alpha = T\alpha \cdot e_{TX} \geq T\alpha \cdot Te_X \simeq 1_{TX}.$$

However, a left adjoint  $\alpha : TX \rightarrow X$  to  $e_X$  is in general only a pseudo-algebra structure on  $X$ , that is,

$$\alpha \cdot e_X \simeq 1_X \quad \text{and} \quad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

For every representable  $(\mathbb{T}, V)$ -category  $(X, a)$ , the structure  $a : TX \rightarrow X$  can be decomposed as  $a = a_0 \cdot \alpha$ , where  $a_0 = a \cdot e_X$  denotes the underlying  $V$ -category structure.

A  $(\mathbb{T}, V)$ -category  $X$  is *injective* whenever, for each fully faithful  $h : A \rightarrow B$  in  $(\mathbb{T}, V)\text{-Cat}$  and each  $(\mathbb{T}, V)$ -functor  $f : A \rightarrow X$ , there is a  $(\mathbb{T}, V)$ -functor  $g : B \rightarrow X$  with  $g \cdot h \simeq f$ .

**Proposition 4.1.** *Every injective  $(\mathbb{T}, V)$ -category is representable.*

*Proof.* Let  $X$  be an injective  $(\mathbb{T}, V)$ -category. The  $(\mathbb{T}, V)$ -functor  $e_X : (X, a) \rightarrow (TX, Ta \cdot m_X^\circ \cdot m_X)$  is an embedding. Indeed,  $e_X$  is injective because the monad  $T$  is non-trivial, and it is fully faithful:

$$e_X^\circ \cdot Ta \cdot m_X^\circ \cdot m_X \cdot Te_X \leq a \cdot Ta \cdot m_X^\circ \leq a \cdot m_X \cdot m_X^\circ \leq a.$$

Hence, there is a  $(\mathbb{T}, V)$ -functor  $\alpha : TX \rightarrow X$  with  $\alpha \cdot e_X \simeq 1_X$ , and so  $X$  is representable.  $\square$

## 5. INJECTIVE $(\mathbb{T}, V)$ -CATEGORIES ARE EXPONENTIABLE

In Section 6 we will show that, under some conditions,  $(\mathbb{T}, V)\text{-Cat}$  is weakly cartesian closed. Notably, we will use that every  $(\mathbb{T}, V)$ -category can be embedded into an injective one; which, by the main result of this section, implies that every  $(\mathbb{T}, V)$ -category can be embedded into an exponentiable one. We hasten to remark that this is easily seen to be fulfilled for  $\mathbb{T}$  being the identity monad, witnessed by the *Yoneda embedding* (see [Law73])

$$y_X : X \rightarrow PX := V^{X^{\text{op}}}.$$

Here  $PX$  is the free cocompletion of  $X$ ; being cocomplete,  $PX$  is injective.

To treat the general case, we *will consider from now on only* extensions of the monad  $\mathbb{T}$  to  $V\text{-Rel}$  given by a  $\mathbb{T}$ -algebra structure  $\xi : TV \rightarrow V$  on  $V$ , so that we are dealing with a *strict topological theory* in the sense of [Hof07]. In this case, the extension of  $T : \mathbf{Set} \rightarrow \mathbf{Set}$  to  $V\text{-Rel}$  is defined by

$$Tr : TX \times TY \rightarrow V$$

$$(\mathfrak{x}, \mathfrak{y}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \mid \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each  $V$ -relation  $r : X \times Y \rightarrow V$ .

In order to obtain a Yoneda embedding, we consider the  $\mathbb{T}$ -algebra  $(V, \text{hom}, \xi)$  which is mapped by  $K$  into the important  $(\mathbb{T}, V)$ -category  $(V, \text{hom}_\xi)$ , where  $\text{hom}_\xi = \text{hom} \cdot \xi$  (see Section 4). The proof of the following result can be found in [CH09] and [Hof11].

**Theorem 5.1.** *If the extension of  $\mathbb{T}$  to  $V\text{-Rel}$  is induced by a strict topological theory, then, for every  $(\mathbb{T}, V)$ -category  $(X, a)$ , the  $V$ -relation  $a : TX \rightarrow X$  defines a  $(\mathbb{T}, V)$ -functor*

$$a : X^{\text{op}} \otimes X \rightarrow (V, \text{hom}_\xi).$$

Moreover, the  $\otimes$ -exponential mate  $y_X = \lceil a \rceil : X \rightarrow V^{X^{\text{op}}}$  of  $a$  is fully faithful, and the  $(\mathbb{T}, V)$ -category  $PX = V^{X^{\text{op}}}$  is injective.

In fact, this construction defines a functor  $P : (\mathbb{T}, V)\text{-Cat} \rightarrow (\mathbb{T}, V)\text{-Cat}$  and  $y = (y_X)_X$  is a natural transformation  $y : 1_{(\mathbb{T}, V)\text{-Cat}} \rightarrow P$ .

Since  $y_X$  is fully faithful, when  $X$  is injective there exists a  $(\mathbb{T}, V)$ -functor  $\text{Sup}_X : PX \rightarrow X$  such that  $\text{Sup}_X \cdot y_X \simeq 1_X$ . As shown in [Hof11, Theorem 2.7],  $\text{Sup}_X \dashv y_X$ . Moreover, for each  $(\mathbb{T}, V)$ -category  $(X, a)$ ,  $y_X$  is one-to-one if, and only if,  $(X, a)$  is *separated*, i.e. for every  $f, g : (Y, b) \rightarrow (X, a)$ ,  $f \simeq g$  only if  $f = g$  (see [HT10], for example). It follows immediately that, for an injective  $(\mathbb{T}, V)$ -functor  $f : X \rightarrow Y$  where  $Y$  is separated, also  $X$  is.

**Lemma 5.2.** *The  $\otimes$ -exponential  $Y^X$  is separated, for every separated  $(\mathbb{T}, V)$ -category  $Y$  and every representable  $(\mathbb{T}, V)$ -category  $X$ ; in particular,  $PX$  is separated, for every  $(\mathbb{T}, V)$ -category  $X$ .*

*Proof.* See [HT10, Corollary 4.12 (2)].  $\square$

**Corollary 5.3.** *Every separated  $(\mathbb{T}, V)$ -category embeds into an injective  $(\mathbb{T}, V)$ -category.*

In Section 2 we introduced the tensor product  $X \otimes Y$  of  $(\mathbb{T}, V)$ -graphs  $X$  and  $Y$ . We remark that, in the setting of a strict topological theory,  $X \otimes Y$  is a  $(\mathbb{T}, V)$ -category provided that  $X$  and  $Y$  are so (see [Hof07]).

The result promised in the title of this section was shown in [Hof14, Proposition 2.7] for the special case of  $\otimes = \wedge$ :

**Proposition 5.4.** *If the quantale  $V$  is a frame and (3.i) commutes for all  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ , then every representable  $(\mathbb{T}, V)$ -category is exponentiable. In particular, in this case every injective  $(\mathbb{T}, V)$ -category is exponentiable.*

To treat the general case, we will make use of the following conditions:

**Assumptions 5.5.** (1) The diagram (3.i) commutes, for all  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ .

(2) For all  $u, v, w \in V$ ,

$$w \wedge (u \otimes v) = \bigvee \{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\};$$

or, equivalently, every injective  $V$ -category is exponentiable: see [HR13, Theorem 5.3].

(3) For every  $V$ -relation  $a : X \rightarrowtail Y$  and  $u \in V$ ,

$$T(a \otimes u) = Ta \otimes u,$$

where  $a \otimes u$  is the  $V$ -relation defined by  $(a \otimes u)(x, y) = a(x, y) \otimes u$ .

(4) The maps  $V \otimes V \xrightarrow{\otimes} V$  and  $X \xrightarrow{(-, u)} X \otimes V$  are  $(\mathbb{T}, V)$ -functors, for all  $u \in V$ .

These morphisms induce an interesting action of  $V$  on every injective  $(\mathbb{T}, V)$ -category  $(X, a)$  as follows. The  $(\mathbb{T}, V)$ -functor

$$X^{\text{op}} \otimes X \otimes V \xrightarrow{a \otimes 1} V \otimes V \xrightarrow{\otimes} V$$

induces a  $(\mathbb{T}, V)$ -functor  $\tilde{a} : X \otimes V \rightarrow PX$ . We denote the composite

$$X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\text{Sup}_X} X$$

by  $\oplus$ , and

$$X \xrightarrow{(-, u)} X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\text{Sup}_X} X,$$

assigning to each  $x \in X$  an element  $x \oplus u$  in  $X$ , by  $- \oplus u$ .

Analogously we will write  $\mathfrak{x} \oplus u$  for  $T(- \oplus u)(\mathfrak{x})$ , for every  $\mathfrak{x} \in TX$  and  $u \in V$ . Note that  $(\mathbb{T}, V)$ -functoriality of  $- \oplus u$  can be written as

$$a(\mathfrak{x}, y) \leq a(\mathfrak{x} \oplus u, y \oplus u),$$

for every  $\mathfrak{x} \in TX$  and  $y \in X$ .



**Lemma 5.6.** *Assuming 5.5 (4), for an injective  $(\mathbb{T}, V)$ -category  $(X, a)$ , with  $a = a_0 \cdot \alpha$  as usual, the following holds, for every  $x, y \in X$ ,  $\mathfrak{x} \in TX$  and  $u \in V$ :*

- (1)  $a_0(x \oplus u, y) = \text{hom}(u, a_0(x, y))$ ;
- (2)  $a_0(x, y \oplus u) \geq a_0(x, y) \otimes u$ ;
- (3)  $a(\mathfrak{x} \oplus u, y) \geq \text{hom}(u, a(\mathfrak{x}, y))$ ;
- (4)  $a(\mathfrak{x}, y \oplus u) \geq a(\mathfrak{x}, y) \otimes u$ .

Moreover, if, in addition, 5.5 (3) holds, then, for every  $\mathfrak{X} \in T^2X$ ,  $\mathfrak{y} \in TX$ ,  $u \in V$ ,

- (5)  $Ta(\mathfrak{X}, \mathfrak{y} \oplus u) \geq Ta(\mathfrak{X}, \mathfrak{y}) \otimes u$ .

*Proof.* (1) For every  $x, y \in X$  and  $u \in V$ ,

$$\begin{aligned}
 a_0(x \oplus u, y) &= a_0(\text{Sup}_X(\tilde{a}(x, u)), y) && \text{(by definition of } \oplus) \\
 &= [\tilde{a}(x, u), y_X(y)] && \text{(because } \text{Sup}_X \dashv y_X) \\
 &= \bigwedge_{\mathfrak{x} \in TX} \text{hom}(\tilde{a}(x, u)(\mathfrak{x}), y_X(y)(\mathfrak{x})) && \text{(by definition of } [\cdot, \cdot]) \\
 &= \bigwedge_{\mathfrak{x} \in TX} \text{hom}(a(\mathfrak{x}, x) \otimes u, a(\mathfrak{x}, y)) && \text{(by definition of } \tilde{a} \text{ and } y_X(y)) \\
 &= \text{hom}(u, a_0(x, y)),
 \end{aligned}$$

because, using the fact that  $a = a_0 \cdot \alpha$  and

$$a_0(\alpha(\mathfrak{x}), x) \otimes u \otimes \text{hom}(u, a_0(x, y)) \leq a_0(\alpha(\mathfrak{x}), x) \otimes a_0(x, y) \leq a_0(\alpha(\mathfrak{x}), y),$$

for  $\mathfrak{x} \in TX$ , we can conclude that

$$\text{hom}(u, a_0(x, y)) \leq \bigwedge_{\mathfrak{x} \in TX} \text{hom}(a_0(\alpha(\mathfrak{x}), x) \otimes u, a_0(\alpha(\mathfrak{x}), y)).$$

Taking  $\mathfrak{x} = e_X(x)$ , we see that this inequality is in fact an equality as claimed.

(2) Since, by hypothesis,  $- \oplus u$  is a  $(\mathbb{T}, V)$ -functor, and so, in particular, a  $V$ -functor  $(X, a_0) \rightarrow (X, a_0)$ ,

$$a_0(x, y) \leq a_0(x \oplus u, y \oplus u) = \text{hom}(u, a_0(x, y \oplus u)),$$

and then

$$a_0(x, y) \otimes u \leq \text{hom}(u, a_0(x, y \oplus u)) \otimes u \leq a_0(x, y \oplus u).$$

(3) One has

$$\begin{aligned}
 k &\leq a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{x})) = a(\mathfrak{x}, \alpha(\mathfrak{x})) \\
 &\leq a(\mathfrak{x} \oplus u, \alpha(\mathfrak{x}) \oplus u) \\
 &= a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u).
 \end{aligned}$$

Using (1) we conclude that

$$\begin{aligned}
 \text{hom}(u, a(\mathfrak{x}, y)) &= a_0(\alpha(\mathfrak{x}) \oplus u, y) \\
 &\leq a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u) \otimes a_0(\alpha(\mathfrak{x}) \oplus u, y) \\
 &\leq a_0(\alpha(\mathfrak{x} \oplus u), y) = a(\mathfrak{x} \oplus u, y).
 \end{aligned}$$

(4) follows directly from (2), while (5) follows from (4).  $\square$

**Lemma 5.7.** *Let  $\varphi : V \rightarrow W$  be a surjective quantale homomorphism; that is,  $\varphi$  preserves the tensor, the neutral element, and suprema. Then, if  $V$  satisfies condition 5.5 (2), so does  $W$ .*

**Theorem 5.8.** *Under Assumptions 5.5, every injective  $(\mathbb{T}, V)$ -category is exponentiable in  $(\mathbb{T}, V)$ -Cat.*

*Proof.* Let  $\mathfrak{X} \in T^2X$ ,  $x \in X$  and  $u, v \in V$ . In order to conclude that

$$\bigvee_{\mathfrak{r} \in TX} (Ta(\mathfrak{X}, \mathfrak{r}) \wedge u) \otimes (a(\mathfrak{r}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v),$$

we make use of Hypothesis 5.5 (2). Let  $u', v' \in V$  with  $u' \leq u$ ,  $v' \leq v$  and  $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$ . First we note that

$$\begin{aligned} Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u &\geq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X})) \otimes u') \wedge u && \text{(by 5.6 (5))} \\ &= (Ta_0(T\alpha(\mathfrak{X}), T\alpha(\mathfrak{X})) \otimes u') \wedge u \\ &\geq (k \otimes u') \wedge u = u', \end{aligned}$$

and

$$\begin{aligned} a(T\alpha(\mathfrak{X}) \oplus u', x) &\geq \text{hom}(u', a(T\alpha(\mathfrak{X}), x)) && \text{(by 5.6 (3))} \\ &= \text{hom}(u', a_0(\alpha(T\alpha(\mathfrak{X})), x)) \\ &= \text{hom}(u', a_0(\alpha(m_X(\mathfrak{X})), x)) \\ &= \text{hom}(u', a(m_X(\mathfrak{X}), x)). \end{aligned}$$

Now, from  $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$  and  $v' \leq v$  we get

$$v' \leq \text{hom}(u', a(m_X(\mathfrak{X}), x)) \wedge v \leq a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v,$$

hence

$$u' \otimes v' \leq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u) \otimes (a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v).$$

Therefore  $a(m_X(\mathfrak{X}), x) \wedge (u \otimes v) \leq \bigvee_{\mathfrak{r} \in TX} (Ta(\mathfrak{X}, \mathfrak{r}) \wedge u) \otimes (a(\mathfrak{r}, x) \wedge v)$ .  $\square$

*Remark 5.9.* Under Assumptions 5.5, it follows from Lemma 5.2 that the exponential  $\langle (X, a), (Y, b) \rangle$  is separated, for all separated injective  $(\mathbb{T}, V)$ -categories  $(X, a)$  and  $(Y, b)$ . In fact, with  $a = a_0 \cdot \alpha$ , the epimorphism  $(X, \alpha) \rightarrow (X, a)$  in  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is mapped to the monomorphism

$$\langle (X, a), (Y, b) \rangle \longrightarrow \langle (X, \alpha), (Y, b) \rangle = (Y, b)^{(X, \alpha)},$$

which proves that  $\langle (X, a), (Y, b) \rangle$  is separated.

## 6. $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ IS WEAKLY CARTESIAN CLOSED

Building on the results of the previous section, in this section we show that, under some conditions,  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  is weakly cartesian closed. We start by proving this property for the full subcategory  $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\text{sep}}$  of  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$  of separated  $(\mathbb{T}, V)$ -categories.

**Theorem 6.1.** *Under Assumptions 5.5,  $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\text{sep}}$  is weakly cartesian closed.*

*Proof.* For  $X, Y$  separated  $(\mathbb{T}, V)$ -categories, consider the Yoneda embeddings  $y_X : X \rightarrow PX$  and  $y_Y : Y \rightarrow PY$ , and the exponential  $\langle PX, PY \rangle$ . The elements of its underlying set can be identified with  $(\mathbb{T}, V)$ -functors  $E \times PX \rightarrow PY$  (where  $E = (1, e_1^c)$  is the generator of  $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ ), and the universal morphism  $\text{ev} : \langle PX, PY \rangle \times PX \rightarrow PY$  with the evaluation map:  $\text{ev}(\varphi, \mathfrak{r}) = \varphi(\mathfrak{r})$  (where, for simplicity, we identify the set  $E \times PX$  with  $PX$ ). We can therefore consider

$$\ll X, Y \gg = \{\varphi : E \times PX \rightarrow PY \mid \varphi(y_X(X)) \subseteq y_Y(Y)\},$$

with the initial structure with respect to the inclusion  $\iota : \ll X, Y \gg \rightarrow \langle PX, PY \rangle$ . Moreover, the morphism

$$\ll X, Y \gg \times X \xrightarrow{\iota \times y_X} \langle PX, PY \rangle \times PX \xrightarrow{\text{ev}} PY$$

factors through  $y_Y$  via a morphism

$$\ll X, Y \gg \times X \xrightarrow{\tilde{\text{ev}}} Y.$$

Next we show that this is a weak exponential in  $(\mathbb{T}, V)\text{-Cat}_{\text{sep}}$ .

Given any separated  $(\mathbb{T}, V)$ -category  $Z$ , and a  $(\mathbb{T}, V)$ -functor  $f : Z \times X \rightarrow Y$ , by injectivity of  $PY$  there exists a  $(\mathbb{T}, V)$ -functor  $f' : Z \times PX \rightarrow PY$  making the square below commute. Then, by universality of the evaluation map  $\text{ev}$ , there exists a unique  $(\mathbb{T}, V)$ -functor  $\bar{f} : Z \rightarrow \langle PX, PY \rangle$  making the bottom triangle commute.

$$\begin{array}{ccc} Z \times X & \xrightarrow{f} & Y \\ 1_Z \times y_X \downarrow & & \downarrow y_Y \\ Z \times PX & \xrightarrow{f'} & PY \\ \bar{f} \times 1_{PX} \downarrow & \nearrow \text{ev} & \\ \langle PX, PY \rangle \times PX & & \end{array}$$

The map  $\bar{f} : Z \rightarrow \langle PX, PY \rangle$ , assigning to each  $z \in Z$  a map  $\bar{f}(z) : PX \rightarrow PY$ , is such that  $\bar{f}(z)(y_X(x)) = \text{ev}(\bar{f}(z), y_X(x)) = y_Y(f(z, x))$ ; that is,  $\bar{f}(z)(y_X(X)) \subseteq y_Y(Y)$ , and this means that  $\bar{f}(z) \in \ll X, Y \gg$ . Hence we can consider the corestriction  $\tilde{f}$  of  $\bar{f}$  to  $\ll X, Y \gg$ , which is again a  $(\mathbb{T}, V)$ -functor since  $\ll X, Y \gg$  has the initial structure with respect to  $\langle PX, PY \rangle$ , so that the following diagram commutes.

$$\begin{array}{ccc} \ll X, Y \gg \times X & \xrightarrow{\tilde{\text{ev}}} & Y \\ \tilde{f} \times 1_X \uparrow & \nearrow f & \\ Z \times X & & \end{array}$$

□

In order to show that  $(\mathbb{T}, V)\text{-Cat}$  is weakly cartesian closed, we follow the proof of [Ros99]. Hence, first we show that:

**Proposition 6.2.** *The reflector  $R : (\mathbb{T}, V)\text{-Cat} \rightarrow (\mathbb{T}, V)\text{-Cat}_{\text{sep}}$  preserves finite products.*

*Proof.* We recall that, for any  $(\mathbb{T}, V)$ -category  $(X, a)$ ,  $R(X, a) = (\tilde{X}, \tilde{a})$ , with  $\tilde{X} = X / \sim$ , where  $x \sim y$  if  $k \leq a(e_X(x), y) \wedge a(e_X(y), x)$ , and  $\tilde{a} = \eta_X \cdot a \cdot (T\eta_X)^\circ$ , with  $\eta_X : X \rightarrow \tilde{X}$  the projection. This structure makes  $\eta_X$  both an initial and a final morphism (see [HST14] for details).

Let  $f : R(X \times Y) \rightarrow RX \times RY$  be the unique morphism such that  $f \cdot \eta_{X \times Y} = \eta_X \times \eta_Y$ .

$$\begin{array}{ccc} (X \times Y, c) & \xrightarrow{\eta_{X \times Y}} & (R(X \times Y), \tilde{c}) \\ & \searrow \eta_X \times \eta_Y & \downarrow f \\ & & (RX \times RY, d) \end{array}$$

From  $c(e_{X \times Y}(x, y), (x', y')) = a(e_X(x), x') \wedge b(e_Y(y), y')$  it is immediate that  $(x, y) \sim (x', y')$  in  $X \times Y$  if, and only if,  $x \sim x'$  in  $X$  and  $y \sim y'$  in  $Y$ . Therefore,  $f$  is a bijection. Assuming the Axiom of Choice, so that  $T$  preserves surjections, we have, for every  $\mathfrak{z} \in T(R(X \times Y))$ ,  $(x, y) \in X \times Y$ ,

$$\begin{aligned} \tilde{c}(\mathfrak{z}, [(x, y)]) &= c(\mathfrak{w}, (x, y)) && (\text{for any } \mathfrak{w} \in (T\eta_{X \times Y})^{-1}(\mathfrak{z})) \\ &= d(T(\eta_X \times \eta_Y)(\mathfrak{w}), ([x], [y])) && (\text{because } \eta_X \times \eta_Y \text{ is initial}) \\ &= d(Tf(\mathfrak{z}), ([x], [y])); \end{aligned}$$

that is,  $f$  is initial and therefore an isomorphism.

□

**Theorem 6.3.** *Under Assumptions 5.5,  $(\mathbb{T}, V)\text{-Cat}$  is weakly cartesian closed.*

*Proof.* Given  $(\mathbb{T}, V)$ -categories  $(X, a)$ ,  $(Y, b)$ , to build the weak exponential  $\ll X, Y \gg$  we will show the *cosolution set condition* for the functor  $- \times (X, a)$ .

For each  $(\mathbb{T}, V)$ -functor  $f : (Z, c) \times (X, a) \rightarrow (Y, b)$  we consider its reflection  $Rf : RZ \times RX \cong R(Z \times X) \rightarrow RY$  and we factorise it through the weak evaluation in  $(\mathbb{T}, V)\text{-Cat}_{\text{sep}}$ ,  $Rf = \widetilde{\text{ev}} \cdot (\overline{Rf} \times 1_{RX})$ , so that in the diagram below the outer rectangle commutes.

Then we define  $Z_f = Z / \sim$  by

$$z \sim z' \text{ if both } f(z, x) = f(z', x), \text{ for every } x \in X, \text{ and } \overline{Rf}(\eta_Z(z)) = \overline{Rf}(\eta_Z(z')),$$

and equip it with the final structure for the projection  $q_f : Z \rightarrow Z_f$ . Then  $h_f : Z_f \rightarrow \ll RX, RY \gg$ , with  $h_f([z]) = \overline{Rf}(\eta_Z(z))$ , is a  $(\mathbb{T}, V)$ -functor since its composition with  $q_f$  is  $\overline{Rf} \cdot \eta_Z$  and  $q_f$  is final. Then we factorise  $f$  via the surjection  $q_f \times 1_X : Z \times X \rightarrow Z_f \times X$  as in the diagram below. Moreover, the map  $\hat{f} : Z_f \times X \rightarrow Y$ , with  $\hat{f}([z], x) = f(z, x)$ , is a  $(\mathbb{T}, V)$ -functor because  $\eta_Y \cdot \hat{f} = \widetilde{\text{ev}} \cdot (h_f \times \eta_X)$  is and  $\eta_Y$  is initial.

$$\begin{array}{ccccc}
 Z \times X & \xrightarrow{f} & & & Y \\
 \eta_Z \times 1_X \downarrow & \searrow q_f \times 1_X & & \nearrow \hat{f} & \downarrow \eta_Y \\
 RZ \times X & & Z_f \times X & \xrightarrow{\quad} & (\coprod_g Z_g \times X) \cong (\coprod_g Z_g) \times X \\
 \overline{Rf} \times 1_X \downarrow & \nearrow h_f \times 1_X & & \nwarrow \text{ev} & \\
 \ll RX, RY \gg \times X & \xrightarrow{1 \times \eta_X} & \ll RX, RY \gg \times RX & \xrightarrow{\widetilde{\text{ev}}} & RY
 \end{array}$$

Since the cardinality of  $Z_f$  is bounded by the cardinality of the set  $|\ll RX, RY \gg| \times |Y|^{|X|}$ , as witnessed by the injective map

$$\begin{aligned}
 Z_f &\rightarrow |\ll RX, RY \gg| \times |Y|^{|X|}, \\
 [z] &\mapsto (\overline{Rf}(\eta_Z(z)), f(z, -))
 \end{aligned}$$

there is only a set of possible  $(\mathbb{T}, V)$ -categories  $Z_f$ . Hence we can form its coproduct, as in the diagram above, and consider the induced  $(\mathbb{T}, V)$ -functor  $\text{ev} : (\coprod_g Z_g) \times X \cong \coprod_g (Z_g \times X) \rightarrow Y$  (note that the isomorphism follows from extensivity of  $(\mathbb{T}, V)\text{-Cat}$ ).  $\square$

## 7. EXAMPLES

In this section we use Theorem 6.3 to present examples of weakly cartesian closed categories. Hence, in conjunction with the following theorem established in [Ros99], we obtain examples of categories with cartesian closed exact completion since all other conditions of that theorem are trivially satisfied in these examples.

**Theorem 7.1.** *Let  $\mathbf{C}$  be a complete, infinitely extensive and well-powered category in which every morphism factorizes as a regular epi followed by a mono, and where  $f \times 1$  is an epimorphism for every regular epimorphism  $f : A \rightarrow B$  in  $\mathbf{C}$ . Then, if  $\mathbf{C}$  is weakly cartesian closed, the exact completion  $\mathbf{C}_{\text{ex}}$  of  $\mathbf{C}$  is cartesian closed.*

We note that, in order to conclude that  $(\mathbb{T}, V)\text{-Cat}$  is weakly cartesian closed, we have to check whether  $V$  and  $\mathbb{T}$  satisfy Assumptions 5.5.

First we analyse examples where  $\mathbb{T}$  is the identity monad. In this particular setting we only have to check that 5.5 (2) holds. The category  $V\text{-Cat}$  is always monoidal closed, as shown in [Law73]. Therefore, when  $V$  is a frame considered as a quantale, then  $V\text{-Cat}$  is cartesian closed. This is the case of 2, and so one concludes that **Ord** is cartesian closed. Moreover, for  $V$  the lattice  $([0, \infty], \geq)$  with  $\otimes = \wedge$ ,  $V\text{-Cat}$  is the category of ultrametric spaces, which is therefore also cartesian closed.

When  $V = P_+$ ,  $V\text{-Cat}$  is the category **Met** of Lawvere's metric spaces [Law73], which is not cartesian closed (see [CH06] for details). However, the quantale  $P_+$  satisfies 5.5 (2), and so from Theorem 6.3 it follows that **Met** is weakly cartesian closed.

Metric and ultrametric spaces can be also viewed as categories enriched in a quantale based on the complete lattice  $[0, 1]$  with the usual “less or equal” relation  $\leq$ , which is isomorphic to  $[0, \infty]$  via the map  $[0, 1] \rightarrow [0, \infty]$ ,  $u \mapsto -\ln(u)$  where  $-\ln(0) = \infty$ . More in detail, we consider the following quantale operations on  $[0, 1]$  with neutral element 1.

- (1) For  $\otimes = *$  being the ordinary multiplication, via the isomorphism  $[0, 1] \simeq [0, \infty]$ , this quantale is isomorphic to the quantale  $P_+$ , hence  $[0, 1]\text{-Cat} \simeq \mathbf{Met}$ .
- (2) For the tensor  $\otimes = \wedge$  being infimum, the isomorphism  $[0, 1] \simeq [0, \infty]$  establishes an equivalence between  $[0, 1]\text{-Cat}$  and the category of ultrametric spaces and non-expansive maps.
- (3) Another interesting multiplication on  $[0, 1]$  is the *Łukasiewicz tensor*  $\otimes = \odot$  given by  $u \odot v = \max(0, u + v - 1)$ . Via the lattice isomorphism  $[0, 1] \rightarrow [0, 1]$ ,  $u \mapsto 1 - u$ , this quantale is isomorphic to the quantale  $[0, 1]$  with “greater or equal” relation  $\geq$  and tensor  $u \otimes v = \min(1, u + v)$  truncated addition. Therefore  $[0, 1]\text{-Cat}$  is equivalent to the category of bounded-by-1 metric spaces and non-expansive maps. Moreover, with respect to the “greater or equal” relation and truncated addition on  $[0, 1]$ , the map

$$[0, \infty] \rightarrow [0, 1], u \mapsto \min(1, u)$$

is a surjective quantale morphism; therefore, by Lemma 5.7, also  $[0, 1]$  with the Łukasiewicz tensor satisfies 5.5 (2).

- (4) More generally, every continuous quantale structure  $\otimes$  on the lattice  $[0, 1]$  (with Euclidean topology and the usual “less or equal” relation) with neutral element 1 satisfies 5.5 (2). This can be shown using the fact, proven in [Fau55] and [MS57], that every such tensor  $\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a combination of the three operations on  $[0, 1]$  described above. More precise:

- (a) For  $u, v \in [0, 1]$  and  $e \in [0, 1]$  idempotent with  $u \leq e \leq v$ :  $u \otimes v = \min(u, v) = u$ .
- (b) For every non-idempotent  $u \in [0, 1]$ , there exist idempotents  $e$  and  $f$  with  $e < u < f$  and such that the interval  $[e, f]$  (with the restriction of the tensor on  $[0, 1]$  and with neutral element  $f$ ) is isomorphic to  $[0, 1]$  either with multiplication or Łukasiewicz tensor.

Now let  $w, u, v \in [0, 1]$ . We may assume  $u \leq v$ . If  $u \otimes v \leq w$ , then clearly

$$w \wedge (u \otimes v) = u \otimes v = \bigvee \{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\}.$$

We consider now  $w < u \otimes v \leq u \leq v$ . If  $w$  is idempotent, then

$$w = w \otimes v, \quad w \leq u, \quad v \leq v;$$

otherwise there are idempotents  $e$  and  $f$  with  $e < w < f$  and  $[e, f]$  is isomorphic to  $[0, 1]$  either with multiplication or Łukasiewicz tensor.

**Case 1:**  $v \leq f$ . Then 5.5 (2) holds since  $w, u \otimes v, u, v \in [e, f]$ .

**Case 2:**  $f < v$ . Then  $w = w \wedge v = w \otimes v$ ,  $w \leq u$  and  $v \leq v$ .

We conclude that  $[0, 1]$ -**Cat** is weakly cartesian closed, for every continuous quantale structure  $\otimes$  on the lattice  $[0, 1]$  with neutral element 1.

Now let  $V = \Delta$  be the *quantale of distribution functions* (see [HR13, CH17] for details). As observed in [HR13], it verifies 5.5 (2), and so we can conclude from Theorem 6.3 that *the category  $\Delta$ -Cat of probabilistic metric spaces and non-expansive maps is weakly cartesian closed*.

When  $\mathbb{T}$  is not the identity monad, some further work is need to guarantee Assumptions 5.5.

**Theorem 7.2.** (1) *The tensor product on the quantale  $V$  defines a  $(\mathbb{T}, V)$ -functor  $\otimes : V \otimes V \rightarrow V$ .*

(2) *Let  $u \in V$  satisfying  $u \cdot ! \geq \xi \cdot Tu$ .*

$$\begin{array}{ccc} T1 & \xrightarrow{Tu} & TV \\ ! \downarrow & \geq & \downarrow \xi \\ 1 & \xrightarrow{u} & V \end{array}$$

*Then  $(-, u) : X \rightarrow X \times V$  is a  $(\mathbb{T}, V)$ -functor, for every  $(\mathbb{T}, V)$ -category  $X$ .*

(3) *Let  $u \in V$  satisfying  $u \cdot ! = \xi \cdot Tu$ . Then  $T(r \otimes u) = (Tr) \otimes u$ , for every  $V$ -relation  $r : X \rightarrowtail Y$ .*

*Proof.* The first assertion is [Hof11, Proposition 1.4(1)]. To see (2), assume that  $u \in V$  with  $u \cdot ! \geq \xi \cdot Tu$ . Let  $(X, a)$  be a  $(\mathbb{T}, V)$ -category,  $\mathfrak{x} \in TX$  and  $x \in X$ . Considering the map  $X \xrightarrow{!} 1 \xrightarrow{u} V$ , we have to show that

$$a(\mathfrak{x}, x) \leq a(\mathfrak{x}, x) \otimes \text{hom}(T(u \cdot !)(\mathfrak{x}), u),$$

which follows immediately from  $u \cdot ! \geq \xi \cdot Tu$ . Finally, to prove (3), let  $r : X \rightarrowtail Y$  be a  $V$ -relation and  $u \in V$  with  $u \cdot ! = \xi \cdot Tu$ . Note that the  $V$ -relation  $r \otimes u : X \rightarrowtail Y$  is given by

$$X \times Y \xrightarrow{r} V \xrightarrow{\langle 1_V, u \cdot ! \rangle} V \times V \xrightarrow{\otimes} V.$$

Hence, applying the **Set**-functor  $T$  to the functions  $r : X \times Y \rightarrow V$  and  $r \otimes u : X \times Y \rightarrow V$ , we obtain

$$\begin{aligned} \xi \cdot T(r \otimes u) &= \xi \cdot T(\otimes) \cdot T\langle 1_V, u \cdot ! \rangle \cdot Tr \\ &= \otimes \cdot (\xi \times \xi) \cdot \text{can}_{X,Y} \cdot T\langle 1_V, u \cdot ! \rangle \cdot Tr \\ &= \otimes \cdot \langle \xi, u \cdot ! \cdot \xi \rangle \cdot Tr \\ &= \otimes \cdot \langle 1_V, u \cdot ! \rangle \cdot \xi \cdot Tr. \end{aligned}$$

Therefore, returning to  $V$ -relations, we conclude that  $T(r \otimes u) = (Tr) \otimes u$ . □

*Remark 7.3.* If  $T1 = 1$ , then  $u \cdot ! = \xi \cdot Tu$  for every  $u \in V$ .

In order to guarantee Assumptions 5.5 (1), we need an extra condition on  $\xi$ .

**Proposition 7.4.** *Assume that*

$$\begin{array}{ccc} T(V \times V) & \xrightarrow{T(\wedge)} & TV \\ \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \downarrow & \leq & \downarrow \xi \\ V \times V & \xrightarrow{\wedge} & V. \end{array}$$

Then, for all  $V$ -relations  $r : X \rightarrowtail X'$  and  $s : Y \rightarrowtail Y'$ ,

$$\begin{array}{ccc} T(X \times Y) & \xrightarrow{\text{can}_{X,Y}} & TX \times TY \\ T(r \otimes s) \downarrow & \geq & \downarrow Tr \otimes Ts \\ T(X' \times Y') & \xrightarrow{\text{can}_{X',Y'}} & TX' \times TY'. \end{array}$$

*Proof.* First we note that, from the preservation of weak pullbacks by  $T$ , it follows that the commutative diagram

$$\begin{array}{ccc} T(A \times B) & \xrightarrow{T(f \times g)} & T(X \times Y) \\ \text{can}_{A,B} \downarrow & & \downarrow \text{can}_{X,Y} \\ TA \times TB & \xrightarrow{Tf \times Tg} & TX \times TY \end{array}$$

is also a weak pullback.

Let  $\mathfrak{w} \in T(X \times Y)$ ,  $\mathfrak{x}' \in TX'$  and  $\mathfrak{y}' \in TY'$ . Put  $(\mathfrak{x}, \mathfrak{y}) = \text{can}_{X,Y}(\mathfrak{w})$ . By the definition of the extension of  $T$  and since  $V$  is a Heyting algebra,

$$Tr(\mathfrak{x}, \mathfrak{x}') \wedge Ts(\mathfrak{y}, \mathfrak{y}') = \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}_1) \wedge \xi \cdot Ts(\mathfrak{w}_2) \mid \begin{array}{l} \mathfrak{w}_1 \in T(X \times X') : \mathfrak{w}_1 \mapsto \mathfrak{x}, \mathfrak{w}_1 \mapsto \mathfrak{x}' \\ \mathfrak{w}_2 \in T(Y \times Y') : \mathfrak{w}_2 \mapsto \mathfrak{y}, \mathfrak{w}_2 \mapsto \mathfrak{y}' \end{array} \right\}.$$

Note that in

$$\begin{array}{ccccc} & & T(X \times Y \times X' \times Y') & & \\ & & \cong \downarrow & & \\ T(X \times Y) & \xleftarrow{T(\pi_X \times \pi_Y)} & T(X \times X' \times Y \times Y') & \xrightarrow{T(r \times s)} & T(V \times V) \xrightarrow{T(\wedge)} TV \\ \text{can} \downarrow & & \text{can} \downarrow & & \text{can} \downarrow \quad \downarrow \xi \\ TX \times TY & \xleftarrow{T\pi_X \times T\pi_Y} & T(X \times X') \times T(Y \times Y') & \xrightarrow{Tr \times Ts} & TV \times TV \\ & & & & \downarrow \xi \times \xi \quad \leq \\ & & & & V \times V \xrightarrow{\wedge} V \end{array}$$

the left hand side is a weak pullback, the middle diagram commutes, and in the right hand side we have “lower path”  $\leq$  “upper path” as indicated. Therefore, for such  $\mathfrak{w}_1 \in T(X \times X')$  and  $\mathfrak{w}_2 \in T(Y \times Y')$ , there exists some  $\mathfrak{v} \in T(X \times X' \times Y \times Y')$  which projects to  $\mathfrak{w} \in T(X \times Y)$  and to  $(\mathfrak{w}_1, \mathfrak{w}_2) \in T(X \times X') \times T(Y \times Y')$ . Hence, taking also into account the definition of the  $V$ -relation  $T(r \otimes s)$ ,

$$\begin{aligned} Tr(\mathfrak{x}, \mathfrak{x}') \wedge Ts(\mathfrak{y}, \mathfrak{y}') &\leq \bigvee \left\{ \xi \cdot T(\wedge) \cdot T(r \times s)(\mathfrak{v}) \mid \mathfrak{v} \in T(X \times Y \times X' \times Y'); \begin{array}{l} \mathfrak{v} \mapsto \mathfrak{w} \\ \mathfrak{v} \mapsto \mathfrak{x}', \mathfrak{v} \mapsto \mathfrak{y}' \end{array} \right\} \\ &\leq \bigvee \{ T(r \otimes s)(\mathfrak{w}, \mathfrak{w}') \mid \mathfrak{w}' \in T(X' \times Y'), \text{can}_{X',Y'}(\mathfrak{w}') = (\mathfrak{x}', \mathfrak{y}') \}. \end{aligned}$$

□

*Remark 7.5.* We note that the inequality

$$\begin{array}{ccc} T(V \times V) & \xrightarrow{T(\wedge)} & TV \\ \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \downarrow & \geq & \downarrow \xi \\ V \times V & \xrightarrow{\wedge} & V \end{array}$$

is always true.

**Corollary 7.6.** *If the quantale  $V$  satisfies Assumption 5.5 (2) and the diagrams*

$$\begin{array}{ccc} T(V \times V) & \xrightarrow{T(\wedge)} & TV \\ \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \downarrow & & \downarrow \xi \\ V \times V & \xrightarrow{\wedge} & V \end{array} \quad \text{and} \quad \begin{array}{ccc} T1 & \xrightarrow{Tu} & TV \\ ! \downarrow & & \downarrow \xi \\ 1 & \xrightarrow{u} & V \end{array}$$

*commute, for all  $u \in V$ , then all Assumptions 5.5 are satisfied.*

Let  $\mathbb{T}$  be the ultrafilter monad  $\mathbb{U} = (U, m, e)$ . Then, when  $V$  is any of the quantales listed above but  $\Delta$ , all the needed conditions are satisfied. Therefore, in particular we can conclude that:

- Examples 7.7.** (1) *The category  $\mathbf{Top} = (\mathbb{U}, 2)\text{-Cat}$  of topological spaces and continuous maps is weakly cartesian closed (as shown by Rosický in [Ros99]).*  
(2) *The category  $\mathbf{App} = (\mathbb{U}, P_+)\text{-Cat}$  of approach spaces and non-expansive maps is weakly cartesian closed.*  
(3) *In fact, for each continuous quantale structure on the lattice  $([0, 1], \leq) \simeq ([0, \infty], \geq)$ ,  $(\mathbb{U}, [0, 1])\text{-Cat}$  is weakly cartesian closed. In particular, the category of non-Archimedean approach spaces and non-expansive maps studied in [CVO17] is weakly cartesian closed.*  
(4) *If  $V$  is a completely distributive complete lattice with  $\otimes = \wedge$ , then, with*

$$\xi : UV \rightarrow V, \mathfrak{x} \mapsto \bigwedge_{A \in \mathfrak{x}} \bigvee A,$$

*all the conditions of Theorem 6.3 are satisfied (see [Hof07, Theorem 3.3]) and therefore  $(\mathbb{U}, V)\text{-Cat}$  is weakly cartesian closed. In particular, with  $V = P2$  being the powerset of a 2-element set, we obtain that the category  $\mathbf{BiTop}$  of bitopological spaces and bicontinuous maps is weakly cartesian closed (see [HST14]).*

*Remark 7.8.* For  $V = \Delta$  the quantale of distribution functions, we do not know whether there is an appropriate compact Hausdorff topology  $\xi : UV \rightarrow V$  satisfying the conditions of this section.

Now let  $\mathbb{T}$  be the free monoid monad  $\mathbb{W} = (W, m, e)$ . For each quantale  $V$ , we consider

$$\xi : WV \rightarrow V, (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n, () \mapsto k$$

which induces the extension  $W : V\text{-Rel} \rightarrow V\text{-Rel}$  sending  $r : X \rightarrowtail Y$  to the  $V$ -relation  $Wr : WX \rightarrowtail WY$  given by

$$Wr((x_1, \dots, x_n), (y_1, \dots, y_m)) = \begin{cases} r(x_1, y_1) \otimes \dots \otimes r(x_n, y_n) & \text{if } n = m, \\ \perp & \text{if } n \neq m. \end{cases}$$

The category  $(\mathbb{W}, 2)\text{-Cat}$  is equivalent to the category **MultiOrd** of *multi-ordered sets* and their morphisms (see [HST14]), more generally,  $(\mathbb{W}, V)$ -categories can be interpreted as multi- $V$ -categories and their morphisms. The representable multi-ordered sets are precisely the ordered monoids, which is a special case of [Her00, Her01] describing monoidal categories as representable multi-categories (see also [CCH15]). We recall that the separated injective multi-ordered sets are precisely the quantales (see [LBKR12] and also [Sea10]), and we conclude:

**Proposition 7.9.** *Every quantale is exponentiable in **MultiOrd**.*

**Theorem 7.10.** *If the quantale  $V$  is a frame, then  $(\mathbb{W}, V)\text{-Cat}$  is weakly cartesian closed. In particular, **MultiOrd** is weakly cartesian closed.*



Finally, for a monoid  $(H, \cdot, h)$ , we consider the monad  $\mathbb{H} = (- \times H, m, e)$ , with  $m_X : X \times H \times H \rightarrow X \times H$  given by  $m_X(x, a, b) = (x, a \cdot b)$  and  $e_X : X \rightarrow X \times H$  given by  $e_X(x) = (x, h)$ . Here we consider

$$\xi : V \times H \rightarrow V, (v, a) \mapsto v,$$

which leads to the extension  $- \times H : V\text{-}\mathbf{Rel} \rightarrow V\text{-}\mathbf{Rel}$  sending the  $V$ -relation  $r : X \leftrightarrow Y$  to the  $V$ -relation  $r \times H : X \times H \leftrightarrow Y \times H$  with

$$r \times H((x, a), (y, b)) = \begin{cases} r(x, y) & \text{if } a = b, \\ \perp & \text{if } a \neq b. \end{cases}$$

In particular,  $(\mathbb{H}, 2)$ -categories can be interpreted as  $H$ -labelled ordered sets and equivariant maps.

For every quantale  $V$  and every  $v : 1 \rightarrow V$ , the diagrams

$$\begin{array}{ccc} V \times V \times H & \xrightarrow{\wedge \times 1_H} & V \times H \\ \pi_{1,2} \downarrow & & \downarrow \xi = \pi_1 \\ V \times V & \xrightarrow{\wedge} & V \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 \times H & \xrightarrow{v \times 1_H} & V \times H \\ ! \downarrow & & \downarrow \xi \\ 1 & \xrightarrow{v} & V \end{array}$$

commute, therefore we obtain:

**Theorem 7.11.** *For every quantale  $V$  satisfying Assumption 5.5 (2), the category  $(\mathbb{H}, V)\text{-Cat}$  is weakly cartesian closed.*

## REFERENCES

- [AHS90] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories: The joy of cats*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1990. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507.
- [AR18] Jiří Adámek and Jiří Rosický. How nice are free completions of categories? Technical report, 2018, [arXiv:1806.02524](https://arxiv.org/abs/1806.02524) [math.CT].
- [BBS04] Andrej Bauer, Lars Birkedal, and Dana S. Scott. Equilogical spaces. *Theoretical Computer Science*, 315(1):35–59, 2004.
- [CCH15] Dimitri Chikhladze, Maria Manuel Clementino, and Dirk Hofmann. Representable  $(\mathbb{T}, \mathcal{V})$ -categories. *Applied Categorical Structures*, 23(6):829–858, January 2015, eprint: <http://www.mat.uc.pt/preprints/ps/p1247.pdf>.
- [CH03] Maria Manuel Clementino and Dirk Hofmann. Topological features of lax algebras. *Applied Categorical Structures*, 11(3):267–286, June 2003, eprint: <http://www.mat.uc.pt/preprints/ps/p0109.ps>.
- [CH06] Maria Manuel Clementino and Dirk Hofmann. Exponentiation in  $V$ -categories. *Topology and its Applications*, 153(16):3113–3128, October 2006.
- [CH09] Maria Manuel Clementino and Dirk Hofmann. Lawvere completeness in topology. *Applied Categorical Structures*, 17(2):175–210, August 2009, [arXiv:0704.3976](https://arxiv.org/abs/0704.3976) [math.CT].
- [CH17] Maria Manuel Clementino and Dirk Hofmann. The Rise and Fall of  $V$ -functors. *Fuzzy Sets and Systems*, 321:29–49, August 2017, eprint: <http://www.mat.uc.pt/preprints/ps/p1606.pdf>.
- [CHJ14] Maria Manuel Clementino, Dirk Hofmann, and George Janelidze. The monads of classical algebra are seldom weakly cartesian. *Journal of Homotopy and Related Structures*, 9(1):175–197, November 2014, eprint: <http://www.mat.uc.pt/preprints/ps/p1246.pdf>.
- [CHT03] Maria Manuel Clementino, Dirk Hofmann, and Walter Tholen. Exponentiability in categories of lax algebras. *Theory and Applications of Categories*, 11(15):337–352, 2003, eprint: <http://www.mat.uc.pt/preprints/ps/p0302.pdf>.
- [CT03] Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory—a common approach. *Journal of Pure and Applied Algebra*, 179(1-2):13–47, April 2003.
- [CVO17] Eva Colebunders and Karen Van Opdenbosch. Topological properties of non-Archimedean approach spaces. *Theory and Applications of Categories*, 32(41):1454–1484, 2017.
- [Fau55] William M. Faucett. Compact semigroups irreducibly connected between two idempotents. *Proceedings of the American Mathematical Society*, 6(5):741–747, May 1955.

- [Her00] Claudio Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, May 2000.
- [Her01] Claudio Hermida. From coherent structures to universal properties. *Journal of Pure and Applied Algebra*, 165(1):7–61, November 2001.
- [Hof06] Dirk Hofmann. Exponentiation for unitary structures. *Topology and its Applications*, 153(16):3180–3202, October 2006.
- [Hof07] Dirk Hofmann. Topological theories and closed objects. *Advances in Mathematics*, 215(2):789–824, November 2007.
- [Hof11] Dirk Hofmann. Injective spaces via adjunction. *Journal of Pure and Applied Algebra*, 215(3):283–302, March 2011, [arXiv:0804.0326](#) [math.CT].
- [Hof14] Dirk Hofmann. The enriched Vietoris monad on representable spaces. *Journal of Pure and Applied Algebra*, 218(12):2274–2318, December 2014, [arXiv:1212.5539](#) [math.CT].
- [HR13] Dirk Hofmann and Carla D. Reis. Probabilistic metric spaces as enriched categories. *Fuzzy Sets and Systems*, 210:1–21, January 2013, [arXiv:1201.1161](#) [math.GN].
- [HS15] Dirk Hofmann and Gavin J. Seal. Exponentiable approach spaces. *Houston Journal of Mathematics*, 41(3):1051–1062, 2015, [arXiv:1304.6862](#) [math.GN].
- [HST14] Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors. *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, volume 153 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, July 2014. Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.
- [HT10] Dirk Hofmann and Walter Tholen. Lawvere completion and separation via closure. *Applied Categorical Structures*, 18(3):259–287, November 2010, [arXiv:0801.0199](#) [math.CT].
- [Law73] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43(1):135–166, December 1973. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.
- [LBKR12] Joachim Lambek, Michael Barr, John F. Kennison, and Robert Raphael. Injective hulls of partially ordered monoids. *Theory and Applications of Categories*, 26(13):338–348, 2012.
- [Low97] Robert Lowen. *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1997.
- [MS57] Paul S. Mostert and Allen L. Shields. On the structure of semi-groups on a compact manifold with boundary. *Annals of Mathematics. Second Series*, 65(1):117–143, January 1957.
- [MST06] Mojgan Mahmoudi, Christoph Schubert, and Walter Tholen. Universality of coproducts in categories of lax algebras. *Applied Categorical Structures*, 14(3):243–249, June 2006.
- [Rib18] Willian Ribeiro. On generalized equilogical spaces. Technical Report 18-50, Department of Mathematics, University of Coimbra, 2018, [arXiv:1811.08240](#) [math.CT].
- [Ros99] Jiří Rosický. Cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 142(3):261–270, October 1999.
- [Sea10] Gavin J. Seal. Order-adjoint monads and injective objects. *Journal of Pure and Applied Algebra*, 214(6):778–796, June 2010.
- [Tho09] Walter Tholen. Ordered topological structures. *Topology and its Applications*, 156(12):2148–2157, July 2009.

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