LIMITS IN CATEGORIES OF VIETORIS COALGEBRAS

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Abstract. Motivated by the need to reason about hybrid systems, we study limits in categories of coalgebras whose underlying functor is a Vietoris polynomial one — intuitively, the topological analogue of a Kripke polynomial functor. Among other results, we prove that every Vietoris polynomial functor admits a final coalgebra if it respects certain conditions concerning separation axioms and compactness. When the functor is restricted to some of the categories induced by these conditions the resulting categories of coalgebras are even complete.

As a practical application, we use these developments in the specification and analysis of non-deterministic hybrid systems, in particular to obtain suitable notions of stability, and behaviour.

1. Introduction

1.1. **Motivation and context.** Coalgebras [Rut00, Adá05, Jac12] form a powerful theory of state-based transition systems where definitions and results are formulated at a high level of genericity that covers several families of systems at once, from deterministic automata and Kripke frames to different kinds of probabilistic models. Traditionally, these formulations are elaborated in a set-based context; *i.e.* no further structure in the system's state space than that of a set is assumed. In many cases, however, a switch of context is needed. The projects on the coalgebraic foundations of stochastic systems, where the Giry functor and measurable spaces have a central role (*cf.* [Vig05, Pan09, Dob09]), are evident examples of this. Research on coalgebras over Stone spaces (*e.g.* [KKV04, BFV10, VV14]) and coalgebras over pseudometric spaces [BBKK14] forms equally important cases. In [KKV04, BFV10, VV14], the aim is to provide a suitable coalgebraic semantics for finitary modal logics by taking advantage of a Vietoris functor, while in [BBKK14] is to introduce a notion of distance between states.

In this paper our focus is on coalgebras over arbitrary *topological spaces*, because we believe that they provide important mechanisms to the design and analysis of *hybrid systems* [Tab09, Alu15, Sta01]. Briefly put, hybrid systems are those that possess both discrete and continuous behaviour, a result of the complex interaction between digital devices, and physical processes like velocity, movement, temperature, and time. Two recurring examples are the cruise control system, basically a digital device with influence over velocity, and the *bouncing ball*. In the latter, movement and velocity have a continuous nature, while the impact on the ground is assumed to be a discrete event that instantaneously alters the current velocity. As we will see in the following sections, such an interaction between discrete and continuous behaviour calls for a shift from the set-based setting to richer contexts, in particular to topological ones so that suitable notions of stability, bisimulation, and behaviour can be obtained. These are the practical motivations for the theoretical results that this paper provides. But we stress that coalgebras over topological spaces have the potential for much more – the works [Vig05, Pan09, Dob09, BFV10, VV14, BBKK14, KKV04], for example, elegantly attest this. Our results are therefore applicable to a much broader context than that of hybrid systems.

Each functor $F : \mathsf{C} \to \mathsf{C}$ induces a category of coalgebras $\mathsf{CoAlg}(F)$ that can be seen as a framework for a particular family of state-based transition systems, whose transition type is determined by $F : \mathsf{C} \to \mathsf{C}$ (*cf.* [Rut00]). The powerset $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, for example, often associated with non-deterministic behaviour, gives rise to Kripke frames.

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In such a context, the systematic study of (co)limits in categories of coalgebras is a natural research line. In fact, final coalgebras, which form a specific type of limit, are often searched for, as they encode a canonical notion of behaviour for all F-coalgebras. Another special kind of limit, equalisers of coalgebras, is extensively used in coalgebraic specification (*cf.* [Rut00, Adá05]). It provides a notion of subsystem, and is essential to characterise a system induced by a set of coequations.

1.2. **Contributions and related work.** As mentioned before, this paper concerns coalgebras over arbitrary topological spaces. More concretely, coalgebras whose underlying functor is defined over the category Top of topological spaces and continuous maps. Analogously to what has already been done in Set (*e.g.* [Rut00, GS01]), the aim here is to investigate the existence of limits in categories of coalgebras whose underlying functor is ***Vietoris polynomial*** — the topological analogue of a Kripke polynomial functor. The former is called 'Vietoris polynomial' because it arises from the composition of different Vietoris functors [Vie22, Mic51, CT97] (the topological analogues of the powerset functor) with polynomial functors over Top. To keep the nomenclature simple, we call every coalgebra whose underlying functor is Vietoris polynomial a ***Vietoris coalgebra***.

As composites of constant, (co)product, identity, and powerset functors, Kripke polynomial functors have long since been recognised as a particularly relevant class of functors (*cf.* [Rut00, BRS09, KKV04]). They are intuitive and the corresponding coalgebras subsume several types of state-based systems. Moreover, they are well-behaved in regard to the existence of limits in their categories of coalgebras if the powerset functor is submitted to certain cardinality restrictions. We will see that somewhat similar results hold for Vietoris polynomial functors as well. Actually, an instance of a Vietoris functor, which we call ***compact Vietoris functor***, has already been studied multiple times in the coalgebraic setting (*e.g.* [KKV04, BFV10, VV14, DDG16]), and will appear in a book on coalgebras that is currently in preparation [AMM16]. In particular, [KKV04] shows that compact Vietoris polynomial functors in the category Stone of Stone spaces and continuous maps admit a final coalgebra. Also, document [DDG16] presents a theorem that can be generalised to show that the compact Vietoris functor in the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps, admits a final coalgebra. In fact, this generalised result is also implicitly mentioned in [Eng89, page 245]. Related to this, but in a broader setting, we collect a number of results scattered in coalgebraic and topological literature, and

- add to this collection some results of our own. In particular, we generalise Hughes' theorem (Theorem 2.14) and prove that, under certain conditions, functors between categories of coalgebras are *topological*. Topological functors have powerful properties such as the existence of left and right adjoints, lifting of limits, and lifting of factorisations [AHS90].
- This collection of results allows us to obtain several new results about limits in categories of Vietoris coalgebras. For example, that categories of polynomial coalgebras over Top are complete, and that categories of compact Vietoris coalgebras over $\mathsf{CompHaus}$ are complete as well. Using in particular [Zen70, Lemma B], we also show that categories of compact Vietoris coalgebras are complete in the category Haus of Hausdorff spaces and continuous maps. Moreover we will see that all categories of Vietoris coalgebras over Top have equalisers.
- We then take advantage of the limit-preserving properties of the inclusion functors $\mathsf{CompHaus} \to \mathsf{Top}$ and $\mathsf{Haus} \to \mathsf{Top}$ to show that every compact Vietoris polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ that can be restricted either to $\mathsf{CompHaus}$ or Haus admits a final coalgebra.

Our setting is a broader one also because we consider different instances of Vietoris functors, a particular case being what we call the ***lower Vietoris functor***, studied in a coalgebraic setting in [BKR07].

- We will show that every lower Vietoris polynomial functor behaves well in the category $\mathsf{StablyComp}$ of stably compact spaces and spectral maps. In particular, that its category of coalgebras is complete.
- In order to extend these results to more variants of Vietoris functors, we study the existence of adjunctions between categories of coalgebras. One positive result is that, assuming the existence

of a monomorphic natural transformation between the underlying functors, such an adjunction exists under mild conditions.

To illustrate the practical side of these developments, and, more generally, the potential of coalgebras over Top to the design and analysis of hybrid systems, we argue that the coalgebraic specification in Set of the bouncing ball has some deficiencies. Among them, the incapability to reason about the system's stability, and the non-existence of a suitable final coalgebra if non-determinism is taken into account. We will see that these issues can be solved, to some extent, by adopting the category Top as the underlying semantic universe.

1.3. **Roadmap.** The ensuing section introduces some categorial notions, provides an overview, and extends some results about limits in categories of coalgebras. Then, it formally reviews the concept of Vietoris coalgebra and different instances of Vietoris functors — as already mentioned, our agenda has a broader scope than most coalgebraic literature on Vietoris functors, which mainly focuses on one specific case.

Section 3 starts with our study about polynomial coalgebras over Top, and topological functors between categories of coalgebras. Then, it adds two instances of Vietoris functors (the lower and the compact) to the mix which, as expected, introduce a number of difficulties. A number of topological concepts are recalled at this point to help us achieve some of the results mentioned above.

Section 4 explores the existence of adjunctions between categories of coalgebras induced by natural transformations relating functors on the underlying categories. As already stated, this allows to extend the results of the previous section to subfunctors of Vietoris polynomial ones, thus covering at once several variants of Vietoris functors.

Section 5 illustrates an application of this work to the design of hybrid systems. Finally, Section 6 suggests possible research lines for future work and concludes.

We assume that the reader has basic knowledge of category theory [Mac71, AHS90], topology [Kel55, Gou13], and coalgebras [Rut00, Adá05, Jac12].

2. Preliminaries

2.1. **Categorial notions.** Some categorial notions that the reader may not frequently meet will be used. This section provides a brief overview about them.

Definition 2.1. A diagram $D : I \to \mathsf{C}$ is said to be ***codirected*** whenever I is a codirected partially ordered set, that is, I is non-empty and for all $i, j \in I$ there is some $k \in I$ with $k \to i$ and $k \to j$. A cone for a codirected diagram is called a ***codirected cone***. In particular, a limit of a codirected diagram is called ***codirected***.

Example 2.2. Inverse sequence (or ω^{op}) diagrams, which have the shape depicted below, are codirected.

$$\cdot \longleftarrow \cdot \longleftarrow \cdot \longleftarrow \dots$$

Inverse sequence diagrams have a central role in showing that a given functor admits a final coalgebra (see Theorem 2.10).

Remark 2.3. The codirected limit of a diagram $D : I \to \mathsf{Set}$ is given by the subset

$$\left\{ (x_i)_{i \in I} \in \prod_{i \in I} D(i) \mid \forall j \to i \in I, D(j \to i)(x_j) = x_i \right\}$$

of the product $\prod_{i \in I} D(i)$.

Definition 2.4. A category C is said to be ***connected*** if it is non-empty and every two objects $A, B \in \mathsf{C}$ can be connected by a finite zig-zag of morphisms as depicted below.

$$A \leftarrow \cdot \rightarrow \cdots \leftarrow \cdot \rightarrow B$$

A diagram $D : I \to \mathsf{C}$ is called ***connected diagram*** if I is connected, and a limit of D is called ***connected limit*** if $D : I \to \mathsf{C}$ is connected.

Examples 2.5. Equalisers and codirected limits are two examples of connected limits.

We will see in the following section that polynomial functors over Top preserve connected limits, in particular codirected ones.

Definition 2.6. Let $F : \mathsf{A} \to \mathsf{B}$ be a functor. A cone $\mathcal{C} = (C \to X_i)_{i\in I}$ in A is said to be ***initial with respect to*** F if for every cone $\mathcal{D} = (D \to X_i)_{i\in I}$ and every morphism $h : FD \to FC$ such that $F\mathcal{D} = F\mathcal{C} \cdot h$, there exists a unique A-morphism $\bar{h} : D \to C$ with $\mathcal{D} = \mathcal{C} \cdot \bar{h}$ and $h = F\bar{h}$.

We simply say that the cone is ***initial*** whenever no ambiguities arise.

Examples 2.7.
(1) A cone $(f_i : X \to X_i)$ in Top is initial with respect to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ if and only if X is equipped with the so called initial (weak) topology. Explicitly, the topology generated by the subbasis

$$f_i^{-1}(U) \qquad (i \in I, U \subseteq X_i \text{ open}).$$

(2) In the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps, a monocone is initial in Top (*cf.* [Gou13, Theorem 4.4.27]). Interestingly, the converse also holds, as a initial cone in Top whose domain is a T_0 space is necessarily mono.

Remark 2.8. In Example 2.7(1) the subbasis is actually a basis if the cone is codirected.

Theorem 2.9 ([AHS90, Proposition 13.15])**.** *Let $F : \mathsf{A} \to \mathsf{B}$ be a limit preserving faithful functor and $D : I \to \mathsf{A}$ a diagram. A cone $\mathcal{C}$ for D is a limit of D if and only if the cone $F\mathcal{C}$ is a limit of FD and $\mathcal{C}$ is initial with respect to F.*

2.2. **Limits in categories of coalgebras.** Let $F : \mathsf{C} \to \mathsf{C}$ be an arbitrary functor. Then, dually to the algebraic case, one can easily show that colimits in $\mathsf{CoAlg}(F)$ exist if they do so in C (*cf.* [Rut00, Adá05]). The story about limits in categories of coalgebras is, however, more complex. In this subsection we review some well-known results on this topic, a special focus being given to those more relevant to the paper. We start at a generic level, with the following two theorems (*cf.* [Rut00, Adá05]).

Theorem 2.10. *Let C be a category with a final object 1 and $F : \mathsf{C} \to \mathsf{C}$ a functor. If the category C has a limit L of the diagram*

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots$$

and F preserves this limit, then the canonical isomorphism $L \to FL$ is a final F-coalgebra.

Theorem 2.11. *Assume that $F : \mathsf{C} \to \mathsf{C}$ preserves limits of a certain type. Then the forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{C}$ creates limits of the same type.*

An important consequence of the last theorem is that $\mathsf{CoAlg}(F)$ has all types of limit that C has and that the functor $F : \mathsf{C} \to \mathsf{C}$ preserves. Unfortunately, as we will witness later, this assumption is often too strong. Resorting to the notion of covarietor, the following results will be more helpful.

Definition 2.12. A functor $F : \mathsf{C} \to \mathsf{C}$ is said to be a ***covarietor*** if the canonical forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{C}$ is left adjoint.

This adjoint situation allows to take advantage of the theory of (co)monads regarding (co)completeness of Eilenberg-Moore (co)algebras to derive the following theorem (*cf.* [Lin69]). See [Adá05, Remark 3.12] for more details.

Theorem 2.13. *Let F be a covarietor over a complete category. If $\mathsf{CoAlg}(F)$ has equalisers then $\mathsf{CoAlg}(F)$ is complete.*

Related to this, Hughes proved the following theorem in [Hug01, Theorem 2.4.2].

Theorem 2.14. *Let* C *be regularly wellpowered, cocomplete, and possess equalisers. Moreover, assume that it has an (Epi, RegMono)-factorisation structure, and that the functor* $F : \mathsf{C} \to \mathsf{C}$ *preserves regular monomorphisms. Then* $\mathsf{CoAlg}(F)$ *has equalisers.*

Using Theorem 2.13, one can then easily deduce the following corollary.

Corollary 2.15. *If the conditions in the last theorem hold and, additionally,* C *is complete and* F *is a covarietor, then the category* $\mathsf{CoAlg}(F)$ *is complete.*

We refer the interested reader to other results on limits in categories of coalgebras. In particular, the work of Kurz [Kur01], which shows that $\mathsf{CoAlg}(F)$ is complete whenever it has a suitable factorisation structure, F is a covarietor, and C is complete; document [GS01], where the authors study the existence of equalisers and products in categories of coalgebras over Set; and the documents [PW98, GS01], where the existence of limits is studied under the assumption of F being bounded.

To close this section, we provide an improvement to Hughes' theorem. We start with notation.

Definition 2.16. For a small category I, a ***cone for*** I in a category C is given by a functor $D : I \to \mathsf{C}$ together with a cone $(X \to D(i))_{i\in I}$ for D. Given a class $\mathcal{M}$ of cones for I, the category C is called ***$\mathcal{M}$-wellpowered*** if for every functor $D : I \to \mathsf{C}$ there is up to isomorphism only a set of cones for D in $\mathcal{M}$.

Our first lemma is in the spirit of [AHS90, Section 12] and shows that "cocompleteness almost implies completeness".

Lemma 2.17. *Let* C *be a cocomplete category and* I *a small category. Furthermore, let* E *be a class of* C*-morphisms and* $\mathcal{M}$ *be a class of cones for* I *in* C*. If* C *is* $\mathcal{M}$*-wellpowered and every cone for* I *has a* $(E, \mathcal{M})$*-factorisation, then* C *has limits of shape* I*.*

Proof. We will show that the diagonal functor

$$\Delta : \mathsf{C} \to \mathsf{C}^I$$

has a right adjoint, using Freyd's General Adjoint Functor Theorem (see [Mac71]). By assumption, C is cocomplete and the functor Δ clearly preserves colimits, so we just need to show that the Solution Set Condition holds. In this context it unfolds to the following condition: for every functor $D : I \to \mathsf{C}$, there is a set $\mathcal{S}$ of cones for D such that every cone $(f_i : C \to D(i))_{i\in I}$ for D factors through a cone in $\mathcal{S}$.

Since C is $\mathcal{M}$-wellpowered we have, by assumption, a set $\mathcal{S}$ of representants for D in $\mathcal{M}$. Moreover C has a $(E, \mathcal{M})$-factorisation system for I, which means that the cone $(f_i : C \to D(i))_{i\in I}$ can be factorised as depicted below

$$\begin{tikzcd} C \arrow[rr, "f_i"] \arrow[dr, "e"'] & & D(i) \\ & A \arrow[ur, "g_i"'] & \end{tikzcd}$$

with the cone $(g_i : A \to D(i))_{i\in I}$ in $\mathcal{S}$. □

The factorisation system assumed in this lemma may appear to be rather unconventional, but, as the following remarks will show, it actually emerges from mild conditions.

Remark 2.18. Consider a category C equipped with classes E and M of morphisms so that E is contained in the class of epimorphisms of C, every morphism in C has a (E, M)-factorisation and C is M-wellpowered. Under additional assumptions, such factorisations can be extended to cones for I. To be more concrete:

(1) Assume that C has products. Then we put

$$\mathcal{M} = \left\{ \text{all cones } (f_i : X \to D(i))_{i\in I} \text{ for } I \text{ where } \langle f_i \rangle_{i\in I} : X \to \prod_{i\in I} D(i) \text{ is in } M \right\}.$$

Clearly, every cone for I is $(E, \mathcal{M})$-factorisable (see [AHS90, Proposition 15.19]), and C is $\mathcal{M}$-wellpowered.

(2) In order to relate the previous lemma with Hughes' theorem, assume that $I = \{1 \rightrightarrows 2\}$. The class of cones

$$\mathcal{M} = \{\text{all cones } (f_i : X \to D(i))_{i\in I} \text{ for } I \text{ with } f_1 \text{ in } M\},$$

makes every cone for I $(E, \mathcal{M})$-factorisable and the category C is $\mathcal{M}$-wellpowered.

In both cases, if the category C is (E, M)-structured then it is $(E, \mathcal{M})$-structured as well.

Finally, we apply the results above to categories of coalgebras.

Theorem 2.19. *Let $F : \mathsf{C} \to \mathsf{C}$ be an endofunctor over a cocomplete category C and let I be a small category. If C is $(E, \mathcal{M})$-structured for cones for I, $\mathcal{M}$-wellpowered and F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$, then $\mathsf{CoAlg}(F)$ has limits of shape I.*

Proof. The assumptions guarantee that the factorisation system in C lifts to the category $\mathsf{CoAlg}(F)$ (*cf.* [Adá05, Che14]). The claim then follows from Lemma 2.17. □

Let us now relate in a more precise manner the previous theorem with Hughes' theorem.

Theorem 2.20. *Let $F : \mathsf{C} \to \mathsf{C}$ be an endofunctor over a cocomplete category C. If C is regularly wellpowered, has an (Epi, RegMono)-factorisation structure and $F : \mathsf{C} \to \mathsf{C}$ preserves regular monomorphisms, then $\mathsf{CoAlg}(F)$ has equalisers.*

Proof. Let $I = \{1 \rightrightarrows 2\}$ and use Remark 2.18(2) to provide a $(E, \mathcal{M})$-factorisation system for cones for I. The category C is clearly $\mathcal{M}$-wellpowered and by a simple reasoning one shows that F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$. Now apply Theorem 2.19. □

The last result shows that Hughes' assumption of C having equalisers is not necessary. Another interesting point is the ability that we gain to reason not just about equalisers but any type of limit. We will take advantage of this generalisation in the next section (see Corollary 3.16).

Note also that the following corollaries can be obtained almost for free.

Corollary 2.21. *Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor that preserves monocones of a certain type. Then the category $\mathsf{CoAlg}(F)$ has limits of the same type.*

Recall that Top is an (Epi,initial monocones)-category and an (RegEpi,monocones)-category (*cf.* [AHS90, Examples 15.3 (6)]). The following result can then be derived.

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Corollary 2.22. *Let $F : \mathsf{Top} \to \mathsf{Top}$ be a functor that preserves either small monocones or small initial monocones of a certain type. Then the category $\mathsf{CoAlg}(F)$ has limits of the same type.*

2.3. **Vietoris polynominal functors.** Although traditionally considered in Set (*e.g.* [BRS09, Jac12]), the notion of a polynomial functor can be formally defined at a more generic level.

Definition 2.23. Let C be a category with (co)products. We call a functor $F : \mathsf{C} \to \mathsf{C}$ ***polynomial*** if it can be recursively defined from the grammar below

$$F ::= F + F \mid F \times F \mid A \mid \mathrm{Id}$$

where A corresponds to an object of C.

Remark 2.24. Alternatively, one can define the class of polynomial functors as the smallest class of functors $F : \mathsf{C} \to \mathsf{C}$ that contains the identity functor, all constant functors, and is closed under products and sums of functors. Here, for functors $F, G : \mathsf{C} \to \mathsf{C}$, the product of F and G, and the sum of F and G are, respectively, the composites

$$\mathsf{C} \xrightarrow{\langle F,G\rangle} \mathsf{C} \times \mathsf{C} \xrightarrow{\times} \mathsf{C}, \text{ and } \mathsf{C} \xrightarrow{\langle F,G\rangle} \mathsf{C} \times \mathsf{C} \xrightarrow{+} \mathsf{C}.$$

Note that if the functors $F, G : \mathsf{C} \to \mathsf{C}$ preserve limits of a certain type the functor $F \times G : \mathsf{C} \to \mathsf{C}$ preserves limits of the same type as well. Note also that

Proposition 2.25. *The functor* $(+) : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ *preserves connected limits.*

Proof. It is well-known that the functor $(+) : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$ preserves connected limits. Then observe that $(+) : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ preserves initial cones and apply Theorem 2.9. □

Corollary 2.26. *If the functors* $F, G : \mathsf{Top} \to \mathsf{Top}$ *preserve connected limits the functor* $F + G : \mathsf{Top} \to \mathsf{Top}$ *preserves connected limits as well.*

In the set-based context, the powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is traditionally used in conjunction with polynomial functors to bring non-deterministic behaviour into the scene, the resulting functor being a so called ***Kripke polynomial functor***. The situation is more complex in the topological context because a number of functors can be seen as 'analogues' of the powerset. Most of them have their roots in the Hausdorff metric (*cf.* [Pom05, Hau14]) and in Vietoris' "Bereiche zweiter Ordnung" [Vie22]. Informally, we call them ***Vietoris functors***. The remainder of this section provides some details about them.

Consider a compact Hausdorff space X, the ***classic Vietoris space*** $\mathcal{V}X$ [Vie22] consists of the set of all closed subsets of X, *i.e.*

$$\mathcal{V}X = \{K \subseteq X \mid K \text{ is closed}\}$$

equipped with the 'hit-and-miss topology' generated by the subbasis of sets of the form

$$U^{\Diamond} = \{A \in \mathcal{V}X \mid A \cap U \neq \varnothing\} \qquad (\text{"}A \text{ hits } U\text{"}),$$
$$U^{\Box} = \{A \in \mathcal{V}X \mid A \subseteq U\} \qquad (\text{"}A \text{ misses } X \setminus U\text{"}),$$

where $U \subseteq X$ is open. Nowadays there are several well-studied variants of this archetype that give rise to endofunctors over specific subcategories of Top. The interested reader will find in [Mic51] and [CT97] more details about these constructions. For now, we concentrate on two particular cases, described below.

Examples 2.27.
(1) For a topological space X, define $\mathcal{V}X = \{K \subseteq X \mid K \text{ is compact }\}$ with the topology generated by the sets $U^{\Box}$ and $U^{\Diamond}$, with U ranging over all open subsets $U \subseteq X$. Then, given a continuous map $f : X \to Y$, define $\mathcal{V}f : \mathcal{V}X \to \mathcal{V}Y$ as $\mathcal{V}f(A) = f[A]$. We call this variant ***compact Vietoris functor***. It is well-known that $\mathcal{V}X$ is compact Hausdorff whenever X is. In fact, for compact Hausdorff spaces this construction coincides with the classic one [Vie22].
(2) For a topological space X, define $\mathcal{V}X = \{K \subseteq X \mid K \text{ is closed}\}$ with the topology generated by the sets $U^{\Diamond}$, with U ranging over all open subsets $U \subseteq X$. Then, given a continuous map $f : X \to Y$, define $\mathcal{V}f : \mathcal{V}X \to \mathcal{V}Y$ as $\mathcal{V}f(A) = \overline{f[A]}$, where $\overline{f[A]}$ denotes the closure of $f[A]$. This variant is called ***lower Vietoris functor***.

Remark 2.28. The classic Vietoris construction, with closed sets, does not define an obvious functor on Top. That is, adding the sets $U^{\Box}$ to the subbasis of Example 2.27 (2) does not define a functor. To see why, consider the set $\{1, 2, 3\}$ equipped with the topology generated by the sets $\{1, 2\}$ and $\{2, 3\}$. For the subspace embedding $i : \{1, 2\} \to \{1, 2, 3\}$, $(Vi)^{-1}[\{1, 2\}^{\Box}] = \{\varnothing, \{1\}\}$. However, every open set of $\mathcal{V}\{1, 2\}$ that contains $\{1\}$ contains $\{1, 2\}$.

A number of projects on (coalgebraic) modal logic studied the compact Vietoris functor in the category of Stone spaces (*e.g.* [KKV04, VV14]) and in the category of compact Hausdorff spaces [BBH12]. The second case was explored by [CLP91, Pet96, BKR07] in the context of Priestley spaces.

Definition 2.29. Let $\mathcal{V} : \mathsf{Top} \to \mathsf{Top}$ be the lower Vietoris functor. We call a functor $F : \mathsf{Top} \to \mathsf{Top}$ ***lower Vietoris polynomial*** if it can be recursively defined from the grammar below.

$$F ::= F + F \mid F \times F \mid A \mid \mathrm{Id} \mid \mathcal{V}$$

Similarly, if we consider the compact Vietoris functor $\mathcal{V} : \mathsf{Top} \to \mathsf{Top}$ *in lieu* of the lower one, then we speak of a ***compact Vietoris polynomial*** functor.

3. On limits in categories of Vietoris coalgebras

3.1. **Polynomial functors in Top.** Using standard results, we now show that for a polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ the associated category of coalgebras $\mathsf{CoAlg}(F)$ is complete. A useful fact for this proof is that the category Top is (co)complete (*cf.* [AHS90]). Moreover, note that

Theorem 3.1. *All polynomial functors $F : \mathsf{Top} \to \mathsf{Top}$ preserve connected limits.*

Proof. Clearly the identity functor $\mathrm{Id} : \mathsf{Top} \to \mathsf{Top}$ preserves all limits, and the constant functor $A : \mathsf{Top} \to \mathsf{Top}$ trivially preserves connected limits. The claim now follows from Remark 2.24 and Corollary 2.26. □

From the theorem above one can derive the following results in a straightforward manner.

Proposition 3.2. *All polynomial functors $F : \mathsf{Top} \to \mathsf{Top}$ preserve regular monomorphisms.*

Proof. First note that the diagrams associated with equalisers are connected. Then, recall that a regular monomorphism is an equaliser of a pair of morphisms. □

Theorem 3.3. *All polynomial functors $F : \mathsf{Top} \to \mathsf{Top}$ are covarietors.*

Proof. Since a polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ preserves connected limits (Theorem 3.1) it preserves the codirected ones as well. The claim is then a direct consequence of [Bar93, Theorem 2.1]. □

In regard to equalisers in $\mathsf{CoAlg}(F)$, one can easily show that the necessary requirements to apply Theorem 2.14 are met. Actually, it is well-known that the category Top is regularly wellpowered (*cf.* [AHS90]), and we already saw that it is (co)complete. Moreover, it has an (Epi, RegMono)-factorisation structure (*cf.* [AHS90]). Therefore,

Corollary 3.4. *If $F : \mathsf{Top} \to \mathsf{Top}$ is a polynomial functor, the category $\mathsf{CoAlg}(F)$ has equalisers.*

Proof. A direct consequence of Theorem 2.14 and Proposition 3.2. □

Theorem 3.5. *If $F : \mathsf{Top} \to \mathsf{Top}$ is a polynomial functor, the category $\mathsf{CoAlg}(F)$ is complete.*

Proof. Observe that F is a covarietor (Theorem 3.3), and that the category $\mathsf{CoAlg}(F)$ has equalisers (Corollary 3.4). Then, apply Theorem 2.13. □

We will now use 'less standard' results to go further than the previous theorem. More concretely, we will show that not only is $\mathsf{CoAlg}(F)$ complete but also that there is a functor with powerful properties from $\mathsf{CoAlg}(F)$ to the analogous category of coalgebras over Set. By going further we also mean that the results that we will introduce next may be used in categories different than Top, prime examples are the category of preordered sets Ord and the category of pseudometric spaces PMet.

The general idea is that starting with a category B with good properties and assuming the existence of a functor $\mathsf{A} \to \mathsf{B}$ that lifts these properties to a category A, there will often be a functor $\mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$ with the same lifting properties than $\mathsf{A} \to \mathsf{B}$ for functors $\overline{F} : \mathsf{A} \to \mathsf{A}$, $F : \mathsf{B} \to \mathsf{B}$ making the diagram below commute.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\overline{F}"] \arrow[d, "U"'] & \mathsf{A} \arrow[d, "U"] \\ \mathsf{B} \arrow[r, "F"'] & \mathsf{B} \end{tikzcd}$$

The following definition recalls the notion of topological functor, which lifts several properties of a category.

Definition 3.6. A functor $U : \mathsf{A} \to \mathsf{B}$ is called ***topological*** if every cone $\mathcal{C} = (X \to UX_i)_{i\in I}$ in B has a U-initial lifting, *i.e.* a initial cone $\mathcal{D} = (A \to X_i)_{i\in I}$ with respect to $U : \mathsf{A} \to \mathsf{B}$ such that $\mathcal{C} = U\mathcal{D}$.

Remark 3.7. Every topological functor is both left and right adjoint, lifts limits and certain types of factorisations (see [AHS90]).

Proposition 3.8. *Consider two categories* A, B *a functor* $U : \mathsf{A} \to \mathsf{B}$*, endofunctors* $\overline{F} : \mathsf{A} \to \mathsf{A}$*,* $F : \mathsf{B} \to \mathsf{B}$*, and a natural transformation*

$$\delta : U\overline{F} \to FU.$$

Then, there is a functor $\overline{U} : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(\overline{F})$ *defined by the equations*

$$\overline{U}(X, c) = (UX, \delta_X \cdot Uc), \qquad \overline{U}f = Uf$$

that makes the diagram below commute.

$$\begin{tikzcd}
\mathsf{CoAlg}(\overline{F}) \arrow[r] \arrow[d, "\overline{U}"'] & \mathsf{A} \arrow[d, "U"] \\
\mathsf{CoAlg}(F) \arrow[r] & \mathsf{B}
\end{tikzcd}$$

Moreover,

Proposition 3.9. *If the functor* $U : \mathsf{A} \to \mathsf{B}$ *is faithful, then the induced functor* $\overline{U} : \mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$ *is faithful.*

Lemma 3.10. *Assume that the natural transformation* $\delta : \overline{F}U \to UF$ *is mono and* U *is faithful. Let* $(f_i : (X, c) \to (Y_i, d_i))_{i\in I}$ *be a cone in* $\mathsf{CoAlg}(\overline{F})$*, and* $(f_i : X \to Y_i)_{i\in I}$ *be initial with respect to* $U : \mathsf{A} \to \mathsf{B}$*. Then, the cone* $(f_i : (X, c) \to (Y_i, d_i))_{i\in I}$ *is initial with respect to the functor* $\overline{U} : \mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$*.*

Proof. Let $(f_i : (X, c) \to (Y_i, d_i))_{i\in I}$ be a cone in $\mathsf{CoAlg}(\overline{F})$ and $(f_i : X \to Y_i)_{i\in I}$ be initial with respect to $U : \mathsf{A} \to \mathsf{B}$. Then, consider another cone $(g_i : (Z, e) \to (Y_i, d_i))_{i\in I}$ in $\mathsf{CoAlg}(\overline{F})$ and assume that its $\overline{U}$-image is factorised as shown by the diagram below.

$$\begin{tikzcd}
\overline{U}(Z, e) \arrow[d, "h"'] \arrow[dr, "\overline{U}g_i"] & \\
\overline{U}(X, c) \arrow[r, "\overline{U}f_i"'] & \overline{U}(Y_i, d_i)
\end{tikzcd}$$

The forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{B}$ yields the following factorisation of the cone $(Ug_i : UZ \to UY_i)_{i\in I}$.

$$\begin{tikzcd}
UZ \arrow[d, "h"'] \arrow[dr, "Ug_i"] & \\
UX \arrow[r, "Uf_i"'] & UY_i
\end{tikzcd}$$

Since the cone $(f_i : X \to Y_i)_{i\in I}$ is initial with respect to $U : \mathsf{A} \to \mathsf{B}$, there is a unique arrow $\overline{h} : Z \to X$ in A such that for all $i \in I$ we have

$$g_i = f_i \cdot \overline{h}, \qquad U\overline{h} = h.$$

It remains to show that the arrow $\overline{h} : Z \to X$ is also a coalgebra homomorphism $\overline{h} : (Z, e) \to (X, c)$. For this, consider the diagram below.

$$\begin{tikzcd} Z \arrow[r, "\overline{h}"] \arrow[d, "e"'] & X \arrow[r, "f_i"] \arrow[d, "c"] & Y_i \arrow[d, "d_i"] \\ \overline{F}Z \arrow[r, "\overline{F}\,\overline{h}"'] & \overline{F}X \arrow[r, "\overline{F}f_i"'] & \overline{F}Y_i \end{tikzcd}$$

By assumption, the equation $Fh \cdot \delta_Z \cdot Ue = \delta_X \cdot Uc \cdot h$ holds. Then reason in the following manner:

$$\begin{aligned} Fh \cdot \delta_Z \cdot Ue = \delta_X \cdot Uc \cdot h &\iff FU\overline{h} \cdot \delta_Z \cdot Ue = \delta_X \cdot Uc \cdot U\overline{h} \\ &\iff \delta_X \cdot U\overline{F}\,\overline{h} \cdot Ue = \delta_X \cdot Uc \cdot U\overline{h} \\ &\implies U\overline{F}\,\overline{h} \cdot Ue = Uc \cdot U\overline{h} \\ &\iff U(\overline{F}\,\overline{h} \cdot e) = U(c \cdot \overline{h}) \\ &\implies \overline{F}\,\overline{h} \cdot e = c \cdot \overline{h}. \end{aligned}$$

□

Theorem 3.11. *Assume that $\overline{F} : \mathsf{A} \to \mathsf{A}$ preserves initial cones and that $U\overline{F} = FU$. Then if the functor $U : \mathsf{A} \to \mathsf{B}$ is topological, the functor $\overline{U} : \mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$ is topological as well.*

Proof. Let $(f_i : (X, c) \to \overline{U}(Y_i, d_i))_{i \in I}$ be a cone in $\mathsf{CoAlg}(F)$. Since the functor $U : \mathsf{A} \to \mathsf{B}$ is topological, the induced cone $(f_i : X \to UY_i)_{i \in I}$ admits a U-initial lifting

$$(\overline{f}_i : A \to Y_i)_{i \in I}.$$

By assumption, the cone $(\overline{F}\,\overline{f}_i : \overline{F}A \to \overline{F}Y_i)_{i \in I}$ is also initial. Moreover, note that the following equations hold,

$$\begin{aligned} U\left(A \xrightarrow{\overline{f}_i} Y_i \xrightarrow{d_i} \overline{F}Y_i\right) &= \left(X \xrightarrow{f_i} UY_i \xrightarrow{Ud_i} FUY_i\right) \\ U\left(\overline{F}A \xrightarrow{\overline{F}\,\overline{f}_i} \overline{F}Y_i\right) &= \left(FX \xrightarrow{Ff_i} FUY_i\right) \qquad (i \in I) \end{aligned}$$

and that we have the factorisation below.

$$\begin{tikzcd} X \arrow[d, "c"'] \arrow[dr, "Ud_i \cdot f_i"] & \\ FX \arrow[r, "Ff_i"'] & FUY_i \end{tikzcd}$$

This provides an arrow $\overline{c} : A \to \overline{F}A$ such that $U\overline{c} = c$, and that makes the diagram below to commute.

$$\begin{tikzcd} A \arrow[d, "\overline{c}"'] \arrow[dr, "d_i \cdot \overline{f}_i"] & \\ \overline{F}A \arrow[r, "\overline{F}\,\overline{f}_i"'] & \overline{F}Y_i \end{tikzcd}$$

We thus have a cone $(\overline{f}_i : (A, \overline{c}) \to (Y_i, d_i))_{i \in I}$ in $\mathsf{CoAlg}(\overline{F})$. To finish the proof recall that the cone $(\overline{f}_i : A \to Y_i)_{i \in I}$ is initial with respect to the functor $U : \mathsf{A} \to \mathsf{B}$ and apply Lemma 3.10. □

Corollary 3.12. *Let $U : \mathsf{A} \to \mathsf{B}$ be a topological functor and consider two functors $\overline{F} : \mathsf{A} \to \mathsf{A}$, $F : \mathsf{B} \to \mathsf{B}$ such that $\overline{F} : \mathsf{A} \to \mathsf{A}$ preserves initial cones. Moreover assume that $U\overline{F} = FU$. Then the category $\mathsf{CoAlg}(\overline{F})$ is complete iff $\mathsf{CoAlg}(F)$ is complete.*

The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ is topological (*cf.* [AHS90]) and it is straightforward to show that all polynomial functors over Top preserve initial cones. Using the previous corollary this entails that all categories of coalgebras of a polynomial functor over Top are complete.

As hinted before, Corollary 3.12 has stronger consequences than Theorem 3.5: it considers all functors in Top that preserve initial cones (and not just the polynomial ones) and it does not make any assumption about the category A being Top. In fact, the only assumption about the category A is that it has a topological functor $\mathsf{A} \to \mathsf{B}$. We invite the reader to examine in [AHS90] several examples of such categories.

3.2. **Some notes about Vietoris functors.** The last corollary is a positive result of our study of limits in categories of polynomial coalgebras. On the other hand, the addition of Vietoris functors to the mix brings a whole new level of difficulty that calls for a number of topological concepts, an investigation of Vietoris functors and some of their preservation properties. The study of such properties is the main goal of this section.

Lemma 3.13. *Let X be a topological space and $\mathcal{B}$ a base for the topology of X.*

*(1) The set $\{B^\Diamond \mid B \in \mathcal{B}\}$ is a subbase for the lower Vietoris space $\mathcal{V}X$ (*cf. *Example 2.27(2)).*
*(2) If $\mathcal{B}$ is closed under finite unions, then the set $\{B^\Diamond \mid B \in \mathcal{B}\} \cup \{B^\Box \mid B \in \mathcal{B}\}$ is a subbase for the compact Vietoris space $\mathcal{V}X$ (*cf. *Example 2.27(1)).*

Proof. Let $\mathcal{S}$ be a set of open subsets of X. First note that, for both the lower and the compact Vietoris space,

$$\left(\bigcup \mathcal{S}\right)^\Diamond = \bigcup \{S^\Diamond \mid S \in \mathcal{S}\}.$$

This proves the first statement. To see that the second one is also true, observe that

$$\left(\bigcup \mathcal{S}\right)^\Box = \bigcup \left\{ \left(\bigcup \mathcal{F}\right)^\Box \mid \mathcal{F} \subseteq \mathcal{S} \text{ finite} \right\}$$

since we only consider compact subsets of X. □

Lemma 3.14. *Both the compact and the lower Vietoris functor $\mathcal{V} : \mathsf{Top} \to \mathsf{Top}$ preserve initial codirected cones.*

Proof. Let $(f_i : X \to X_i)_{i \in I}$ be an initial codirected cone in Top. Then the set

$$\left\{ f_i^{-1}(U) \mid i \in I, U \subseteq X_i \text{ open} \right\}$$

is a base for the topology of X (Remark 2.8). Moreover, the base is closed under finite unions. Therefore, by the lemma above, the proof follows from the equations

$$((f_i)^{-1}(U))^\Box = (\mathcal{V}f_i)^{-1}(U^\Box) \qquad\qquad ((f_i)^{-1}(U))^\Diamond = (\mathcal{V}f_i)^{-1}\left(U^\Diamond\right),$$

for all $i \in I$ and $U \subseteq X_i$ open, which are straightforward to show. □

Theorem 3.15. *The lower Vietoris functor preserves initial codirected monocones. The compact Vietoris functor preserves initial codirected monocones of Hausdorff spaces.*

Proof. First note that for a topological space X the lower Vietoris space $\mathcal{V}X$ is T_0, and if X is Hausdorff the compact Vietoris space $\mathcal{V}X$ is Hausdorff as well (*cf.* [Mic51]). Then recall that a initial cone in Top whose domain is T_0 (or T_2) is necessarily mono and apply Lemma 3.14. □

Together with Proposition 3.2 it follows:

Corollary 3.16. *Every compact polynomial functor and every lower polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ preserves regular monomorphisms.*

Proof. We already saw that all polynomial functors preserve regular monomorphisms (Proposition 3.2), and that the lower Vietoris functor preserves them as well (Theorem 3.15). Moreover, we saw that the compact Vietoris functor preserves initial monomorphisms (Lemma 3.14) and it is straightforward to show that it preserves monomorphisms. □

From Theorem 3.15 and Corollary 2.19 we obtain the following results.

Corollary 3.17. *For every lower Vietoris polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ the category $\mathsf{CoAlg}(F)$ has codirected limits. For every compact Vietoris polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ the category $\mathsf{CoAlg}(F)$ has codirected limits of Hausdorff spaces.*

Corollary 3.18. *For every Vietoris polynomial functor $F : \mathsf{Top} \to \mathsf{Top}$ the category $\mathsf{CoAlg}(F)$ has equalisers.*

Proof. Direct consequence of Theorem 2.20 and Corollary 3.16. □

Remark 3.19. The assumption above about codirectedness is essential: neither the compact nor the lower Vietoris functor $\mathcal{V} : \mathsf{Top} \to \mathsf{Top}$ preserve monocones in general. Take, for instance, a compact Hausdorff space X with at least two elements. Then $A = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$, and A is different from $B = X \times X$. However, with $\pi_1 : X \times X \to X$ and $\pi_2 : X \times X \to X$ denoting the projection maps,

$$\mathcal{V}\pi_1(A) = \mathcal{V}\pi_1(B) = X = \mathcal{V}\pi_2(A) = \mathcal{V}\pi_2(B);$$

which shows that the cone $(\mathcal{V}\pi_1 : \mathcal{V}(X \times X) \to \mathcal{V}X, \mathcal{V}\pi_2 : \mathcal{V}(X \times X) \to \mathcal{V}X)$ is not mono.

Theorem 3.15 shows some good behaviour with respect to codirected initial monocones. However, none of the functors of Examples 2.27 preserves codirected limits in Top.

Examples 3.20.

(1) We consider $I = \mathbb{N}$ with the natural order, and the functor $D : \mathbb{N} \to \mathsf{Set}$ which sends $n \leq m$ to the inclusion map $\{0, \dots n\} \hookrightarrow \{0, \dots, m\}$. Clearly, the set of natural numbers $\mathbb{N}$ is a colimit of this directed diagram. Then, the composite $\mathsf{Set}(-, \mathbb{N}) \cdot D^{\mathrm{op}} : \mathbb{N}^{\mathrm{op}} \to \mathsf{Set}$ yields a codirected diagram with limit $\mathsf{Set}(\mathbb{N}, \mathbb{N})$, the limit projections $p_n : \mathsf{Set}(\mathbb{N}, \mathbb{N}) \to \mathsf{Set}(D(n), \mathbb{N})$ being given by restriction. Equipping all sets with the indiscrete topology, we obtain a codirected limit in Top. The compact Vietoris functor does not send this limit to a monocone since $(Vp_n)_{n \in \mathbb{N}}$ cannot distinguish between the sets $\mathsf{Set}(\mathbb{N}, \mathbb{N})$ and

$$\{f : \mathbb{N} \to \mathbb{N} \mid \{n \in \mathbb{N} \mid f(n) \neq 0\} \text{ is finite}\}.$$

(2) The next example is based on the "empty inverse limit" of [Wat72]. Here I is the set of all finite subsets of $\mathbb{R}$, with order being containment $\supseteq$. For $F \in I$, let $D(F)$ be the discrete space of all injective functions $F \to \mathbb{N}$, and the map $D(G \supseteq F)$ is given by restriction. Note that each connecting map $D(G \supseteq F)$ is surjective. Then the limit of this diagram in Top is empty since an element of this limit would define an injective function $\mathbb{R} \to \mathbb{N}$. The lower Vietoris functor sends the limit cone for D to a monocone but not to a limit cone since the limit of $\mathcal{V}D$ has at least two elements: $(\varnothing)_{F \in I}$ and $(D(F))_{F \in I}$. Using the indiscrete topology instead of the discrete one shows that the lower Vietoris functor does not preserve codirected limits of diagrams of compact spaces and closed maps.

(3) In the example above we can use other topologies to show that the lower or the compact Vietoris functor does not preserve certain codirected limits. As an example, we consider here $\mathbb{N}$ equipped with the topology

$$\{\uparrow n \mid n \in \mathbb{N}\} \cup \{\varnothing\};$$

where $\uparrow n = \{k \in \mathbb{N} \mid n \leq k\}$. Note that $\mathbb{N}$ is T_0 and every non-empty collection of open subsets of $\mathbb{N}$ has a largest element with respect to inclusion $\subseteq$. The latter implies that, for every finite

set F, every subset of $\mathbb{N}^F$ is compact. To see this, let $C \subseteq \mathbb{N}^F$ and assume that C is covered by subbasic open subsets of $\mathbb{N}^F$:

$$C \subseteq \bigcup_{\lambda \in \Lambda} \pi_{i_\lambda}^{-1}[\uparrow n_\lambda].$$

Note that the set $K = \{i_\lambda \mid \lambda \in \Lambda\} \subseteq F$ is finite. For every $i \in K$, let $k_i = \min\{n_\lambda \mid \lambda \in \Lambda, i_\lambda = i\}$. Then

$$C \subseteq \bigcup_{i \in K} \pi_i^{-1}[\uparrow k_i].$$

Using now Alexander's Subbase Theorem (*cf.* [Kel55]), we conclude that C is compact.

With I being as in the previous example, we consider now $D(F)$ as a subspace of $\mathbb{N}^F$. Then, for every $G \supseteq F$, the map $D(G \supseteq F) : D(G) \to D(F)$ is continuous. Hence, this construction defines a codirected diagram $D : I \to \mathsf{Top}$ where each $D(F)$ is T_0, compact, and locally compact; and the limit of this diagram is empty. With the same argument as above, neither the lower nor the compact Vietoris functor preserve this limit.

3.3. **Vietoris polynomial functors.** Section 3.1 studied limits in categories of polynomial coalgebras, essentially by analysing the preservation of connected limits in Top and by providing sufficient conditions for the existence of topological functors between categories of coalgebras. In the current section our focus is on Vietoris coalgebras. In fact, Examples 3.20 already showed that it is highly problematic to consider all topological spaces, because the lower and the compact Vietoris functors do not preserve codirected limits in Top. Hence, we will restrict our attention to different subcategories of Top where more positive results appear.

Definition 3.21. A topological space X is called ***stably compact*** whenever X is T_0, locally compact, well-filtered and every finite intersection of compact saturated subsets is compact [Jun04]. A continuous map between stably compact spaces is called ***spectral*** whenever the inverse image of compact saturated subsets is compact. Stably compact spaces and spectral maps form a category which we denote by $\mathsf{StablyComp}$.

Remark 3.22. Note that every stably compact space is compact. More information on this type of space can be found in [GHK$^+$03] and [Jun04].

Theorem 3.23. *The category* $\mathsf{StablyComp}$ *is complete and regularly wellpowered. The inclusion functor* $\mathsf{StablyComp} \to \mathsf{Top}$ *preserves limits and finite coproducts.*

Proof. It is straightforward to check that the finite coproduct of stably compact spaces is stably compact (*cf.* [Gou13, Proposition 9.2.1]). The other claims follow from monadicity of $\mathsf{StablyComp} \to \mathsf{Top}$ which is shown in [Sim82]. We note that [Sim82] uses the designation *well-compacted* instead of stably compact. □

Further properties of $\mathsf{StablyComp}$ can be easily derived if ones uses a order-theoretic perspective.

Definition 3.24. A ***partially orderered compact space*** is a triple $(X, \leq, \tau)$ consisting of a set X, a partial order $\leq$ on X and a compact topology τ on X so that the set

$$\{(x, y) \in X \times X \mid x \leq y\}$$

is closed with respect to the product topology.

Remark 3.25. Every partially ordered compact space $(X, \leq, \tau)$ is necessarily Hausdorff as the antisymmetry property of the relation $\leq$ implies that the diagonal $\{(x, x) \mid x \in X\}$ is closed in $X \times X$.

The category $\mathsf{StablyComp}$ is isomorphic to the category $\mathsf{PosComp}$ of partially orderered compact spaces and monotone continuous maps (*cf.* [GHK$^+$80]). The isomorphism $\mathsf{PosComp} \to \mathsf{StablyComp}$ commutes with the underlying forgetful functors to Set, sending a partially ordered compact space $(X, \leq, \tau)$ to the stably compact space with the same underlying set and the topology defined by the upper-open

sets of $(X, \leq, \tau)$. Its inverse functor uses the ***specialisation order*** of a topological space, defined by $x \leq y \iff x \in \overline{\{y\}}$. It maps a stably compact space (X, τ) into a space $(X, \tau', \leq)$ where the relation $\leq$ is the specialisation ordering and τ' the ***patch topology*** of (X, τ), *i.e.* the topology generated by the complements of compact saturated subsets and also the opens in (X, τ).

Remark 3.26. The canonical forgetful functor $\mathsf{PosComp} \to \mathsf{CompHaus}$ has a left adjoint which equips a compact Hausdorff space with the discrete order. Using the isomorphism above, the adjunction

$$\begin{tikzcd} \mathsf{PosComp} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathsf{CompHaus} \arrow[l, bend left, "\text{discrete}"] \end{tikzcd}$$

reads in the language of stably compact spaces as

$$\begin{tikzcd} \mathsf{StablyComp} \arrow[r, bend left, "\text{patch}"] \arrow[r, phantom, "\top"] & \mathsf{CompHaus}. \arrow[l, bend left, "\text{inclusion}"] \end{tikzcd}$$

In the sequel we will freely jump between both perspectives.

Theorem 3.27. *The category* $\mathsf{PosComp}$ *is cocomplete and the epimorphisms of* $\mathsf{PosComp}$ *are precisely the surjective morphisms.*

Proof. Cocompleteness of $\mathsf{PosComp}$ follows from [Tho09, Corollary 2]. Combining several results of [Nac65], it is shown in [HNN18] that every epimorphism in $\mathsf{PosComp}$ is surjective. □

Clearly, (Surjections,Substructure) is a factorisation structure for morphisms in $\mathsf{PosComp}$. Since the surjections are precisely the epimorphisms in $\mathsf{PosComp}$, we conclude that $\mathsf{PosComp}$ is (Epi,RegMono)-structured, and thus also the category $\mathsf{StablyComp}$. Moreover, the regular monomorphisms in $\mathsf{StablyComp}$ are precisely the topological subspace embeddings.

Let us turn our attention back to the study of Vietoris functors with the isomorphism $\mathsf{StablyComp} \simeq \mathsf{PosComp}$ in mind. The lower Vietoris functor on Top restricts to a functor $\mathcal{V} : \mathsf{StablyComp} \to \mathsf{StablyComp}$ (*cf.* [Sch93]). Its counterpart on $\mathsf{PosComp}$ can be described in the following manner.

Proposition 3.28. *Under the isomorphism* $\mathsf{StablyComp} \simeq \mathsf{PosComp}$*, the lower Vietoris functor* $\mathcal{V} : \mathsf{StablyComp} \to \mathsf{StablyComp}$ *corresponds to the functor*

$$\mathsf{PosComp} \to \mathsf{PosComp}$$

which sends a partially ordered compact space X to the space of all lower-closed subsets of X, with order inclusion $\subseteq$, and compact topology generated by the sets

$$\begin{aligned} &\{A \subseteq X \mid A \text{ lower-closed and } A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ upper-open}), \\ &\{A \subseteq X \mid A \text{ lower-closed and } A \cap K = \varnothing\} \quad (K \subseteq X \text{ upper-closed}). \end{aligned} \tag{3.i}$$

Given a map $f : X \to Y$ in $\mathsf{PosComp}$*, the functor returns the map that sends a lower-closed subset $A \subseteq X$ to the down-closure $\downarrow f[A]$ of $f[A]$.*

Proof. Let $(X, \leq, \tau)$ be a partially ordered compact space with corresponding stably compact space (X, σ). Clearly, the underlying set of $\mathcal{V}(X, \sigma)$ is the set of all lower-closed subsets of X. We will show that the patch topology of $\mathcal{V}(X, \sigma)$ coincides with the topology defined by (3.i). First note that every set of the form

$$\{A \subseteq X \mid A \text{ lower-closed and } A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ upper-open}),$$

is open in $\mathcal{V}(X, \sigma)$ and therefore is also in the patch topology. For $K \subseteq X$ upper-closed, the complement of the set

$$\{A \subseteq X \mid A \text{ lower-closed and } A \cap K = \varnothing\}$$

is equal to $K^\Diamond$. Using Alexander's Subbase Theorem, it is straightforward to verify that $K^\Diamond$ is compact in $\mathcal{V}(X,\sigma)$. Since the specialisation order of $\mathcal{V}(X,\sigma)$ is subset inclusion, $K^\Diamond$ is also saturated. Hence, the topology defined by (3.i) is coarser than the patch topology of $\mathcal{V}(X,\sigma)$. Since it is also Hausdorff, by [Jun04, Lemma 2.2], both topologies coincide (*cf.* [Eng89]). In particular, the construction of the proposition defines indeed a partially ordered compact space.

In regard to maps in $\mathsf{PosComp}$, [Nac65, Proposition 4 on page 44] tells that for every map $f : X \to Y$ in $\mathsf{PosComp}$ and every lower-closed subset $A \subseteq X$, the down-closure $\downarrow f[A]$ of $f[A]$ is closed in Y, and therefore coincides with the closure of $f[A]$ in the stably compact topology of Y. □

Recall that the lower Vietoris functor preserves codirected initial monocones (see Theorem 3.15). Hence, for every codirected diagram $D : I \to \mathsf{StablyComp}$ with limit cone $(p_i : L_D \to D(i))_{i\in I}$, the canonical comparison map

$$h : \mathcal{V}L_D \to L_{\mathcal{V}D},\ K \mapsto (\overline{p_i[K]})_{i\in I}$$

is an embedding. To show that $\mathcal{V} : \mathsf{StablyComp} \to \mathsf{StablyComp}$ preserves these limits, we are left with the task of proving that h is also surjective. To do so, we use the fact that $\mathsf{StablyComp}$ inherits a nice characterisation of codirected limits from the category $\mathsf{CompHaus}$. A first hint of the latter characterisation is in [Bou42], but, to the best of our knowledge, is rarely used in the literature. Actually, we were not able to find a proof in the literature, except for [Hof99]; so we sketch a proof below.

Theorem 3.29. *Let $D : I \to \mathsf{CompHaus}$ be a codirected diagram and $\mathcal{C} = (p_i : L \to D(i))_{i\in I}$ a cone for D. The following conditions are equivalent:*

(1) The cone $\mathcal{C}$ is a limit of D.
(2) The cone $\mathcal{C}$ is mono and, for every $i \in I$, the image of p_i contains the intersection of the images of all $D(j \to i)$, in symbols

$$\operatorname{im} p_i \supseteq \bigcap_{j\to i} \operatorname{im} D(j \to i).$$

Proof. Assume first that $(p_i : L \to D(i))_{i\in I}$ satisfies the two conditions and let $(f_i : X \to D(i))_{i\in I}$ be a cone for D. Let $x \in X$, and, for every $i \in I$, put $A_i = p_i^{-1}(f_i(x))$. Clearly, A_i is closed, moreover, A_i is non-empty since

$$\operatorname{im} f_i \subseteq \bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i$$

Since the family $(A_i)_{i\in I}$ is codirected and L is compact, there is some $z \in \bigcap_{i\in I} A_i$. We put $f(x) = z$, this way we define a map $f : X \to L$ with $p_i \cdot f = f_i$, for all $i \in I$. Since $(p_i : L \to D(i))_{i\in I}$ is a monocone, we conclude that $(p_i : L \to D(i))_{i\in I}$ is a limit of D. Conversely, if $(p_i : L \to D(i))_{i\in I}$ is a limit, then it is clearly a monocone. Let now $i_0 \in I$ and $x \in \bigcap_{j\to i_0} \operatorname{im} D(j \to i_0)$. We may assume that i_0 is final in I. For each $i \in I$, we put

$$A_i = \{(x_i)_{i\in I} \in \prod_{i\in I} D(i) \mid x_{i_0} = x \text{ and, for all } i \to j \in I, x_j = D(i \to j)(x_i)\}.$$

Then A_i is non-empty, and it is a closed subset of $\prod_{i\in I} D(i)$ since it is an equaliser of continuous maps between Hausdorff spaces. Furthermore, for $i \to j \in I$, $A_i \subseteq A_j$. Hence there is some $z \in \bigcap_{i\in I} A_i$; by construction, $z \in L$ and $p_{i_0}(z) = x$. □

Remark 3.30. For every cone $(p_i : C \to D(i))_{i\in I}$ the inequality $\operatorname{im} p_i \subseteq \bigcap_{j\to i} \operatorname{im} D(j \to i)$ holds. Hence, in the theorem above, the reverse inequality, distinguishes monocones from limit cones.

Proposition 3.31. *Let $\mathcal{A}$ be a codirected set of closed subsets of a partially ordered compact space X. Then, $\downarrow\bigcap_{A\in\mathcal{A}} A = \bigcap_{A\in\mathcal{A}} \downarrow A$.*

Proof. Clearly, $\downarrow\bigcap_{A\in\mathcal{A}} A \subseteq \bigcap_{A\in\mathcal{A}} \downarrow A$. To show that the reverse inequality holds, consider $z \in \bigcap_{A\in\mathcal{A}} \downarrow A$. Then, for every $A \in \mathcal{A}$, the set $\uparrow z \cap A$ is non-empty, and closed because $\{z\}$ is compact (*cf.*[Nac65, Proposition 4 on page 44]). Moreover, since $\mathcal{A}$ is codirected, the set $\{\uparrow z \cap A \mid A \in \mathcal{A}\}$ has the finite

intersection property. Therefore, by compactness, it follows that $\uparrow z \cap \bigcap_{A\in\mathcal{A}} A \neq \varnothing$, which implies that $z \in \downarrow\bigcap_{A\in\mathcal{A}} A$. □

Proposition 3.32. *Let $D : I \to \mathsf{PosComp}$ be a codirected diagram, $(p_i : L_D \to D(i))_{i\in I}$ a limit for D and $(L_{\mathcal{V}D} \to \mathcal{V}D(i))_{i\in I}$ a limit for $\mathcal{V}D : I \to \mathsf{PosComp}$. Then the function $h : \mathcal{V}L_D \to L_{\mathcal{V}D}$ defined by $K \mapsto (\downarrow p_i[K])_{i\in I}$ is surjective.*

Proof. Let $(K_i)_{i\in I} \in L_{\mathcal{V}D}$. For every $i \in I$, $K_i \subseteq D(i)$ is closed, hence, $K_i \in \mathsf{PosComp}$. For every $i \in I$ and $j \to i \in I$, take $K(i)$ as K_i and $K(j \to i)$ as the continuous and monotone map of type $K_j \to K_i$ given by the restriction of $D(j \to i)$ to K_j. This way, by Remark 2.3, we obtain a codirected diagram $K : I \to \mathsf{PosComp}$ such that for every $j \to i \in I$, $\downarrow K(j \to i)[K(j)] = [K(i)]$.

Let $(p_i : L_K \to K(i))_{i\in I}$ be a limit for K. By construction, $L_K \subseteq L_D$ is lower-closed. Thus, $L_K \in \mathcal{V}L_D$. We claim that $h(L_k) = (K_i)_{i\in I}$. Let $i_0 \in I$. Since the following diagram of forgetful functors

$$\begin{tikzcd} \mathsf{PosComp} \arrow[rr] \arrow[dr] & & \mathsf{CompHaus} \arrow[dl] \\ & \mathsf{Set} & \end{tikzcd}$$

commutes and the functor $\mathsf{PosComp} \to \mathsf{CompHaus}$ preserves limits, from Theorem 3.29 we obtain

$$p_{i_0}[L_K] = \bigcap_{j\to i_0} K(j \to i_0)[K_j].$$

Therefore, by Proposition 3.31,

$$\downarrow p_{i_0}[L_K] = \downarrow \bigcap_{j\to i_0} K(j \to i_0)[K(j)] = \bigcap_{j\to i_0} \downarrow K(j \to i_0)[K(j)] = K_{i_0}.$$

□

As expected, we obtain the following results.

Corollary 3.33. *The lower Vietoris functor $\mathcal{V} : \mathsf{StablyComp} \to \mathsf{StablyComp}$ preserves codirected limits.*

Proof. A direct consequence of the previous proposition. □

Corollary 3.34. *All lower Vietoris polynomial functors $F : \mathsf{StablyComp} \to \mathsf{StablyComp}$ preserve codirected limits.*

Proof. Analogous to that of Theorem 3.1. □

Theorem 3.35. *For every lower Vietoris polynomial functor $F : \mathsf{StablyComp} \to \mathsf{StablyComp}$, the category $\mathsf{CoAlg}(F)$ is complete.*

Proof. Firstly, observe that Theorems 3.23, 3.27 and Corollary 3.16 guarantee the hypothesis of Theorem 2.14, therefore the category $\mathsf{CoAlg}(F)$ has equalisers. Then the assertion follows from Corollary 3.34 and [Bar93, Theorem 2.1]. □

In regard to final coalgebras, there is still room to improve the theorem above. Indeed, the inclusion functor $I : \mathsf{StablyComp} \to \mathsf{Top}$ is well-behaved with respect to limits, in particular it preserves and reflects them (*cf.* [Sim82]); this allows us to derive the following theorem.

Theorem 3.36. *Every lower Vietoris polynomial functor in* Top *that can be restricted to* $\mathsf{StablyComp}$ *admits a final coalgebra.*

The lower and the compact Vietoris functors on Top are seemingly unrelated, notwithstanding, these functors are closely related when restricted, respectively, to $\mathsf{StablyComp}$ and $\mathsf{CompHaus}$. From the description of the lower Vietoris functor $\mathcal{V}$ on $\mathsf{PosComp}$ we obtain that the compact Vietoris functor $\mathcal{V} : \mathsf{CompHaus} \to \mathsf{CompHaus}$ is the composite

$$\mathsf{CompHaus} \xrightarrow{\text{discrete}} \mathsf{PosComp} \xrightarrow{\mathcal{V}} \mathsf{PosComp} \xrightarrow{\text{forgetful}} \mathsf{CompHaus}.$$

Being right adjoint, the functor $\mathsf{PosComp} \xrightarrow{\text{forgetful}} \mathsf{CompHaus}$ preserves limits, but also the inclusion functor $\mathsf{CompHaus} \to \mathsf{PosComp}$ does so. As an interesting consequence, studying preservation of limits by the lower Vietoris functor in $\mathsf{StablyComp} \simeq \mathsf{PosComp}$ encompasses studying preservation of limits by the compact Vietoris in $\mathsf{CompHaus}$. In particular, the following results come for free.

Corollary 3.37. *The compact Vietoris $\mathcal{V} : \mathsf{CompHaus} \to \mathsf{CompHaus}$ preserves codirected limits.*

Corollary 3.38. *All compact Vietoris polynomial functors $F : \mathsf{CompHaus} \to \mathsf{CompHaus}$ preserve codirected limits.*

By taking advantage of the fact that a compact subspace of an Hausdorff space is a compact Hausdorff space, [Zen70] proves this property of the compact Vietoris functor even for Hausdorff spaces.

Theorem 3.39. *The compact Vietoris functor $\mathcal{V} : \mathsf{Haus} \to \mathsf{Haus}$ preserves codirected limits.*

The following results then emerge in a straightforward manner.

Theorem 3.40. *All compact Vietoris polynomial functors $F : \mathsf{Haus} \to \mathsf{Haus}$ preserve codirected limits.*

Proof. Follows from the previous theorem and the fact that all polynomial functors $F : \mathsf{Haus} \to \mathsf{Haus}$ preserve codirected limits. □

Corollary 3.41. *Let $F : \mathsf{Haus} \to \mathsf{Haus}$ be a compact Vietoris polynomial functor. The associated category of coalgebras $\mathsf{CoAlg}(F)$ is complete.*

Proof. Being an epireflective subcategory of Top, the category Haus is complete and cocomplete, and regularly wellpowered. Furthermore, Haus is (Epi,RegMono)-structured; but note that $f : X \to Y$ in Haus is a regular monomorphism if and only if f is a closed embedding. It is straightforward to prove that the compact Vietoris functor preserves closed embeddings; therefore, by Theorem 2.14, $\mathsf{CoAlg}(F)$ has equalisers. As an alternative, Haus is also (Surjection, Embedding)-structured; and now use Corollary 2.19 and Corollary 3.16 to conclude that $\mathsf{CoAlg}(F)$ has equalisers. Then the assertion follows from Theorem 3.40 and [Bar93, Theorem 2.1]. □

Theorem 3.42. *Let $F : \mathsf{Top} \to \mathsf{Top}$ be a Vietoris polynomial functor that can be restricted to Haus. Then, the category $\mathsf{CoAlg}(F)$ has a final coalgebra.*

Proof. A consequence of the fact that $I : \mathsf{Haus} \to \mathsf{Top}$ preserves and reflects limits (*cf.* [AHS90]). □

To close this section we will relate its results with the works [KKV04, BKR07]. Recall that the former considers compact Vietoris polynomial functors over Stone. The latter consider coalgebras for the lower Vietoris functor in the category Spec of spectral spaces and spectral maps.

The categories Stone and Spec have a close relation with some of the categories we considered so far, in particular $\mathsf{CompHaus}$ and $\mathsf{StablyComp}$. By taking advantage of this relation we will see that the fact that every compact Vietoris functor $F : \mathsf{Stone} \to \mathsf{Stone}$ admits a final coalgebra (as shown in [KKV04]) is actually a consequence of Corollary 3.38, and the fact that every lower Vietoris polynomial functor $F : \mathsf{Spec} \to \mathsf{Spec}$ admits a final coalgebra is a direct consequence of Theorem 3.35.

Remark 3.43. Recall that a Stone space X is a compact Hausdorff space with a basis of clopen sets. This is equivalent to saying that X is compact Hausdorff and that the cone of continuous maps $(X \to 2)$ to the discrete two-point-space is initial.

Lemma 3.44. *Let $(X \to X_i)_{i \in I}$ be a initial cone in $\mathsf{CompHaus}$ where X_i is a Stone space for every $i \in I$. Then X is a Stone space as well.*

Proof. Follows from the fact that each space X_i defines a initial cone of continuous maps $(X_i \to 2)$ and that initial cones are closed under composition. □

Corollary 3.45. *The canonical forgetful functor* $\mathsf{Stone} \to \mathsf{CompHaus}$ *creates limits. Hence, the category* Stone *is complete, and the functor* $\mathsf{Stone} \to \mathsf{CompHaus}$ *preserves and reflects limits.*

Theorem 3.46. *Every compact Vietoris polynomial functor* $F : \mathsf{Stone} \to \mathsf{Stone}$ *preserves codirected limits.*

Proof. Observe that every compact Vietoris polynomial functor $F : \mathsf{Stone} \to \mathsf{Stone}$ is also a functor $F : \mathsf{CompHaus} \to \mathsf{CompHaus}$ and that the diagram below commutes. The claim then follows directly from the fact that the functor $\mathsf{Stone} \to \mathsf{CompHaus}$ preserves and reflects limits.

$$\begin{tikzcd} \mathsf{Stone} \arrow[r, "F"] \arrow[d] & \mathsf{Stone} \arrow[d] \\ \mathsf{CompHaus} \arrow[r, "F"'] & \mathsf{CompHaus} \end{tikzcd}$$

□

Corollary 3.47. *Every compact Vietoris polynomial functor* $F : \mathsf{Stone} \to \mathsf{Stone}$ *admits a final coalgebra.*

Analogous results can be achieved for the category Spec. To see this let us start a remark akin to Remark 3.43.

Remark 3.48. Recall that a spectral space X is a stably compact space with a basis of compact open subsets. This is equivalent to saying that X is stably compact and that the cone of spectral maps $(X \to 2)$ to the Sierpiński space is initial.

Lemma 3.49. *Let* $(X \to X_i)_{i \in I}$ *be a initial cone in* $\mathsf{StablyComp}$ *where* X_i *is a spectral space for every* $i \in I$*. Then* X *is a spectral space as well.*

Proof. Follows from the fact that each space X_i defines a initial cone of continuous maps $(X_i \to 2)$ to the Sierpiński space and that initial cones are closed under composition. □

Corollary 3.50. *The canonical forgetful functor* $\mathsf{Spec} \to \mathsf{StablyComp}$ *creates limits. Hence, the category* Spec *is complete, and the functor* $\mathsf{Spec} \to \mathsf{StablyComp}$ *preserves and reflects limits.*

Theorem 3.51. *Every lower Vietoris polynomial functor* $F : \mathsf{Spec} \to \mathsf{Spec}$ *preserves codirected limits.*

Proof. Observe that every lower Vietoris polynomial functor $F : \mathsf{Spec} \to \mathsf{Spec}$ is also a functor $F : \mathsf{StablyComp} \to \mathsf{StablyComp}$ and that the diagram below commutes. The claim then follows directly from the fact that the functor $\mathsf{Spec} \to \mathsf{StablyComp}$ preserves and reflects limits.

$$\begin{tikzcd} \mathsf{Spec} \arrow[r, "F"] \arrow[d] & \mathsf{Spec} \arrow[d] \\ \mathsf{StablyComp} \arrow[r, "F"'] & \mathsf{StablyComp} \end{tikzcd}$$

□

Corollary 3.52. *Every lower Vietoris polynomial functor* $F : \mathsf{Spec} \to \mathsf{Spec}$ *admits a final coalgebra.*

4. Limits via adjunction

In this section we extend the results of the previous section to subfunctors of (Vietoris) polynomial functors, by making use of adjunction. To achieve this we introduce a number of conditions which guarantee that a functor $\mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ induced by a natural transformation $F \to G$ has a right adjoint: note that if the functor $\mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ is also fully faithful, then we can easily show that $\mathsf{CoAlg}(F)$ is "as complete as" $\mathsf{CoAlg}(G)$. A key property we use here is a straightforward generalisation of the notion of taut natural transformation originally introduced in [Möb83] and [Man02].

We start with the definition below.

Definition 4.1. Every natural transformation $\sigma : F \to G$ induces a functor $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$, defined by

$$I(X, c) = (X, \sigma_X \cdot c), \qquad If = f.$$

Note that the functor $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ is faithful. Moreover,

Proposition 4.2. *If $\sigma : F \to G$ is a monomorphic natural transformation, then the functor $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ is also full.*

Proof. Take a homomorphism $f : I(X, c) \to I(Y, d)$. By assumption, the equation $Gf \cdot \sigma_X \cdot c = \sigma_Y \cdot d \cdot f$ holds. Then, use naturality and the fact that $\sigma_Y : FY \to GY$ is a monomorphism to show that $Ff \cdot c = d \cdot f$. □

We will now show that, under some assumptions on the natural transformation $\sigma : F \to G$, the functor above has a right adjoint.

Assumption 4.3. In the remainder of this section the letter C denotes a category with an (E, M)-factorisation structure where M is included in the class of monomorphisms. We assume that C is M-wellpowered, that $\sigma : F \to G$ is a natural transformation between endofunctors on C where every component σ_X is in M, and that G sends morphisms in M to morphisms in M.

Theorem 4.4. *Under Assumption 4.3 with C cocomplete, the functor $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ is left adjoint.*

Proof. We will show that the assumptions of the General Adjoint Functor Theorem hold. Since C is cocomplete, the category $\mathsf{CoAlg}(F)$ is cocomplete as well. Moreover, $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ preserves colimits, as $UI : \mathsf{CoAlg}(F) \to \mathsf{C}$ preserves colimits, and the forgetful functor $U : \mathsf{CoAlg}(G) \to \mathsf{C}$ reflects them. It remains to verify the Solution Set Condition. For this, take a coalgebra $d : Y \to GY$. Let $\mathcal{S}_0$ be a set of representatives of the collection of all C-objects Q admitting an M-morphism $Q \to Y$, and let $\mathcal{S}$ be the set of all F-coalgebras based on an object in $\mathcal{S}_0$. Let now (X, c) be an F-coalgebra and $f : (X, \sigma_X \cdot c) \to (Y, d)$ be a homomorphism of G-coalgebras. By hypothesis, $f : X \to Y$ factorises as $f = m \cdot e$

$$X \xrightarrow{e} Q \xrightarrow{m} Y$$

with $e \in E$ and $m \in M$. Since $\sigma_Q : FQ \to GQ$ and $Gm : GQ \to GY$ are in M, there is a diagonal $q : Q \to FQ$ so that the right hand square and the lower-left square in

$$\begin{tikzcd}
GX \arrow[r, "Ge"] & GQ \arrow[r, "Gm"] & GY \\
FX \arrow[u, "\sigma_X"] \arrow[r, "Fe"'] & FQ \arrow[u, "\sigma_Q"'] & \\
X \arrow[u, "c"] \arrow[r, "e"'] & Q \arrow[u, dotted, "q"'] \arrow[r, "m"'] & Y \arrow[uu, "d"']
\end{tikzcd}$$

commute; the upper-left square commutes since σ is a natural transformation. This proves that $f : (X, \sigma_X \cdot c) \to (Y, d)$ factorises via the image of an object in $\mathcal{S}$. □

Corollary 4.5. *The category $\mathsf{CoAlg}(F)$ has all (co)limits of a certain type if $\mathsf{CoAlg}(G)$ does so.*

Corollary 4.6. *Let $F : \mathsf{Top} \to \mathsf{Top}$ be a compact Vietoris polynomial functor that can be restricted to Haus. Every subfunctor of F admits a final coalgebra.*

Remark 4.7. The Corollary above applies to various interesting variants of the compact Vietoris functor that were not yet mentioned. In particular,

- the one that discards the empty set,

- analogously to the finitary powerset functor, the one that takes infinite sets out of comission, and
- the one which considers only compact and connected subsets (*cf.* [Dud72]).

All these variants are subfunctors of the compact Vietoris functor. In conjunction with the polynomial ones, they form a family of subfunctors of compact Vietoris polynomial functors.

Corollary 4.8. *Let $F : \mathsf{Top} \to \mathsf{Top}$ be a lower Vietoris polynomial functor that can be restricted to* $\mathsf{StablyComp}$. *Every subfunctor of F admits a final coalgebra.*

The proof of Theorem 4.4 gives us also a hint on how to construct a coreflection of a G-coalgebra (Y, d): take the "largest M-subcoalgebra of (Y, d)". In the sequel we make this idea more precise. To do so, motivated by [Möb83] and [Man02], we introduce the following notion.

Definition 4.9. A natural transformation $\sigma : F \to G$ is ***M-taut*** if each naturality square induced by a morphism in M is a pullback square; that is, for every morphism $m : X \to Y$ in M, the diagram below is a pullback square.

$$\begin{tikzcd} FX \arrow[r, "Fm"] \arrow[d, "\sigma_X"'] & FY \arrow[d, "\sigma_Y"] \\ GX \arrow[r, "Gm"'] & GY \end{tikzcd}$$

Recall from [AHS90, Definition 7.79] that, for monomorphisms $m_1 : M_1 \to X$ and $m_2 : M_2 \to X$ in a category, m_1 is ***smaller than*** m_2 (written as $m_1 \leq m_2$) whenever there is some $m : M_1 \to M_2$ with $m_2 \cdot m = m_1$. Note that m is necessarily a monomorphism. Assuming that C has pullbacks, take a G-coalgebra (Y, d) and consider the pullback square

(4.i)
$$\begin{tikzcd} S \arrow[r] \arrow[d, "i"'] & FY \arrow[d, "\sigma_Y"] \\ Y \arrow[r, "d"'] & GY \end{tikzcd}$$

in C. Note that $i : S \to Y$ is in M, by [AHS90, Proposition 14.15].

Lemma 4.10. *(1) For every F-coalgebra (X, c) and every homomorphism $m : (X, \sigma_X \cdot c) \to (Y, d)$ with $m \in M$, m is smaller than $i : S \to Y$.*

(2) Assume now that the natural transformation $\sigma : F \to G$ is M-taut and let $m : (Q, q) \to (Y, d)$ be a homomorphism in $\mathsf{CoAlg}(G)$ where $m \in M$ and $m \leq i$. Then there is a F-coalgebra structure $q' : Q \to FQ$ on Q with $\sigma_Q \cdot q' = q$.

Proof. An easy calculation, and [AHS90, Proposition 14.9] show that the first claim is true. In regard to the second one, let $\bar{m} : Q \to S$ be the arrow in C with $i \cdot \bar{m} = m$. Then, since in the diagram

$$\begin{tikzcd} & Y \arrow[rr, "d"] & & GY \\ & S \arrow[u, "i"] \arrow[rr] & & FY \arrow[u, "\sigma_Y"'] \\ Q \arrow[uuu, bend left, "m"] \arrow[uu, "\bar{m}"] \arrow[rr, dotted] \arrow[rrrr, bend right, "c"'] & & FQ \arrow[uu, "Fm"] \arrow[r, "\sigma_Q"] & GQ \arrow[uu, "Gm"'] \end{tikzcd}$$

the right hand parallelogram is a pullback square and the outer diagram and the top parallelogram commute. This provides the desired arrow $Q \to FQ$. □

For a G-coalgebra (Y, d), the class of all subcoalgebras $m : (X, c) \to (Y, d)$ with $m \in M$ is preordered under the smaller-than relation. Since C is M-wellpowered, this class is equivalent to an ordered set; and by a slight abuse of language we will speak of the ordered set of M-subobjects of (Y, d).

Recall from Proposition 4.2 that the induced functor $I : \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(G)$ is fully faithful since $\sigma : F \to G$ is a monomorphic natural transformation. Hence, we can consider $\mathsf{CoAlg}(F)$ as a full subcategory of $\mathsf{CoAlg}(G)$. From the results above we obtain:

Theorem 4.11. *In addition to Assumption 4.3, assume that C has pullbacks, σ is M-taut and, for every G-coalgebra (Y, d), the ordered set of M-subobjects is complete. Then, for every G-coalgebra (Y, d), the coreflection of (Y, d) is given by the supremum $(\bar{Q}, \bar{q}) \to (Y, d)$ of all G-homomorphisms $(Q, q) \to (Y, d)$ with (Q, q) in $\mathsf{CoAlg}(F)$ and $m : Q \to Y$ in M smaller than $i : S \to Y$ (defined by the pullback square* (4.i)*).*

Remark 4.12. If C has coproducts, then also $\mathsf{CoAlg}(G)$ has coproducts which guarantees completeness of the ordered set of M-subobjects of (Y, d). In fact, let $(m_i : (X_i, c_i) \to (Y, d))_{i \in I}$ a family of subcoalgebras of (Y, d) with $m_i \in M$, for every $i \in I$. Then the supremum $m : (X, c) \to (Y, d)$ of this family is given by the (E, M)-factorisation of the canonical map $f : \coprod_{i \in I}(X_i, c_i) \to (Y, d)$ induced by this family.

$$\begin{tikzcd}
\coprod_{i\in I}(X_i, c_i) \arrow[r, "e"] \arrow[rr, bend left, "f"] & (X, c) \arrow[r, "m"] & (Y, d) \\
(X, c_i) \arrow[u, "k_i"] \arrow[urr, "m_i"'] & &
\end{tikzcd}$$

5. Vietoris coalgebras at work

Moving to the more practical side, recall the bouncing ball system mentioned in the introduction. Formally, it consists of a ball that is dropped at a certain height (p), and with an initial velocity (v). Due to the gravitional effect (g), it falls into the ground and then bounces back up, losing, for example, half of its kinetic energy. As the documents [NBHM16, NB16, NB17] show, such a behaviour can be described coalgebraically, with the help of the functor defined below.

Definition 5.1. Let T denote the topological space $\mathbb{R}_{\geq 0}$. Then define $\mathcal{H} : \mathsf{Top} \to \mathsf{Top}$ as the functor such that for any topological space X, and any continuous map $g : X \to Y$,

$$\mathcal{H}X = \{(f, d) \in X^{\mathsf{T}} \times \mathsf{D} \mid f \cdot \curlywedge_d = f\}, \qquad \mathcal{H}g = g^{\mathsf{T}} \times \mathrm{id}$$

where D is the one-point compactification of T and $\curlywedge_d = \min(\_\ , d)$.

Intuitively, the functor $\mathcal{H} : \mathsf{Top} \to \mathsf{Top}$ captures continuous behaviour as considered in hybrid systems, *i.e.* the continuous evolutions of physical processes, such as the movement of a plane, or the temperature of a room. Document [NB16] provides the following specification for the bouncing ball described above.

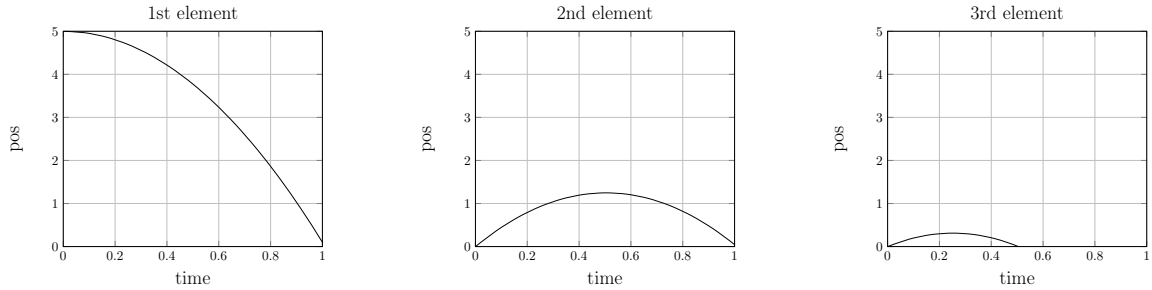
Definition 5.2. Use S, O as shorthand to $\mathbb{R}_{\geq 0} \times \mathbb{R}$, and $\mathbb{R}$, respectively. The bouncing ball is given by the Set-coalgebra $\langle \mathsf{nxt}, \mathsf{out} \rangle : S \to S \times U\mathcal{H}O$

$$\langle \mathsf{nxt}, \mathsf{out} \rangle\ (p, v) = \big((0, u), (\mathsf{mov}(p, v,\ \_\ ), d)\big)$$

where variable u corresponds to the (abrupt) change of velocity due to the collision with the ground, function $\mathsf{mov}(p, v,\ \_\ ) : \mathsf{T} \to O$ describes the ball's movement between jumps, and d denotes the time that the ball takes to reach the ground. In symbols,

$$u = (v + gd) \times -0.5, \quad \mathsf{mov}(p, v, t) = p + vt + \tfrac{1}{2}gt^2, \quad d = \tfrac{\sqrt{2gp + v^2} + v}{g}.$$

Recall that for each set A the functor $(\ \_\ \times A) : \mathsf{Set} \to \mathsf{Set}$ has a final coalgebra (*cf.* [Rut00]), thus providing a notion of behaviour for the ball. To be more concrete, the coalgebra $\big(S, \langle \mathsf{nxt}, \mathsf{out}\rangle\big)$ has a canonical homomorphism $[\![\ \_\ ]\!] : S \to (U\mathcal{H}O)^\omega$ to the final coalgebra $\big((U\mathcal{H}O)^\omega, \langle \mathsf{tl}, \mathsf{hd}\rangle\big)$, where $\mathsf{tl} : (U\mathcal{H}O)^\omega \to (U\mathcal{H}O)^\omega$, and $\mathsf{hd} : (U\mathcal{H}O)^\omega \to U\mathcal{H}O$ are the 'tail' and 'head' functions, respectively. The map $\langle \mathsf{tl}, \mathsf{hd}\rangle$ computes the behaviour of the ball for a given height and velocity. For example, the first three elements of $[\![(0,5)]\!]$ yield the following plots.



In order to bring non-determinism into the scene, suppose, for example, that when the ball hits the ground it loses part of its kinetic energy non-deterministically. In this context, one may consider the coalgebra $\langle \mathsf{nxt}, \mathsf{out}\rangle : S \to \mathcal{P}S \times U\mathcal{H}O$

$$\langle \mathsf{nxt}, \mathsf{out}\rangle\,(p, v) = \big(U, (\mathsf{mov}(p, v,\ \_\ ), d)\big)$$

with $U = \big\{\ \big(0, (v + gd) \times c\big) \in S \mid c \in [-0.7, -0.5]\ \big\}$. However, the functor $(\mathcal{P} \times U\mathcal{H}O) : \mathsf{Set} \to \mathsf{Set}$ has no final coalgebra (*cf.* [Rut00]), and thus there is no canonical notion of behaviour for the non-deterministic bouncing ball specified above. We will show that the issue can be fixed by shifting to Top. For this, the following result is useful.

Proposition 5.3. *Let $\mathcal{V} : \mathsf{Top} \to \mathsf{Top}$ be the compact Vietoris functor. The family $\tau = (\tau_{X,Y})$ of maps*

$$\begin{aligned} \tau_{X,Y} : (\mathcal{V}X) \times Y &\to \mathcal{V}(X \times Y) \\ (S, y) &\mapsto S \times \{y\} \end{aligned}$$

defines a natural transformation

$$\begin{tikzcd} \mathsf{Top} \times \mathsf{Top} \arrow[r, "\times"] \arrow[d, "\mathcal{V}\times\mathrm{Id}"'] \arrow[dr, phantom, "\tau \Nearrow"] & \mathsf{Top} \times \mathsf{Top} \arrow[d, "\mathcal{V}"] \\ \mathsf{Top} \times \mathsf{Top} \arrow[r, "\times"'] & \mathsf{Top}. \end{tikzcd}$$

Proof. Let X and Y be topological spaces. For all $S \in \mathcal{V}X$ and $y \in Y$, since S is compact, the product $S \times \{y\}$ is also compact, which entails that $S \times \{y\} \in \mathcal{V}(X \times Y)$. Then, continuity of the map $\tau_{X,Y}$ is a direct consequence of the equalities below.

$$\tau_{X,Y}^{-1}\Big[\Big(\bigcup_{i\in I} U_i \times V_i\Big)^{\Diamond}\Big] = \bigcup_{i\in I} (U_i)^{\Diamond} \times V_i$$

$$\tau_{X,Y}^{-1}\Big[\Big(\bigcup_{i\in I} U_i \times V_i\Big)^{\Box}\Big] = \bigcup\Big\{\Big(\bigcup_{i\in F} U_i\Big)^{\Box} \times \bigcap_{i\in F} V_i \mid F \subseteq I \text{ finite}\ \Big\}$$

The proof that all naturality squares commute is straightforward. □

Remark 5.4. When the compact Vietoris functor is equipped with the natural transformation above it becomes a strong functor. The latter concept was introduced in [Koc72] and is widely adopted in monadic programming.

With the natural transformation above, it becomes straightforwad to consider the non-deterministic bouncing ball in a topological setting. Actually, it can be shown to be a coalgebra

$$\langle \mathsf{nxt}, \mathsf{out} \rangle : S \to \mathcal{V}S \times \mathcal{H}O$$

First, the map $\mathsf{out} : S \to \mathcal{H}O$ was already shown to be continuous in [NBHM16]. Then, observe that the map $\mathsf{nxt} : S \to \mathcal{V}S$ can be rewritten as a composite

$$S \xrightarrow{\langle f,g \rangle} \mathcal{V}S \times S^S \xrightarrow{\tau} \mathcal{V}(S \times S^S) \xrightarrow{\mathcal{V}\,\mathrm{ev}} \mathcal{V}S$$

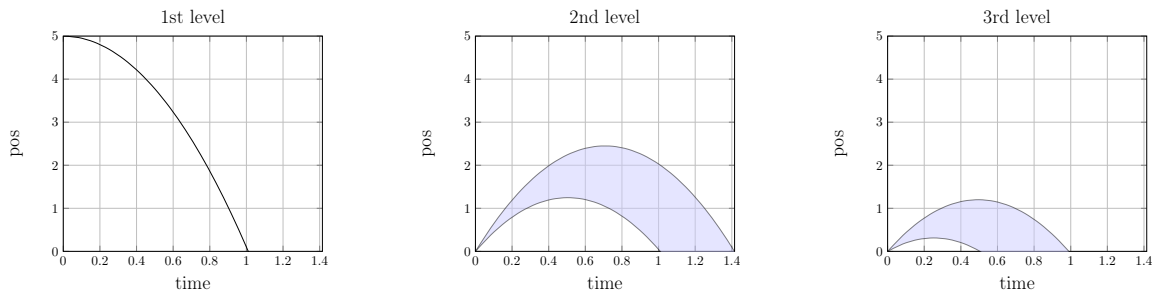
$$f\,(p,v) = \{0\} \times [0.5, 0.7]$$
$$g\,(p,v) = \lambda(x,y) \in S.\,(0, (v + gd) \times -y)$$

which proves our claim. One more result is needed.

Theorem 5.5. *The functor $\mathcal{H} : \mathsf{Top} \to \mathsf{Top}$ can be restricted to the category of Hausdorff spaces.*

Proof. Let X be a locally compact space and Y an Hausdorff space. Then, the function space Y^X equipped with the compact-open topology is Hausdorff (*cf.* [Kel55]). The claim now follows from Hausdorff spaces being closed under products, and subspaces. □

As discussed in the previous sections, every compact Vietoris polynomial functor that can be restricted to the category of Hausdorff spaces has a final coalgebra, which, according to Theorem 5.5, is the case for $\mathcal{V} \times \mathcal{H}O : \mathsf{Top} \to \mathsf{Top}$. Intuitively, the elements of the final $(\mathcal{V} \times \mathcal{H}O)$-coalgebra can be seen as compactly branching trees, *i.e.* trees where the set of sons of each node is compact. This is similar to the property imposed to finitely branching trees, which occur in the final coalgebras involving the *finite* powerset functor (*cf.* [Rut00]). Interestingly, the functor $\mathcal{V} \times \mathcal{H}O : \mathsf{Top} \to \mathsf{Top}$ admits an alternative representation: superimpose the evolutions of each level of the tree. To illustrate this, the non-deterministic bouncing ball yields the following plots for the first two bounces, with the pair $(5, 0)$ as the initial state.



The notion of ***stability*** [Sta01] is another important aspect in the development of hybrid systems. Roughly put, the term 'stability' refers to a system's stability in regard to its behaviour against perturbations; the system is called stable if small changes in its state (or input) only produce small changes in its behaviour — such a notion is directly related to that of distance between behaviours, which was already studied in a coalgebraic setting [BBKK14].

In a Set-based context it is difficult to reason about the stability of a system, because its state space, which is assumed to be just a set, lacks sufficient structure. In the topological setting, however, the issue can be better handled. To start with, observe that topological spaces already carry a notion of proximity, given by the open sets. Moreover, note that the notion of a stable system is closely related to that of a continuous map, as discussed, for example, in [Sta01]. This relation can be precisely described in a coalgebraic context: take a functor $F : \mathsf{Top} \to \mathsf{Top}$, and assume that $\mathsf{CoAlg}(F)$ has a final coalgebra (ν_F, ω_F). Then, for any F-coalgebra (S, c) there is a continuous map $[\![\ \_\ ]\!] : S \to \nu_F$ such that for each state $s \in S$, $[\![s]\!]$ is the associated behaviour. Since the map is continuous, 'close' states must have 'close' behaviours, which coincides with our notion of system stability. This suggests the following coalgebraic definition of stability.

Definition 5.6. Let $F : \mathsf{Top} \to \mathsf{Top}$ be a functor that admits a final coalgebra. Then a (not necessarily continuous) map $c : X \to FX$ is called stable if it is a member of $\mathsf{CoAlg}(F)$. In other words, if it is a continuous map.

Examples 5.7. The bouncing balls $\langle \mathsf{nxt}, \mathsf{out} \rangle : S \to S \times \mathcal{H}O$, and $\langle \mathsf{nxt}, \mathsf{out} \rangle : S \to \mathcal{V}S \times \mathcal{H}O$ are continuous maps, and, consequently, stable systems. In this case calling either of the bouncing balls stable, is to say that a small change in their initial position and velocity does not drastically alter their (possible) trajectories over time.

Finally, note that the systems considered here jump between states discretely, as opposed to their outputs which are, essentially, evolutions in time of specific values. One possible way to accomodate the evolution of states as well is to consider coalgebras in $\mathsf{CoAlg}(\mathcal{H})$. We will use the results of the previous sections to show that this category is also complete.

Definition 5.8. Let $F : \mathsf{C} \to \mathsf{C}$ be a functor over a category C with (co)products. We call F ***exponent polynomial*** if it can be recursively defined from the grammar below, with letters A and B denoting, respectively, an arbitrary object and an exponentiable object of C.

$$F ::= F + F \mid F \times F \mid A \mid \mathrm{Id} \mid F^B$$

Since all exponential functors $(\,\_\,)^B : \mathsf{C} \to \mathsf{C}$ are right adjoints, the following results come almost for free.

Proposition 5.9. *All exponent polynomial functors $F : \mathsf{Top} \to \mathsf{Top}$ preserve connected limits.*

Corollary 5.10. *The categories of coalgebras of all exponent polynomial functors over Top are complete.*

Theorem 5.11. *The category of coalgebras $\mathsf{CoAlg}(\mathcal{H})$ is complete.*

Proof. The previous corollary assures that the category $\mathsf{CoAlg}\left((\,\_\,)^{\mathsf{T}} \times \mathsf{D}\right)$ is complete. Then, observe that the functor $\mathcal{H} : \mathsf{Top} \to \mathsf{Top}$ is a subfunctor of $(\,\_\,)^{\mathsf{T}} \times \mathsf{D} : \mathsf{Top} \to \mathsf{Top}$, and apply Theorem 4.4. □

The previous theorem takes advantage of the adjoint situation below.

$$\begin{tikzcd} \mathsf{CoAlg}\left((\,\_\,)^{\mathsf{T}} \times \mathsf{D}\right) \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathsf{CoAlg}\left(\mathcal{H}\right) \arrow[l, bend left] \end{tikzcd}$$

Then, with the theorem below, and using the results of the previous sections, we obtain a specific method to construct coreflections of $((\,\_\,)^{\mathsf{T}} \times \mathsf{D})$-coalgebras.

Theorem 5.12. *The 'inclusion' natural transformation $\iota : \mathcal{H} \to (\,\_\,)^{\mathsf{T}} \times \mathsf{D}$ is mono-taut.*

Proof. Consider a monomorhism $m : X \to Y$ in Top. We will show that the diagram below is a pullback square.

$$\begin{tikzcd} \mathcal{H}X \arrow[r, "\mathcal{H}m"] \arrow[d, "\iota_X"'] & \mathcal{H}Y \arrow[d, "\iota_Y"] \\ X^{\mathsf{T}} \times \mathsf{D} \arrow[r, "m^{\mathsf{T}} \times id"'] & Y^{\mathsf{T}} \times \mathsf{D} \end{tikzcd}$$

Thus, take two morphisms $f : Z \to X^{\mathsf{T}} \times \mathsf{D}$, $g : Z \to \mathcal{H}Y$, and assume that the equation below holds.

$$(m^{\mathsf{T}} \times id) \cdot f = \iota_Y \cdot g$$

Let $z \in Z$ and put $(x, y) = f(z)$ and $(a, b) = g(z)$. Then, by the definition of $\mathcal{H}$, $a = a \cdot \curlywedge_b$ since $\operatorname{im} g \subseteq \mathcal{H}Y$. Using $(m^{\mathsf{T}} \times id) \cdot f = \iota_Y \cdot g$, one gets $m \cdot x = m \cdot x \cdot \curlywedge_y$; and from this, one obtains $x = x \cdot \curlywedge_y$ since $m : X \to Y$ is a monomorphism. This shows that the condition $\operatorname{im} f \subseteq \mathcal{H}X$ holds. Then, since the map $\iota_X : \mathcal{H}X \to X^{\mathsf{T}} \times \mathsf{D}$ is an embedding, and $\operatorname{im} f \subseteq \operatorname{im} \iota_X$, there must be a unique arrow $h : Z \to \mathcal{H}X$ such that $\iota_X \cdot h = f$. It remains to show that $g = \mathcal{H}m \cdot h$. This is a direct consequence of the diagram above being commutative, and the map $\iota_Y : \mathcal{H}Y \to Y^{\mathsf{T}} \times \mathsf{D}$ mono. □

6. Conclusions and future work

Even if most coalgebraic literature takes Set as the base category, state-based transition systems often call for a shift to other categories, where mechanisms that suitably handle their intricacies are available. Such was the case in [Pan09, Dob09], two research lines on the topic of stochastic systems, and in [KKV04, BFV10, VV14], where the category of Stone spaces and continuous maps plays a key role in setting an appropriate coalgebraic semantics for finitary modal logics.

In our case the base category adopted was Top. As discussed in the previous section, this was because the Set-based context proved to be insufficient for the design of (non-deterministic) hybrid systems, namely in what concerns canonical representations of behaviour and stability. The shift to the topological setting provided, almost for free, a notion of stability (in the spirit of [Sta01]), and showed that a number of non-deterministic hybrid systems in Top have an associated final coalgebra, even if in Set they do not. Both results were achieved using this paper's theoretical developments. But again, we stress that the latter can be applied to other contexts as well.

The relevance of Vietoris coalgebras for different topics is further witnessed by the common existence of important limits in their categories of coalgebras. We saw that every compact Vietoris polynomial functor admits a final coalgebra if it can be restricted to the category Haus while every lower Vietoris polynomial functor admits a final coalgebra if it can be restricted to StablyComp. Moreover, we saw that several variants of such functors also inherit these results and that all categories of Vietoris coalgebras have equalisers.

However, several theoretical questions concerning limits in categories of Vietoris coalgebras still remain open. For example, we studied codirected limit preservation by Vietoris functors under different topological contexts (see Section 3), showing cases in which they were preserved, and cases in which they were not. But we are still not precisely sure what is the 'weakest' context in which they are preserved. Another example concerns the existence of products in categories of Vietoris coalgebras. Recall also our study of topological functors between categories of coalgebras. Among other things, it provides a full characterisation of situations in which it is possible to systematically lift well-known results about coalgebras over Set to coalgebras over other categories. We saw that this is indeed the case between coalgebras of polynomial functors over Set and their counterparts in Top, but we are also interested in other situations. Two prime examples that we will explore in future work pertain coalgebras over the category Ord and coalgebras over the category PMet. These coalgebras have significant relevance within the coalgebraic community (*e.g.* [BBKK14, BK11, BKV13]) and we believe that our study can contribute to the topic.

On a note closer to practice, the use of topologies to specify and analyse (non-deterministic) hybrid systems brings a number of benefits, which were just barely grasped in this paper. Our main goal is to further explore them in the near future. The plan is to do so in a coalgebraic component-based approach [Bar03, HJ11], where simple hybrid systems can be composed to form more complex ones. The results that this paper reports provide an interesting step in this direction.

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