THE RISE AND FALL OF V-FUNCTORS

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Abstract. In this article we study function spaces (rise) and descent (fall) in quantale-enriched categories, paying particular attention to enrichment in the non-negative reals, the quantale of distribution functions and the unit interval equipped with a continuous t-norm.

Introduction

As implicitly suggested by the title, the two main themes of this paper are the study of function spaces (or exponentiation) and descent theory in certain categories of interest to topologists. The quest for appropriate function space topologies goes back to at least the work of Fox [14], the motivation coming from algebraic topology ("a path of paths should be a path"). Roughly speaking, the problem is to topologize Y^X in such a way that, for all spaces Z, there is a natural 1-1 correspondence between continuous maps $Z \to Y^X$ and continuous maps $Z \times X \to Y$. Such a topology does not always exist, and those spaces X where it does for all spaces Y were eventually identified as the core-compact spaces by Day and Kelly in [9]. Curiously, [9] does not mention function spaces at all, but it does characterise those spaces X so that the functor $- \times X$ preserves quotients. However, the original question about function spaces actually asks for a right adjoint of $- \times X$, and then the well-known Special Adjoint Functor Theorem tells us that both questions are equivalent. A nice overview of this development can be found in [19]. The second theme, descent theory, is probably less known in topology. This topic has its roots in Galois theory à la Grothendieck and its categorical presentation was developed by Janelidze (see [2]). In a nutshell, for a morphism $f : E \to B$, one asks when is it possible to describe bundles over B as algebras of bundles over E; in other words, when one can "descend" from bundles over E to bundles over B along f.

Our interest in categories of monad-quantale-enriched categories started 15 years ago, with the main goal of providing a unified setting for the study of exponentiation and descent theory in general topology. This framework was suggested by various results on topological spaces via convergence: the characterization of effective descent continuous maps given in [30] and [20], and the characterization of exponentiable topological spaces of [28] and [25, 26]. With this motivation, we presented a detailed study of effective descent morphisms of categories enriched in a quantale V in [4], and a characterization of exponentiable V-functors in [5, 7]. The purpose of this article is to complete the results obtained in these papers in the realm of quantale-enriched categories, with particular focus on specific examples such as probabilistic metric spaces and categories enriched in the unit interval equipped with a continuous t-norm.

In Section 1 we provide the necessary background on quantales and quantale-enriched categories. We introduce important examples, such as the quantale Δ of distribution functions, we recall the classification of continuous quantale structures on the unit interval $[0, 1]$ due to [12] and [27], and the

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2010 *Mathematics Subject Classification.* 18D10, 18D15, 18D20, 18B35, 18A20, 18A25, 54E35, 54E70.

Key words and phrases. quantale, t-norm, enriched category, (probabilistic) metric space, exponentiation, bounded enriched category, effective descent morphism.

Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT – UID/MAT/04106/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

notion of complete distributive complete lattice, and introduce the notion of cancellable quantale which proves to be useful in our investigation of effective descent V-functors.

In Section 2 we recall the characterization of exponentiable V-categories obtained in [5, 7] and refine this description for the specific choices of quantales introduced in the previous section. Section 2 can be seen as a "warm-up" for Section 3 where we consider the less familiar notion of exponentiable V-functor; similarly to the object-case, we provide customised descriptions for our main examples. Furthermore, we show that perfect and étale V-functors (naturally imported from topology) are exponentiable.

The interpretation of metric spaces as categories advocated in [23] puts the emphasis on the triangular inequality and the "self-distance is zero" axiom, and, consequently, on the study of generalised metric spaces which are not necessarily symmetric, separated and finitary. In Section 4 we show how our results about generalised (probabilistic) metric spaces lead to characterizations of exponentiable morphisms between classical spaces. In particular, we introduce finitary and bounded V-categories relative to a choice of "finite" elements of the quantale V.

Finally, Section 5 is devoted to the study of effective descent V-functors. Following the pattern of the previous sections, we recall and improve the results of [4] and specialise these results in our main examples.

1. V-categories

Throughout V is a *commutative and unital quantale*; that is, V is a complete lattice equipped with a symmetric tensor product $\otimes$, with unit $k \neq \bot$ and with right adjoint hom; that is, for each $u, v, w \in V$,

$$u \otimes v \leq w \iff v \leq \hom(u, w).$$

As a category, V is a *symmetric monoidal closed category.* We recall now the notion of a V-enriched category when V is a quantale. The general notion of V-category goes back to [11], [23] and [22].

Definition 1.1. A *V-enriched category* (or simply *V-category*) is a pair (X, a) with X a set and $a : X \times X \to V$ a map satisfying the following conditions:

(R) for each $x \in X$, $k \leq a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leq a(x, x'')$.

A *V-functor* $f : (X, a) \to (Y, b)$, between the V-categories (X, a), (Y, b), is a map $f : X \to Y$ such that

(C) for each $x, x' \in X$, $a(x, x') \leq b(f(x), f(x'))$.

We will denote by V-**Cat** the category of V-categories and V-functors. Without assuming condition (T), the pair (X, a) is said to be a *V-graph*; together with *V-functors* (i.e., maps between V-graphs satisfying (C)), they form the category V-**Gph**.

Examples 1.2. (1) For $V = 2 = (\{0 < 1\}, \wedge, 1)$, a 2-category is an *ordered set* (not necessarily antisymmetric) and a 2-functor is a *monotone map.*

(2) For the complete lattice $[0, \infty]$ ordered by the "greater or equal" relation $\geq$ (so that the infimum of two numbers is their maximum and the supremum of $S \subseteq [0, \infty]$ is given by $\inf S$) with tensor $\otimes = +$, a $[0, \infty]$-category is a *(generalised) metric space* (see [23]) and a $[0, \infty]$-functor is a *non-expansive map.* We note that

$$\hom(u, v) = v \ominus u := \max\{v - u, 0\},$$

for all $u, v \in [0, \infty]$.

(3) The complete lattice $[0, 1]$ with the usual "less or equal" relation $\leq$ is isomorphic to $[0, \infty]$ via the map $[0, 1] \to [0, \infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. Under this isomorphism, the operation $+$ on $[0, \infty]$ corresponds to multiplication $*$ on $[0, 1]$. We denote this quantale as $[0, 1]_*$; hence, $[0, 1]_*$-**Cat** is isomorphic to the category $[0, \infty]$-**Cat** (with tensor $\otimes = +$ on

$[0,\infty]$) of (generalised) metric spaces and non-expansive maps. To keep notation simple, we will often write uv instead of $u * v$, for $u, v \in [0,1]$.

Since $[0,1]$ is a frame, we can also consider it as a quantale with $\otimes = \wedge$ given by infimum, and we denote it by $[0,1]_\wedge$. The category $[0,1]_\wedge$-**Cat** is isomorphic to the category of *(generalised) ultrametric spaces* and *non-expansive maps*.

Quantale operations on $[0,1]$ are usually called *t-norms*. Another interesting t-norm is given by the *Łukasiewicz tensor* where $u \oplus v = \max(0, u+v-1)$; here $\hom(u,v) = \min(1, 1-u+v)$. In fact, it is shown in [12] and [27] that every *continuous* t-norm $\otimes : [0,1] \times [0,1] \to [0,1]$ with neutral element 1 is a combination of the three operations on $[0,1]$ described above. To describe this result, we need some notation: an element $u \in [0,1]$ is called *idempotent* whenever $u \otimes u = u$, and $v \in [0,1]$ is called *nilpotent* whenever $v \neq 0$ and $v^n = 0$, for some $n \in \mathbb{N}$. We have now the following facts.

Theorem. *(a) If* 0 *and* 1 *are the only idempotent elements of* $[0,1]$ *and* $[0,1]$ *has no nilpotent element, then* $\otimes = *$ *is the multiplication.*

(b) If 0 *and* 1 *are the only idempotent elements of* $[0,1]$ *and* $[0,1]$ *has a nilpotent element, then* $\otimes = \oplus$ *is the Łukasiewicz tensor.*

(c) For $u, v \in [0,1]$ *and* $e \in [0,1]$ *idempotent with* $u \le e \le v$*:* $u \otimes v = \min(u,v) = u$.

(d) For every non-idempotent $u \in [0,1]$*, there exist idempotents* e *and* f *with* $e < u < f$ *and such that the interval* $[e,f]$ *(with the restriction of the tensor on* $[0,1]$ *and with neutral element* f*) is isomorphic to* $[0,1]$ *with either multiplication or Łukasiewicz tensor.*

Throughout we will study the behaviour of $[0,1]_\otimes$-**Cat** for a general continuous tensor product $\otimes$ on $[0,1]$.

(4) We consider now the set

$$\Delta = \{\varphi : [0,\infty] \to [0,1] \mid \text{for all } \alpha \in [0,\infty]\text{: } \varphi(\alpha) = \bigvee_{\beta<\alpha} \varphi(\beta)\},$$

of distribution functions. Δ is a complete lattice with the pointwise order. For $\varphi, \psi \in \Delta$ and $\alpha \in [0,\infty]$, define $\varphi \otimes \psi \in \Delta$ by

$$\varphi \otimes \psi(\alpha) = \bigvee_{\beta+\gamma\le\alpha} \varphi(\beta) * \psi(\gamma).$$

One easily verifies that the operation $\otimes : \Delta \times \Delta \to \Delta$ is associative and commutative, and that

$$\kappa : [0,\infty] \to [0,1],\ \alpha \mapsto \begin{cases} 0 & \text{if } \alpha = 0,\\ 1 & \text{else}\end{cases}$$

is a neutral element for $\otimes$. Finally, $\psi \otimes - : \Delta \to \Delta$ preserves suprema since $u * - : [0,1] \to [0,1]$ does so, for all $u \in [0,1]$. For the quantale Δ just described, a Δ-category is a *(generalised) probabilistic metric space* and a Δ-functor is a *probabilistic non-expansive map*. Probabilistic metric spaces are introduced in [24] and extensively described in [32]; a presentation as enriched categories can be found in [3] and [17]. There are two different ways of embedding the category of metric spaces in the category of probabilistic metric spaces, corresponding to the two different descriptions of metric spaces as $[0,\infty]$-**Cat** and $[0,1]_*$-**Cat**:

$$\begin{aligned} \sigma : [0,\infty]\text{-}\mathbf{Cat} &\to \Delta\text{-}\mathbf{Cat} \\ (X,a) &\mapsto (X,\tilde{a}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tau : [0,1]_*\text{-}\mathbf{Cat} &\to \Delta\text{-}\mathbf{Cat} \\ (X,a) &\mapsto (X,\hat{a}) \end{aligned} \tag{1.i}$$

with $\tilde{a}(x,y)(\beta) = 1$ if $\beta > a(x,y)$ and $\tilde{a}(x,y)(\beta) = 0$ elsewhere, while $\hat{a}(x,y)(\beta) = a(x,y)$ if $\beta \neq 0$ and $\hat{a}(x,y)(0) = 0$.

(5) If $(M, \cdot, e)$ is a commutative monoid, then the lattice $(PM, \subseteq)$, with the tensor product defined by

$$M' \times N' = \{m \cdot n \mid m \in M', n \in N'\},$$

for $M', N' \subseteq M$, with unit $\{e\}$, is a commutative and unital quantale[1]; in fact, it is the free quantale over $(M, \cdot, e)$. Categories enriched in such a quantale are extensively studied in [18].

We discuss now some properties of V which will be useful in the sequel. Denoting by $\mathrm{Dn}\, V$ the lattice of down-sets of V ordered by inclusion, the monotone map (=functor)

$$\begin{aligned} \downarrow : V &\longrightarrow \mathrm{Dn}\, V \\ v &\longmapsto \downarrow v = \{u \in V \mid u \leq v\} \end{aligned}$$

has a left adjoint

$$\bigvee : \mathrm{Dn}\, V \longrightarrow V$$

because V is a complete lattice (in fact, the existence of this adjoint is equivalent to completeness of V). The lattice V is said to be *constructively completely distributive (ccd)* (see [29] and [13]) if this left adjoint has itself a left adjoint

$$\Downarrow : V \longrightarrow \mathrm{Dn}\, V,$$

so that

$$\Downarrow v \subseteq S \iff v \leq \bigvee S. \tag{1.ii}$$

Defining the *totally below relation* $\ll$ on V by

$$\begin{aligned} u \ll v \quad &\iff \quad u \in \Downarrow v \\ &\iff \quad \forall S \in \mathrm{Dn}\, V \ v \leq \bigvee S \implies u \in S \\ &\iff \quad \forall A \subseteq V \ v \leq \bigvee A \implies \exists w \in A \ : \ u \leq w \end{aligned}$$

condition (1.ii) gives that, for every $v \in V$,

$$v = \bigvee \{u \in V \mid u \ll v\}.$$

(See [35] for details.)

Examples 1.3. (1) The lattice 2 is ccd and its totally below relation $\ll$ is given by $0 \ll 1$ and $1 \ll 1$.

(2) The lattice $[0, \infty]$ (and so also $[0, 1]$) is ccd, with the totally below relation $\ll$ given by $>$ (respectively by $<$).

(3) The lattice Δ is ccd. To describe its totally below relation, it is useful to consider some special elements of Δ: the *step functions* $\varphi_{\alpha,u}$ (where $\alpha \in [0, \infty]$ and $u \in [0, 1]$) defined by

$$\varphi_{\alpha,u}(\beta) = \begin{cases} 0 & \text{if } \beta \leq \alpha, \\ u & \text{if } \beta > \alpha. \end{cases}$$

One has the following facts.

Lemma. *For every $\psi, \chi \in \Delta$:*
(a) $\psi = \bigvee \{\varphi_{\alpha,u} \mid u < \psi(\alpha)\}$;
(b) for every $\alpha \in [0, \infty]$ and $u \in [0, 1]$, $\varphi_{\alpha,u} \ll \psi \iff u < \psi(\alpha)$;
(c) $\chi \ll \psi \iff \exists \alpha \in [0, \infty] \ : \ \chi(\alpha) = 0$ and $\chi(\infty) < \psi(\alpha)$.

Proof. The equality of (a) is straightforward. To prove (b), let $\alpha \in [0, \infty]$ and $u \in [0, 1]$. Assume first that $u < \psi(\alpha)$ and let $(\psi_i)_{i \in I}$ be a family in Δ such that $\psi \leq \bigvee_{i \in I} \psi_i$. Then there exists $j \in I$ with $\psi_j(\alpha) > u$ and this is enough to conclude that $\varphi_{\alpha,u} \leq \psi_j$. Assume now $\varphi_{\alpha,u} \ll \psi$; by (a) there is some $\beta \in [0, \infty]$ and $v \in [0, 1]$ with $v < \psi(\beta)$ so that $\varphi_{\alpha,u} \leq \varphi_{\beta,v}$, that is, $\alpha \geq \beta$ and $u \leq v$. We conclude that $\psi(\alpha) \geq \psi(\beta) > v \geq u$. To see (c) note that, by (a) and (b), $\chi \ll \psi \iff \exists \alpha, u \ \chi \leq \varphi_{\alpha,u} \ll \psi$, and this is equivalent to the conditions $\chi(\alpha) = 0$ and $\chi(\infty) < \psi(\alpha)$. □

[1]For simplicity here we assume commutativity of the tensor, although it is not essential for most of the results obtained.

(4) PM is also ccd: $S \ll N \iff S = \{x\}$ for some $x \in N$, for $S, N \subseteq M$.

Now we will analyse a condition on V that plays a crucial role in the study of descent. We say that *the tensor product* $\otimes$ *in* V *is cancellable* (or simply that V *is cancellable*) if, for any $u, v \in V \setminus \{\bot\}$, for any families $(u_i)_{i\in I}$, $(v_i)_{i\in I}$ in V with $u_i \le u$ and $v_i \le v$ for every $i \in I$,

$$\bigvee_{i\in I}(u_i \otimes v_i) \ge u \otimes v \text{ in } V \implies \forall u' \ll u,\, v' \ll v\ \ \exists i \in I \,:\, u' \le u_i \text{ and } v' \le v_i.$$

Proposition 1.4. *Let V be a ccd lattice where the set $\{w \in V \mid w \ll k\}$ is directed. If, for any $v \in V \setminus \{\bot\}$, $v \otimes - : V \to V$*

(1) preserves the totally below relation,
(2) is full, i.e. $v \otimes w \le v \otimes w' \implies w \le w'$,

then V is cancellable.

Proof. Let $u, v \in V \setminus \{\bot\}$, $(u_i)_{i\in I}$, $(v_i)_{i\in I}$ be families in V with $u_i \le u$ and $v_i \le v$ for every $i \in I$, such that

$$\bigvee_{i\in I}(u_i \otimes v_i) \ge u \otimes v.$$

Let $u' \ll u$ and $v' \ll v$. Since $u = u \otimes \bigvee_{w\ll k} w = \bigvee_{w\ll k} u \otimes w$, $v = \bigvee_{w\ll k} v \otimes w$ and $\{w \in V \mid w \ll k\}$ is directed, there exists $w \ll k$ with $u' \le u \otimes w$ and $v' \le v \otimes w$. Condition (1) guarantees that $u \otimes v \otimes w \ll u \otimes v$, and then, by definition of $\ll$, there exists $j \in I$ such that $u \otimes v \otimes w \le u_j \otimes v_j$. Therefore, from

$$u \otimes v \otimes w \le u_j \otimes v_j \le u \otimes v_j$$

and (2) we conclude that $v \otimes w \le v_j$, and, analogously, that $u \otimes w \le u_j$. Hence,

$$\bigvee_{i\in I}(u_i, v_i) \ge \bigvee_{w\ll k}(u \otimes w, v \otimes w) = (u \otimes \bigvee_{w\ll k} w, v \otimes \bigvee_{w\ll k} w) = (u, v). \qquad \square$$

Examples 1.5. (1) It follows directly from Proposition 1.4 that the quantales 2, $[0,\infty]$, $[0,1]_*$ are cancellable.

(2) The quantale $[0,1]_\oplus$ is not cancellable; for instance, if $u_i = 0 = v_i$ and $u = v = \frac{1}{2}$, then $0 = \bigvee(u_i \oplus v_i) = u \oplus v$ although $\bigvee u_i = 0 \ne u$.

(3) In Δ we do not know whether, for any $\psi \in \Delta$, $\psi \otimes -$ is full, that is, $\psi \otimes \chi \le \psi \otimes \chi'$ implies $\chi \le \chi'$. We know, however, that $\{\psi \in \Delta \mid \psi \ll \kappa\}$ is directed, but in general $\psi \otimes -$ does not preserve the totally below relation: indeed, for $\alpha \in [0,\infty]$ and $u \in V$, $\varphi_{\alpha,u} \ll k \iff \alpha > 0$ and $u < 1$, while $\psi \otimes \varphi_{\alpha,u} \ll \psi \iff \exists \beta \le \alpha \,:\, \psi(\beta) < u\psi(\infty)$.

To remedy this problem we will study *cancellability* for the subset Δ_0 of Δ consisting of the step-functions. First we show that:

(a) $\varphi_{\beta,v} \ll k \implies \varphi_{\alpha,u} \otimes \varphi_{\beta,v} \ll \varphi_{\alpha,u}$ for any $\varphi_{\alpha,u} \in \Delta_0$;
(b) $\psi \otimes - : \Delta_0 \to \Delta$ is full, for any $\psi \in \Delta$.

To show (a), let $\varphi_{\beta,v} \ll k$, that is, $\beta > 0$ and $v < 1$. Then $\varphi_{\alpha,u} \otimes \varphi_{\beta,v} = \varphi_{\alpha+\beta,uv} \ll \varphi_{\alpha,u}$ since $uv < \varphi_{\alpha,u}(\alpha + \beta) = u$.

For (b) first we compute $\psi \otimes \varphi_{\alpha,u}(\beta) = u\psi(\beta - \alpha)$ for $\beta > \alpha$, and 0 elsewhere. Then, if $\psi \otimes \varphi_{\alpha,u} \le \psi \otimes \varphi_{\beta,v}$, $(\psi \otimes \varphi_{\alpha,u})(\beta) = (\psi \otimes \varphi_{\beta,v})(\beta) = 0$, and so $\beta \le \alpha$. Moreover, for $\gamma \in \,]0,\infty]$,

$$(\psi \otimes \varphi_{\alpha,u})(\gamma + \alpha) = u\psi(\alpha) \le v\psi(\gamma + \alpha - \beta) = (\psi \otimes \varphi_{\beta,v})(\gamma + \alpha),$$

and then

$$\psi(\gamma + n(\alpha - \beta)) \ge \psi(\alpha)\left(\frac{u}{v}\right)^n.$$

Since ψ is bounded by 1, $\frac{u}{v}$ is necessarily less than or equal to 1, i.e. $u \le v$ as claimed.

Now, adapting the proof that V is cancellable provided (a) and (b) hold, it is routine to show that Δ_0 is cancellable in Δ.

2. Exponentiable V-categories

An object X of a finitely complete category $\mathbf{C}$ is called *exponentiable* whenever the functor $-\times X:\mathbf{C}\to\mathbf{C}$ is left adjoint; its right adjoint is typically denoted as $(-)^X:\mathbf{C}\to\mathbf{C}$. Similarly, a morphism $f:X\to Y$ in $\mathbf{C}$ is called *exponentiable* if f is an exponentiable object in the slice category $\mathbf{C}\downarrow Y$. The category $\mathbf{C}$ is called *cartesian closed* whenever every object of $\mathbf{C}$ is exponentiable; and $\mathbf{C}$ is called *locally cartesian closed* if every morphism of $\mathbf{C}$ is exponentiable.

From now on we assume that V, as a category, is cartesian closed. The existence of a right adjoint to $-\wedge v:V\to V$ for every $v\in V$ is in fact equivalent to the existence of a Heyting operation on V. Therefore, our assumption means that *from now on V is a Heyting algebra.* We remark that in all examples treated here this assumption is fulfilled.

The categories V-**Gph** and V-**Cat** are both complete, in fact, V-**Cat** is closed under limits in V-**Gph**. In particular, for V-functors $f:(X,a)\to(Y,b)$ and $g:(Z,c)\to(Y,b)$, their pullback can be taken as the V-category $(X\times_Y Z,d)$, where $X\times_Y Z$ is the pullback in **Set** and $d((x,y),(z,w))=a(x,z)\wedge c(y,w)$, equipped with the canonical projections. Hence, one can investigate whether a V-category, or a V-functor, is exponentiable; and in this section we consider exponentiable V-categories, reserving the next one for the study of exponentiable V-functors.

The following characterization of exponentiable objects in V-**Cat** was proved in [5, Corollary 3.5] under the condition that k is the top element of V, and later, in [7, Section 5], without this extra condition.

Theorem 2.1. *A V-category (X,a) is exponentiable in V-**Cat** if, and only if, for all $x_0,x_1\in X$ and $v_0,v_1\in V$,*
$$\bigvee_{x\in X}(a(x_0,x)\wedge v_0)\otimes(a(x,x_1)\wedge v_1)\geq a(x_0,x_1)\wedge(v_0\otimes v_1). \tag{2.i}$$

When $\otimes=\wedge$ is the categorical product in V, then the condition above reduces to
$$\bigvee_{x\in X}a(x_0,x)\wedge a(x,x_1)\geq a(x_0,x_1),$$
which is trivially true, and in this case V-**Cat** is a cartesian closed category.

In [17] it was shown that:

Proposition 2.2. *The V-category V is exponentiable if, and only if, for all $u,v,w\in V$,*
$$(u\otimes v)\wedge w=\bigvee\{u'\otimes v'\,|\,u'\leq u,v'\leq v,u'\otimes v'\leq w\}.$$

Examples 2.3. (1) As observed after Theorem 2.1, since in 2 the tensor product is $\wedge$, the category 2-**Cat** of ordered sets and monotone maps is cartesian closed. The same argument applies to $[0,1]_\wedge$-**Cat**.

(2) As shown in [5], *a metric space (X,a) is exponentiable in $[0,\infty]$-**Cat** if, and only if, for each $x_0,x_1\in X$, $\alpha_0,\alpha_1\in[0,\infty[$ with $\alpha_0+\alpha_1=a(x_0,x_1)$, and $\varepsilon>0$,*
$$\exists x\in X\ :\ a(x_0,x)<\alpha_0+\varepsilon \text{ and } a(x,x_1)<\alpha_1+\varepsilon,$$
based on the fact that it is enough to consider in (2.i) α_0,α_1 with $\alpha_0+\alpha_1=a(x_0,x_1)$.

(3) The same happens in $[0,1]_\oplus$-**Cat**. Indeed:

Proposition. *A $[0,1]_\oplus$-category (X,a) is exponentiable in $[0,1]_\oplus$-**Cat** if, and only if, for each $x_0,x_1\in X$, $v_0,v_1\in[0,1]$ with $v_0\oplus v_1=a(x_0,x_1)\neq 0$, and $\varepsilon>0$,*
$$\exists x\in X\ :\ a(x_0,x)+\varepsilon>v_0 \text{ and } a(x,x_1)+\varepsilon>v_1. \tag{2.ii}$$

Proof. If $a(x_0,x_1)=0$, (2.i) is trivially valid. If $v_0\oplus v_1=a(x_0,x_1)\neq 0$, then from (2.i) we conclude that
$$\bigvee_{x\in X}(a(x_0,x)\wedge v_0)+(a(x,x_1)\wedge v_1)=v_0+v_1,$$
and so we get (2.ii).

To prove the converse, for $v_0 \oplus v_1 < a(x_0,x_1)$ consider $v_0' \geq v_0$ and $v_1' \geq v_1$ such that $v_0' \oplus v_1' = a(x_0,x_1)$ and use (2.ii) to derive (2.i). For $v_0 \oplus v_1 > a(x_0,x_1)$ we consider $v_0' \leq v_0$ and $v_1' \leq v_1$ such that $v_0' \oplus v_1' = a(x_0,x_1)$ and use again (2.ii) to conclude (2.i). $\square$

(4) The characterization given in (2) transfers to $[0,1]_*$-**Cat** via the isomorphisms $[0,1] \to [0,\infty]$, $u \mapsto -\ln u$, and $[0,\infty] \to [0,1]$, $\alpha \mapsto \exp(-\alpha)$, so that *a $[0,1]_*$-category (X,a) is exponentiable in $[0,1]_*$-**Cat** if, and only if, for each $x_0,x_1 \in X$, $u_0,u_1 \in]0,1]$ with $u_0u_1 = a(x_0,x_1)$, and for each $\varepsilon > 0$,*

$$\exists x \in X \ : \ a(x_0,x) + \varepsilon > u_0 \text{ and } a(x,x_1) + \varepsilon > u_1.$$

(5) We consider now $V = \Delta$, the quantale of distribution functions. We omit the proof of the following result because it follows from the corresponding result for Δ-functors: see the proof of Proposition 3.7 (3).

Proposition. *A Δ-category X is exponentiable in Δ-**Cat** if, and only if, for each $x_0,x_1 \in X$, $\alpha_0,\alpha_1,\beta \in [0,\infty]$ with $\alpha_0+\alpha_1 < \beta$, $u_0,u_1 \in]0,1]$ with $u_0u_1 = a(x_0,x_1)(\beta)$, $\varepsilon > 0$, there exists $x \in X$, $\alpha_0',\alpha_1' \in [0,\infty]$ such that $\alpha_0' > \alpha_0$, $\alpha_1' > \alpha_1$, $\alpha_0'+\alpha_1' = \beta$ and $a(x_0,x)(\alpha_0') + \varepsilon > u_0$, $a(x,x_1)(\alpha_1') + \varepsilon > u_1$.*

From this characterization one can conclude easily that both embeddings of metric spaces into probabilistic metric spaces of (1.i) preserve and reflect exponentiable objects:

Corollary. *A metric space $(X,a) \in [0,\infty]$-**Cat** is exponentiable in $[0,\infty]$-**Cat** if, and only if, the probabilistic metric space $\sigma(X,a)$ is exponentiable in Δ-**Cat**, and a metric space $(X,a) \in [0,1]_*$-**Cat** is exponentiable in $[0,1]_*$-**Cat** if, and only if, $\tau(X,a)$ is exponentiable in Δ-**Cat**.*

(6) It is easy to conclude, from Theorem 2.1, that *a PM-category (X,a) is exponentiable in PM-**Cat** if, and only if, for each $x_0,x_1 \in X$, $m,n \in M$, whenever $mn \in a(x_0,x_1)$ there exists $x \in X$ such that $m \in a(x_0,x)$ and $n \in a(x,x_1)$.*

(7) We present now an example showing that the condition of Proposition 2.2 is not always satisfied; that is, V is not always exponentiable in V-**Cat**. This example also shows that the claim of [17] that every linearly ordered V is an exponentiable V-category is false. Indeed, if we take $N = \{0, \frac{1}{n},\, n \in \mathbb{N}\}$ with the usual order and multiplication, then, for $u = \frac{1}{2}$, $v = \frac{1}{3}$ and $w = \frac{1}{7}$, the condition does not hold. Indeed, there are very few exponentiable N-categories, as we show next. *An N-category (X,a) is exponentiable if, and only if, for all $x_0,x_1 \in X$,*
 (a) $\forall n_0,n_1 \in \mathbb{N} \ : \ a(x_0,x_1) = \frac{1}{n_0n_1} \implies \exists x \in X \ : \ a(x_0,x) = \frac{1}{n_0}$ *and* $a(x,x_1) = \frac{1}{n_1}$*;*
 (b) $a(x_0,x_1) \in \{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$*, and*
 (c) $a(x_0,x_1) = \frac{1}{8} \implies \exists x \in X \ : \ a(x_0,x) = a(x,x_1) = \frac{1}{3}$.
Condition (a) follows directly from (2.i) when $v_0 = \frac{1}{n_0}$ and $v_1 = \frac{1}{n_1}$.

To conclude (b) we split the problem in several cases:
- If $a(x_0,x_1) = \frac{1}{p}$ with p prime > 3, take $v_0 = v_1 = \frac{1}{2}$ in (2.i); then it is clear that there is no $x \in X$ such that $(a(x_0,x) \wedge \frac{1}{2})(a(x,x_1) \wedge \frac{1}{2}) = \frac{1}{p}$.
- If $a(x_0,x_1) = \frac{1}{2^4}$, take $v_0 = \frac{1}{3}$ and $v_1 = \frac{1}{5}$. Then, again, there is no possible x satisfying (2.i): $(a(x_0,x) \wedge \frac{1}{3})(a(x,x_1) \wedge \frac{1}{5}) \neq \frac{1}{16}$.
- If $a(x_0,x_1) = \frac{1}{3^2}$, take $v_0 = \frac{1}{2}$ and $v_1 = \frac{1}{4}$ and argue analogously.
- If $a(x_0,x_1) = \frac{1}{2^3 3}$, take $v_0 = v_1 = \frac{1}{5}$; the only way of having $(a(x_0,x) \wedge \frac{1}{5})(a(x,x_1) \wedge \frac{1}{5}) = \frac{1}{25}$ is when $a(x_0,x) = a(x,x_1) = \frac{1}{5}$ and we know already that this value is never attained.

If $a(x_0,x_1) = \frac{1}{8}$, then the only case in (2.i) that is not guaranteed by (a) is when $v_0 = v_1 = \frac{1}{3}$, and that is why condition (c) is necessary.

If $a(x_0,x_1)$ is either $\frac{1}{4}, \frac{1}{3}$ or $\frac{1}{2}$, then (a) is enough to assure that (2.i) holds.

3. Exponentiable V-functors

In order to identify exponentiable morphisms in V-**Cat**, we shall make use of the characterization of exponentiability in slice categories given in [10]: $f : X \to Y$ is exponentiable in $\mathbf{C} \downarrow Y$ if, and only if, for every object Z in $\mathbf{C}$, the partial product of Z over f exists.

Theorem 3.1 ([8])**.** *The category V-**Gph** is locally cartesian closed.*

For $f:(X,a)\to(Y,b)$ and (Z,c) in V-**Gph**, the partial product of (Z,c) over f can be constructed as follows. For $y\in Y$, consider the V-graph

$$X_y=\{x\in X\mid f(x)=y\}$$

with structure a_y defined by

$$a_y(x_0,x_1)=k\wedge a(x_0,x_1),$$

and put

$$P=\{(s,y)\mid y\in Y, s:(X_y,a_y)\to(Z,c)\text{ in }V\text{-}\mathbf{Gph}\}.$$

The set P becomes a V-graph when equipped with the largest map $d:P\times P\to V$ (w.r.t. the pointwise order induced by V) making the maps

$$p:P\to Y,\qquad\qquad \mathrm{ev}:P\times_Y X\to Z$$
$$(s,y)\mapsto y\qquad\qquad (s,y,x)\mapsto s(x)$$

V-functors.

$$\begin{tikzcd}
Z & P\times_Y X \arrow[l, "\mathrm{ev}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\
 & P \arrow[r, "p"'] & Y
\end{tikzcd}$$

We note that for an element (s,y,x) of the pullback $P\times_Y X$ one has $y=f(x)$, hence we will simply write (s,x). Explicitly, for (s_0,y_0) and (s_1,y_1) in P,

$$d((s_0,y_0),(s_1,y_1))=\bigvee\{u\le b(y_0,y_1)\mid \forall x_0\in f^{-1}(y_0),x_1\in f^{-1}(y_1),$$
$$(u\wedge a(x_0,x_1)\le c(s_0(x_0),s_1(x_1)))\}.$$

Regarding the connection with exponentiability in V-**Cat**, we recall (see [31, Theorem 2.3]):

Theorem 3.2. *For a V-functor $f:(X,a)\to(Y,b)$ the following assertions are equivalent.*

*(i) f is exponentiable in V-**Cat**.*
*(ii) For every V-category (Z,c), the partial product of (Z,c) over f constructed in V-**Gph** is actually a V-category.*
*(iii) The partial product of $(V,\hom)$ over f constructed in V-**Gph** is a V-category.*

Finally, the following characterization of exponentiable morphisms in V-**Cat** can be found in [5] and [7].

Theorem 3.3. *A V-functor $f:X\to Y$ is exponentiable in V-**Cat** if, and only if, for any $x_0,x_1\in X$, $y\in Y$, $v_0,v_1\in V$ such that $v_0\le b(f(x_0),y)$ and $v_1\le b(y,f(x_1))$,*

$$\bigvee_{x\in f^{-1}(y)}(a(x_0,x)\wedge v_0)\otimes(a(x,x_1)\wedge v_1)\ge a(x_0,x_1)\wedge(v_0\otimes v_1).\tag{3.i}$$

To show that the condition above is necessary, the proof of [5] makes use of the V-functors

$$s:X_y\to V,\qquad s':X_y\to V,\tag{3.ii}$$
$$z\mapsto a(x_0,z)\wedge k\qquad z\mapsto a(x_0,z)\wedge v_0$$
$$s'':X_y\to V;$$
$$z\mapsto\bigvee_{x\in f^{-1}(y)}(a(x_0,x)\wedge v_0)\otimes(a(x,z)\wedge v_1)$$

and in the sequel we will adapt this argument to certain subcategories of V-**Cat**.

Corollary 3.4. *When $\otimes = \wedge$ is the categorical product in V, then a V-functor $f : (X, a) \to (Y, b)$ is exponentiable in V-**Cat** if, and only if, for any $x_0, x_1 \in X$ and $y \in Y$,*

$$\bigvee_{x \in f^{-1}(y)} a(x_0, x) \wedge a(x, x_1) \geq a(x_0, x_1) \wedge b(f(x_0), y) \wedge b(y, f(x_1)).$$

We recall from [4, Definition 4.1] that a V-functor $f : (X, a) \to (Y, b)$ (in V-**Gph** or V-**Cat**) is called *proper* if, for all $x_0 \in X$ and $y \in Y$,

$$b(f(x_0), y) = \bigvee_{x \in f^{-1}(y)} a(x_0, x);$$

dually, f is called *open* if

$$b(y, f(x_1)) = \bigvee_{x \in f^{-1}(y)} a(x, x_1),$$

for all $x_1 \in X$ and $y \in Y$.

Clearly, $f : (X, a) \to (Y, b)$ is open if, and only if, the V-functor $f^{\mathrm{op}} : (X, a)^{\mathrm{op}} \to (Y, b)^{\mathrm{op}}$ is proper, where $(X, a)^{\mathrm{op}} = (X, a^\circ)$, with $a^\circ(x, y) = a(y, x)$, is the dual V-category of (X, a). Furthermore, $f : (X, a) \to (Y, b)$ is called *perfect* (*étale*) whenever both f and the canonical map

$$\delta_f : (X, a) \to (X, a) \times_{(Y,b)} (X, a),\ x \mapsto (x, x)$$

are proper (open). One easily verifies that $\delta_f : (X, a) \to (X, a) \times_{(Y,b)} (X, a)$ is proper if, and only if, for all $x, x_0, x_1 \in X$ with $f(x_0) = f(x_1)$ and $x_0 \neq x_1$,

$$a(x, x_0) \wedge a(x, x_1) = \bot.$$

Theorem 3.5. *Every perfect V-functor is exponentiable in V-**Cat**.*

Proof. Let $f : (X, a) \to (Y, b)$ in V-**Cat** be perfect, and let (Z, c) be a V-category. We prove that the partial product (P, d) of (Z, c) over f formed in V-**Gph** is a V-category. To this end, let (s_0, y_0), (s, y) and (s_1, y_1) be in P. To conclude

$$d((s_0, y_0), (s, y)) \otimes d((s, y), (s_1, y_1)) \leq d((s_0, y_0), (s_1, y_1)),$$

we show that

$$(d((s_0, y_0), (s, y)) \otimes d((s, y), (s_1, y_1))) \wedge a(x_0, x_1) \leq c(s_0(x_0), s_1(x_1)),$$

for all $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$. Recall from [4, Proposition 4.2 (1)] that proper maps are pullback-stable in V-**Cat**. Therefore, since with f also the V-functor $\pi_1 : P \times_Y X \to P$ is proper, we get (with $\tilde{d}$ denoting the structure on $P \times_Y X$)

$$\begin{aligned} d((s_0, y_0), (s, y)) \otimes d((s, y), (s_1, y_1)) &= \bigvee_{x \in f^{-1}(y)} \tilde{d}((s_0, x_0), (s, x)) \otimes d((s, y), (s_1, y_1)) \\ &= \bigvee_{x \in f^{-1}(y), x_1' \in f^{-1}(y_1)} \tilde{d}((s_0, x_0), (s, x)) \otimes \tilde{d}((s, x), (s_1, x_1')). \end{aligned}$$

Now let $x \in f^{-1}(y)$ and $x_1' \in f^{-1}(y_1)$. If $x_1 \neq x_1'$, then

$$(\tilde{d}((s_0, x_0), (s, x)) \otimes \tilde{d}((s, x), (s_1, x_1'))) \wedge a(x_0, x_1) \leq a(x_0, x_1') \wedge a(x_0, x_1) = \bot;$$

otherwise we obtain

$$\begin{aligned} &(\tilde{d}((s_0, x_0), (s, x)) \otimes \tilde{d}((s, x), (s_1, x_1))) \wedge a(x_0, x_1) \\ &\qquad \leq c(s_0(x_0), s(x)) \otimes c(s(x), s_1(x_1)) \leq c(s_0(x_0), s_1(x_1)). \qquad \square \end{aligned}$$

The dual version of the result above reads as:

Corollary 3.6. *Every étale V-functor is exponentiable in V-**Cat**.*

Examples 3.7. (1) As stated in [5], from Theorem 3.3 it follows that *a monotone map* $f : (X, \leq) \to (Y, \leq)$ *is exponentiable in* 2-**Cat** *if, and only if, for each* $x_0, x_1 \in X$ *and* $y \in Y$ *with* $x_0 \leq x_1$ *and* $f(x_0) \leq y \leq f(x_1)$*, there exists* $x \in f^{-1}(y)$ *such that* $x_0 \leq x \leq x_1$ (see also [33]).

(2) In [5] it was shown that *a non-expansive map* $f : (X, a) \to (Y, b)$ *is exponentiable in* $[0, \infty]$-**Cat** *if, and only if, for each* $x_0, x_1 \in X$*,* $y \in Y$*,* $\alpha_0, \alpha_1 \in [0, \infty[$ *with* $\alpha_0 \geq b(f(x_0), y)$*,* $\alpha_1 \geq b(y, f(x_1))$ *and* $\alpha_0 + \alpha_1 = \max\{a(x_0, x_1), b(f(x_0), y) + b(y, f(x_1))\}$*, and* $\varepsilon > 0$*,*

$$\exists x \in f^{-1}(y) \;:\; a(x_0, x) < \alpha_0 + \varepsilon \text{ and } a(x, x_1) < \alpha_1 + \varepsilon.$$

(3) For $V = \Delta$, we have:

Proposition. *A* Δ*-functor* $f : X \to Y$ *is exponentiable in* Δ-**Cat** *if, and only if, for each* $x_0, x_1 \in X$*,* $\alpha_0, \alpha_1, \beta \in [0, \infty]$ *with* $\alpha_0 + \alpha_1 < \beta$*,* $u_0, u_1 \in]0, 1]$ *with* $u_0 u_1 \leq a(x_0, x_1)(\beta)$*,* $u_0 \leq b(f(x_0), y)(\alpha)$ *for all* $\alpha > \alpha_0$ *and* $u_1 \leq b(y, f(x_1))(\alpha)$ *for all* $\alpha > \alpha_1$*, and* $\varepsilon > 0$*,*

$$\begin{gathered}\exists x \in f^{-1}(y),\ \alpha_0' > \alpha_0,\ \alpha_1' > \alpha_1,\ \alpha_0' + \alpha_1' = \beta \;: \\ a(x_0, x)(\alpha_0') + \varepsilon > u_0,\ a(x, x_1)(\alpha_1') + \varepsilon > u_1.\end{gathered} \tag{3.iii}$$

Proof. We split the proof in three parts: (a) (3.i) $\Longrightarrow$ (3.iii); (b) (3.iii) $\Longrightarrow$ (3.i) when $v_0, v_1 \in \Delta_0$; (c) (3.i) for $\Delta_0 \implies$ (3.i).

(a) Let $x_0, x_1, y, \alpha_0, \alpha_1, \beta, u_0, u_1, \varepsilon$ be as in (3.iii). In (3.i) let $v_i = \varphi_{\alpha_i, u_i}$ for $i = 0, 1$. Note that $\varphi_{\alpha_0, u_0} \leq b(f(x_0), y)$ and $\varphi_{\alpha_1, u_1} \leq b(y, f(x_1))$. Denote $a(x_0, x) \wedge \varphi_{\alpha_0, u_0}$ by $\psi_{0,x}$ and $\varphi_{\alpha_1, u_1} \wedge a(x, x_1)$ by $\psi_{1,x}$. Then (3.i) gives:

$$\bigvee_{x \in f^{-1}(y)} \bigvee_{\alpha_0' + \alpha_1' = \beta} \psi_{0,x}(\alpha_0')\, \psi_{1,x}(\alpha_1') \geq (a(x_0, x_1) \wedge \varphi_{\alpha_0 + \alpha_1, u_0 u_1})(\beta) = u_0 u_1.$$

Let $\delta = u_0 u_1 \varepsilon > 0$; there exist $x \in f^{-1}(y)$, $\alpha_0', \alpha_1' \in [0, \infty]$ with $\alpha_0' + \alpha_1' = \beta$ such that

$$\psi_{0,x}(\alpha_0') \psi_{1,x}(\alpha_1') + \delta > u_0 u_1,$$

and therefore

$$u_0 \psi_{1,x}(\alpha_1') + u_0 u_1 \varepsilon \geq \psi_{0,x}(\alpha_0') + \psi_{1,x}(\alpha_1') + u_0 u_1 \varepsilon > u_0 u_1 \implies$$

$$\psi_{1,x}(\alpha_1') + \varepsilon \geq \psi_{1,x}(\alpha_1') + u_1 \varepsilon > u_1,$$

and this implies $a(x, x_1)(\alpha_1') + \varepsilon > u_1$. The condition for u_0 is proved analogously.

(b) Let x_0, x_1, y, v_0, v_1 as in (3.i) with $v_i = \varphi_{\alpha_i, u_i}$. Then $u_0 \leq b(f(x_0), y)(\alpha)$ for every $\alpha > \alpha_0$ and $u_1 \leq b(y, f(x_1))$ for every $\alpha > \alpha_1$. We want to show that, for any $\beta \in [0, \infty]$,

$$\bigvee_{x \in f^{-1}(y)} (a(x_0, x) \wedge \varphi_{\alpha_0, u_0}) \otimes (a(x, x_1) \wedge \varphi_{\alpha_1, u_1})(\beta) \geq (a(x_0, x_1) \wedge \varphi_{\alpha_0 + \alpha_1, u_0 u_1})(\beta).$$

For $\beta \leq \alpha_0 + \alpha_1$ the condition is trivially satisfied. Let $\beta > \alpha_1 + \alpha_0$. If $u_0 u_1 \leq a(x_0, x_1)(\beta)$, so that the right side of the inequality is equal to $u_0 u_1$, then, applying (3.iii) for $\varepsilon > 0$ we obtain α_0', α_1' with $\alpha_0' + \alpha_1' = \beta$ and $a(x_0, x)(\alpha_0') + \varepsilon > u_0$, $a(x, x_1)(\alpha_1') + \varepsilon > u_1$. Therefore the left side of the inequality is necessarily larger or equal to $u_0 u_1$. If $u_0 u_1 > a(x_0, x_1)(\beta)$, then take u_1' such that $u_0 u_1' = a(x_0, x_1)(\beta)$ and use the previous argument for u_0, u_1'; the conclusion follows.

(c) To conclude one has to observe that every element of Δ is the join of step functions and that both $\wedge$ and $\otimes$ commute with joins. $\square$

(4) *A* PM*-functor* $f : (X, a) \to (Y, b)$ *is exponentiable in* PM-**Cat** *if, and only if, for each* $x_0, x_1 \in X$*,* $y \in Y$*, and* $m \in b(f(x_0), y)$*,* $n \in b(y, f(x_1))$ *with* $mn \in a(x_0, x_1)$*, there exists* $x \in f^{-1}(y)$ *such that* $m \in a(x_0, x)$ *and* $n \in a(x, x_1)$*.*

4. Restricting to classical spaces

In this paper we considered so far (probabilistic) metric spaces in a generalised sense; classically these spaces are also assumed to be symmetric, separated and finitary. In a metric space, the latter property requires all distances to be less than ∞; whereby for a probabilistic metric space (X, a) this is usually expressed as "the distance from x to y is less than ∞ with probability 1":

$$a(x, y)(\infty) = 1.$$

For a general V-category, the notions of symmetry and separatedness can be introduced in a straightforward way; but it is less obvious what finitary should mean. Here we adopt a relative approach and introduce $\mathcal{F}$-finitary V-categories, for a given choice $\mathcal{F} \subseteq V$ of "finite values".

Definition 4.1. Let (X, a) be a V-graph. Then (X, a) is called *symmetric* whenever $a(x, y) = a(y, x)$, for all $x, y \in X$; and (X, a) is called *separated* if $k \leq a(x, y)$ and $k \leq a(y, x)$ imply $x = y$, for all $x, y \in X$.

The definitions above are formulated at the level of V-graphs; however, our main interest is in V-categories. The full subcategory of V-**Cat** defined by all symmetric, separated, and separated and symmetric V-categories is denoted by

$$V\text{-}\mathbf{Cat}_{\mathrm{sym}}, \qquad V\text{-}\mathbf{Cat}_{\mathrm{sep}}, \qquad V\text{-}\mathbf{Cat}_{\mathrm{sym,sep}},$$

respectively. We note that symmetric V-categories have played a significant role in [34, 1, 16]. The V-category $(V, \hom)$ is separated, and we also consider its *symmetrization* $(V, \hom_s)$ where $\hom_s(u, v) = \hom(u, v) \wedge \hom(v, u)$.

It is easy to see that all categories above are closed under limits in V-**Cat**, therefore one might wonder about exponentiable morphisms in these categories.

Proposition 4.2. *(1) Let $f : (X, a) \to (Y, b)$ be a morphism in V-**Gph** where (X, a) and (Y, b) are symmetric (separated). Then, for every symmetric (separated) V-graph (Z, c), the partial product of (Z, c) over f is symmetric (separated).*

*(2) Let $f : (X, a) \to (Y, b)$ and (Z, c) be in V-$\mathbf{Cat}_{\mathrm{sym}}$ (V-$\mathbf{Cat}_{\mathrm{sep}}$, V-$\mathbf{Cat}_{\mathrm{sym,sep}}$). If the partial product of (Z, c) over f exists in V-$\mathbf{Cat}_{\mathrm{sym}}$ (V-$\mathbf{Cat}_{\mathrm{sep}}$, V-$\mathbf{Cat}_{\mathrm{sym,sep}}$), then it coincides with the partial product of (Z, c) over f in V-**Gph**.*

*(3) A morphism $f : (X, a) \to (Y, b)$ is exponentiable in V-$\mathbf{Cat}_{\mathrm{sym}}$ (V-$\mathbf{Cat}_{\mathrm{sep}}$, V-$\mathbf{Cat}_{\mathrm{sym,sep}}$) if, and only if, $f : (X, a) \to (Y, b)$ is exponentiable in V-**Cat**.*

Proof. The first two statements are easy to prove, where for the second one the same argument as in [5, Proposition 3.3] is used. It follows immediately that every exponentiable V-functor $f : (X, a) \to (Y, b)$ between symmetric (separated) V-categories is also exponentiable in V-$\mathbf{Cat}_{\mathrm{sym}}$, V-$\mathbf{Cat}_{\mathrm{sep}}$ and V-$\mathbf{Cat}_{\mathrm{sym,sep}}$. For the reverse implication one can use the same argument as in [5, Theorem 3.4], but with $(V, \hom_s)$ as codomain of the maps (3.ii) in the symmetric case. $\square$

In order to deal with the notion of "finitary", we will now consider certain subsets of V thinking of their elements as "finite elements".

Definition 4.3. A subset $\mathcal{F} \subseteq V$ is called a $\otimes$*-filter* if $\mathcal{F}$ is a filter of the lattice V, $k \in \mathcal{F}$ and $u \otimes v \in \mathcal{F}$ whenever $u \in \mathcal{F}$ and $v \in \mathcal{F}$.

Examples 4.4. (1) For every quantale V, $\mathcal{F}_k = \{v \in V \mid k \leq v\}$ is a $\otimes$-filter. More generally, if $u \leq k$ is idempotent, then $\mathcal{F}_u = \{v \in V \mid u \leq v\}$ is a $\otimes$-filter. Note that, in particular, $\mathcal{F}_\perp = V$ is a $\otimes$-filter. Certainly, for every $\otimes$-filter $\mathcal{F}$ one has $\mathcal{F}_k \subseteq \mathcal{F} \subseteq \mathcal{F}_\perp$.

(2) If V satisfies

$$\forall u, v \in V\, (u \otimes v = \perp \implies (u = \perp \text{ or } v = \perp)),$$

then $\mathcal{F} = \{v \in V \mid v \neq \perp\}$ is a $\otimes$-filter. The condition above is certainly satisfied by the quantales 2, $[0, 1]_*$, $[0, 1]_\wedge$ and Δ, but fails in $[0, 1]_\oplus$.

(3) In the quantale Δ we have two natural choices of $\otimes$-filters:

$$\mathcal{F}_1 = \{\psi \in \Delta \mid \psi(\infty) = 1\} \qquad \text{and} \qquad \mathcal{F}_2 = \{\psi \in \Delta \mid \exists \alpha < \infty \, \psi(\alpha) = 1\}.$$

Cleary, $\mathcal{F}_2 \subsetneq \mathcal{F}_1$.

Definition 4.5. Let $\mathcal{F} \subseteq V$ be a $\otimes$-filter. A V-graph (X, a) is called $\mathcal{F}$-*finitary* whenever $a(x, x') \in \mathcal{F}$, for all $x, x' \in X$; and a V-graph (X, a) is called $\mathcal{F}$-*bounded* whenever there exists some $u \in \mathcal{F}$ so that $u \leq a(x, x')$, for all $x, x' \in X$.

One observes immediately that (X, a) is $\mathcal{F}$-bounded if and only if $\bigwedge_{x,y \in X} a(x, y) \in \mathcal{F}$. Choosing in the metric case

$$\mathcal{F} = \{u \in [0, \infty] \mid u < \infty\}$$

leads to the usual notions of finitary, respectively bounded, metric space. Similarly to the situation for metric spaces, also probabilistic metric spaces are usually assumed to have only "finite" distances, more specifically, they are assumed to be $\mathcal{F}_1$-finitary. To identify $\mathcal{F}_1$-bounded probabilistic metric spaces, we note first that infima in Δ are in general not calculated pointwise. However, infima are calculated pointwise in the ordered set

$$\mathbf{Ord}([0, \infty], [0, 1]) = \{\varphi : [0, \infty] \to [0, 1] \mid \varphi \text{ is monotone}\};$$

and the inclusion function $i : \Delta \to \mathbf{Ord}([0, \infty], [0, 1])$ has a right adjoint $c : \mathbf{Ord}([0, \infty], [0, 1]) \to \Delta$ sending φ to the distribution function

$$c(\varphi) : [0, \infty] \longrightarrow [0, 1], \; \alpha \longmapsto \sup_{\beta < \alpha} \varphi(\beta).$$

Since c sends infima to infima, we conclude that a probabilistic metric space (X, a) is $\mathcal{F}_1$-bounded if and only if

$$\left(\bigwedge_{x,y \in X} a(x, y) \right)(\infty) = \sup_{\alpha < \infty} \inf_{x,y \in X} a(x, y)(\alpha) = 1,$$

that is, (X, a) is *probabilistic bounded* (see [15]).

We have the following obvious facts.

Lemma 4.6. *Let $\mathcal{F} \subseteq V$ be a $\otimes$-filter.*

(1) The empty V-graph is $\mathcal{F}$-bounded.
(2) Every $\mathcal{F}$-bounded V-graph is $\mathcal{F}$-finitary.
(3) Let $f : (X, a) \to (Y, b)$ be a surjective V-functor. If (X, a) is $\mathcal{F}$-finitary ($\mathcal{F}$-bounded), then so is (Y, b).
(4) Let (X, a) be a V-category with $X \neq \varnothing$ and let $x_0 \in X$. Then (X, a) is $\mathcal{F}$-bounded if, and only if, there exist some $u, u' \in \mathcal{F}$ such that, for all $x \in X$, $u \leq a(x_0, x)$ and $u' \leq a(x, x_0)$.

Clearly, in the last statement above one can always choose $u = u'$. We write $V\text{-}\mathbf{Cat}_{\mathcal{F}}$ for the full subcategory of $V\text{-}\mathbf{Cat}$ defined by all $\mathcal{F}$-finitary V-categories. Since $\mathcal{F} \subseteq V$ is closed under finite infima, $V\text{-}\mathbf{Cat}_{\mathcal{F}}$ is closed in $V\text{-}\mathbf{Cat}$ under subspaces and finite products, hence $V\text{-}\mathbf{Cat}_{\mathcal{F}}$ is finitely complete. We study now, inspired by [5, Subsection 4.3], exponentiation in $V\text{-}\mathbf{Cat}_{\mathcal{F}}$.

Proposition 4.7. *Let $\mathcal{F} \subseteq V$ be a $\otimes$-filter.*

(1) Let $f : (X, a) \to (Y, b)$ be a morphism in $V\text{-}\mathbf{Cat}_{\mathcal{F}}$. Then the following assertions are equivalent.
   (i) For every $\mathcal{F}$-finitary V-category (Z, c), the partial product of (Z, c) over f in $V\text{-}\mathbf{Gph}$ is $\mathcal{F}$-finitary.
   (ii) All fibres of f are $\mathcal{F}$-bounded.
(2) Let $f : (X, a) \to (Y, b)$ and (Z, c) be in $V\text{-}\mathbf{Cat}_{\mathcal{F}}$. If the partial product of (Z, c) over f exists in $V\text{-}\mathbf{Cat}_{\mathcal{F}}$, then it coincides with the one in $V\text{-}\mathbf{Gph}$.

Proof. (1) Assume first that all fibres of f are $\mathcal{F}$-bounded. Let (Z,c) be a V-category and let $(s,y),(s',y')$ be elements of the partial product (P,d) of (Z,c) over f. If either $f^{-1}(y)=\varnothing$ or $f^{-1}(y')=\varnothing$, then $d((s,y),(s',y'))=b(y,y')\in\mathcal{F}$. Let $x_0\in f^{-1}(y)$, $x_0'\in f^{-1}(y')$ and put $v:=c(s(x_0),s(x_0'))\in\mathcal{F}$; and let $u,u'\in\mathcal{F}$ so that, for all $x\in f^{-1}(y)$ and $x'\in f^{-1}(y')$, $a(x,x_0)\geq u$ and $a(x',x_0')\geq u'$. Then, for all $x\in f^{-1}(y)$ and $x'\in f^{-1}(y')$,

$$c(s(x),s'(x'))\geq c(s(x),s(x_0))\otimes c(s(x_0),s'(x_0'))\otimes c(s'(x_0'),s'(x'))\geq u\otimes v\otimes u';$$

which implies $d((s,y),(s',y'))\geq u\otimes v\otimes u'\in\mathcal{F}$. To see the reverse implication, take $y\in Y$ and $x_0\in X_y$, and consider $s:X_y\to X_y$, $x\mapsto x$ and $s':X_y\to X_y, x\mapsto x_0$. Then

$$u:=d((s,y),(s',y'))\in\mathcal{F},\qquad\qquad u':=d((s',y'),(s,y))\in\mathcal{F}$$

hence, for all $x,x'\in X_y$,

$$u\wedge a(x,x')\leq a_y(x,x_0),\qquad\qquad u'\wedge a(x,x')\leq a_y(x_0,x').$$

Taking $x=x'$ gives $\mathcal{F}\ni u\wedge k\leq a_y(x,x_0)$ and $\mathcal{F}\ni u'\wedge k\leq a_y(x_0,x)$, which shows that X_y is $\mathcal{F}$-bounded.
(2) As in [5, Proposition 4.3]. □

As before, one can now deduce that every $f:(X,a)\to(Y,b)$ in V-$\mathbf{Cat}_{\mathcal{F}}$ which is exponentiable in V-$\mathbf{Cat}$ and has $\mathcal{F}$-bounded fibres is also exponentiable in V-$\mathbf{Cat}_{\mathcal{F}}$. However, this is not an equivalence; for instance, taking $\mathcal{F}=\{1\}$ for $V=2$ gives V-$\mathbf{Cat}_{\mathcal{F}}\simeq\mathbf{Set}$ where every morphism is exponentiable, while f in V-$\mathbf{Cat}_{\mathcal{F}}$ is exponentiable in V-$\mathbf{Cat}$ if, and only if, it is surjective. To understand better exponentiability in V-$\mathbf{Cat}_{\mathcal{F}}$, we shall assume that $\mathcal{F}$ is *closed under* hom: for all $u,v\in\mathcal{F}$, also $\hom(u,v)\in\mathcal{F}$. We note that every $\otimes$-filter $\mathcal{F}\subseteq V$ satisfies this condition if $k=\top$ since then $\hom(u,v)\geq\hom(k,v)=v$. Under this condition, we can consider $\mathcal{F}$ as the V-category $(\mathcal{F},\hom)$.

Theorem 4.8. *Let $\mathcal{F}\subseteq V$ be a $\otimes$-filter closed under* hom. *Let $f:(X,a)\to(Y,b)$ be a morphism in V-$\mathbf{Cat}_{\mathcal{F}}$ where $X\neq\varnothing$. Then f is surjective and exponentiable in V-$\mathbf{Cat}_{\mathcal{F}}$ if, and only if, f has $\mathcal{F}$-bounded fibres and, for all $x_0,x_1\in X$, $y\in Y$, $v_0,v_1\in\mathcal{F}$ such that $v_0\leq b(f(x_0),y)$ and $v_1\leq b(y,f(x_1))$,*

$$\bigvee_{x\in f^{-1}(y)}(a(x_0,x)\wedge v_0)\otimes(a(x,x_1)\wedge v_1)\geq a(x_0,x_1)\wedge(v_0\otimes v_1).$$

Proof. We show first that these conditions are sufficient for exponentiability and surjectivity of f. Firstly, since f has $\mathcal{F}$-bounded fibres, the partial product (P,d) in V-$\mathbf{Gph}$ of a V-category (Z,c) over f is bounded; and, analysing the proof of [5, Theorem 3.4], in order to conclude that (P,d) is a V-category one only needs to consider v_0,v_1 in the image of d, hence $v_0,v_1\in\mathcal{F}$. Taking now $x_0=x_1\in X$ and $v_0=v_1=\top$ in the formula above, it follows that $f^{-1}(y)\neq\varnothing$, for every $y\in Y$.
Assume now that $f:(X,a)\to(Y,b)$ is surjective and exponentiable in V-$\mathbf{Cat}_{\mathcal{F}}$. Then f has $\mathcal{F}$-bounded fibres by Proposition 4.7. Furthermore, the maps s,s',s'' of (3.ii) can be restricted to the codomain $(\mathcal{F},\hom)$ and therefore are morphisms in V-$\mathbf{Cat}_{\mathcal{F}}$, now one can use the same argument as in the proof of [5, Theorem 3.4]. □

As we already mentioned, we cannot drop the surjectivity condition in the theorem above. On the other hand, in [5, Theorem 4.2] it is shown that for classical metric spaces exponentiability implies surjectivity; so that *the exponentiable morphisms of classical metric spaces are precisely the morphisms which are exponentiable in* $[0,\infty]$-$\mathbf{Cat}$ *and have bounded fibres.* We do not know whether the same is true for classical probabilistic metric spaces; however, we can still state a characterization of exponentiable morphisms of classical probabilistic metric spaces by combining Theorem 4.8 and Proposition 3.7(3). Before doing so, we have to deal with a slight technical problem: in general, step functions are not elements of $\mathcal{F}_1$.

Lemma 4.9. *Let $f : (X,a) \to (Y,b)$ be a morphism in $\Delta\text{-}\mathbf{Cat}_{\mathcal{F}_1}$ and let $x_0, x_1 \in X$ and $y \in Y$. Then the inequality*

$$\bigvee_{x \in f^{-1}(y)} (a(x_0,x) \wedge \varphi_0) \otimes (a(x,x_1) \wedge \varphi_1) \geq a(x_0,x_1) \wedge (\varphi_0 \otimes \varphi_1). \tag{4.i}$$

holds for all $\varphi_0, \varphi_1 \in \mathcal{F}_1$ with $\varphi_0 \leq b(f(x_0),y)$ and $\varphi_1 \leq b(y, f(x_1))$ if and only if (4.i) *holds for all $\varphi_0, \varphi_1 \in \Delta$ with $\varphi_0 \leq b(f(x_0),y)$ and $\varphi_1 \leq b(y,f(x_1))$.*

Proof. Assume that (4.i) holds for all values in $\mathcal{F}_1$ and let $\varphi_0, \varphi_1 \in \Delta$ with $\varphi_0 \leq b(f(x_0),y)$ and $\varphi_1 \leq b(y,f(x_1))$. Then $\varphi_0(\infty) \leq 1 = a(x_0,x)(\infty)$, $\varphi_1(\infty) \leq 1 = a(x,x_1)$ and $(\varphi_0 \otimes \varphi_1)(\infty) \leq 1 = a(x_0,x_1)(\infty)$; hence (4.i) holds for the argument $\alpha = \infty$ since f is surjective and $\varphi_0(\infty) \otimes \varphi_1(\infty) = (\varphi_0 \otimes \varphi_1)(\infty)$. Let now $\alpha < \infty$. Define $\widetilde{\varphi}_0, \widetilde{\varphi}_1 \in \mathcal{F}_1$ by

$$\widetilde{\varphi}_0(\beta) = \begin{cases} \varphi_0(\beta) & \text{if } \beta \leq \alpha + 1, \\ b(f(x_0),y)(\beta) & \text{if } \beta > \alpha + 1 \end{cases} \quad \text{and} \quad \widetilde{\varphi}_1(\beta) = \begin{cases} \varphi_1(\beta) & \text{if } \beta \leq \alpha + 1, \\ b(y,f(x_1))(\beta) & \text{if } \beta > \alpha + 1. \end{cases}$$

By our hypothesis,

$$\bigvee_{x \in f^{-1}(y)} ((a(x_0,x) \wedge \widetilde{\varphi}_0) \otimes (a(x,x_1) \wedge \widetilde{\varphi}_1))(\alpha) \geq a(x_0,x_1)(\alpha) \wedge (\widetilde{\varphi}_0 \otimes \widetilde{\varphi}_1)(\alpha);$$

and the assertion follows from

$$((a(x_0,x) \wedge \widetilde{\varphi}_0) \otimes (a(x,x_1) \wedge \widetilde{\varphi}_1))(\alpha) = ((a(x_0,x) \wedge \varphi_0) \otimes (a(x,x_1) \wedge \varphi_1))(\alpha)$$

and

$$a(x_0,x_1)(\alpha) \wedge (\widetilde{\varphi}_0 \otimes \widetilde{\varphi}_1)(\alpha) = a(x_0,x_1)(\alpha) \wedge (\varphi_0 \otimes \varphi_1)(\alpha).$$ □

Corollary 4.10. *In the category $\Delta\text{-}\mathbf{Cat}_{\mathrm{sep,sym},\mathcal{F}_1}$ of classical probabilistic metric spaces, a morphism $f : (X,a) \to (Y,b)$ with $X \neq \varnothing$ is surjective and exponentiable if and only if f has probabilistic bounded fibres and satisfies the condition of Proposition 3.7(3).*

Remarks 4.11.
(1) We do not know if there are non-surjective Δ-functors which are exponentiable in the category of classical probabilistic metric spaces but do not satisfy the condition of Proposition 3.7(3).
(2) The arguments used in Lemma 4.9 apply equally to $\mathcal{F}_2$. Therefore: *In the category $\Delta\text{-}\mathbf{Cat}_{\mathrm{sep,sym},\mathcal{F}_2}$, a morphism $f : (X,a) \to (Y,b)$ with $X \neq \varnothing$ is surjective and exponentiable if and only if f has $\mathcal{F}_2$-bounded fibres and satisfies the condition of Proposition 3.7(3).*

5. Effective descent V-functors

In this section we will focus on effective descent morphisms in $V\text{-}\mathbf{Cat}$. First we recall that a V-functor $f : X \to Y$ is said to be *effective for descent* in $V\text{-}\mathbf{Cat}$ if the pullback functor

$$f^* : (V\text{-}\mathbf{Cat}) \downarrow Y \longrightarrow (V\text{-}\mathbf{Cat}) \downarrow X$$

is monadic.

In locally cartesian closed categories effective descent morphisms coincide with *regular epimorphisms*, that is, coequalisers of a pair of morphisms (see [30] for details). This is the case of $V\text{-}\mathbf{Gph}$ when V is a Heyting algebra, as stated in Theorem 3.1.

Proposition 5.1 ([6])**.** *For a V-functor $f : (X,a) \to (Y,b)$ in $V\text{-}\mathbf{Gph}$, the following conditions are equivalent.*

(i) f is a regular epimorphism in $V\text{-}\mathbf{Gph}$.
(ii) f is a pullback-stable regular epimorphism in $V\text{-}\mathbf{Gph}$.
(iii) f is an effective descent morphism in $V\text{-}\mathbf{Gph}$.
(iv) f is final and surjective.

Moreover, if f is a surjective V-functor in $V\text{-}\mathbf{Cat}$, f is a pullback-stable regular epimorphism in $V\text{-}\mathbf{Cat}$ if, and only if, it is final.

We observe that a surjection $f : (X, a) \to (Y, b)$ is final if

$$\forall y_0, y_1 \in Y \;\; b(y_0, y_1) = \bigvee_{x_i \in f^{-1}(y_i)} a(x_0, x_1).$$

Finality of a functor f is in general not sufficient for f being effective for descent in V-**Cat**, reason why we will focus on the following stronger properties. A V-functor $f : (X, a) \to (Y, b)$ is said to be a *$\star$-quotient morphism* if it satisfies the following condition

$$\forall y_0, y_1, y_2 \in Y \;\; b(y_0, y_1) \otimes b(y_1, y_2) = \bigvee_{x_i \in f^{-1}(y_i)} a(x_0, x_1) \otimes a(x_1, x_2)$$

(see [30, 4]).

Theorem 5.2. *For a V-functor $f : (X, a) \to (Y, b)$ in V-**Cat**, (1) $\implies$ (2) $\implies$ (3) $\implies$ (4).*

*(1) f is a pullback-stable $\star$-quotient morphism in V-**Gph**.*
*(2) f is an effective descent morphism in V-**Cat**.*
*(3) f is a pullback-stable $\star$-quotient morphism in V-**Cat**.*
(4) f is a $\star$-quotient morphism.

Proof. (1) $\implies$ (2) $\implies$ (4) are shown in [4, Section 5]; since effective descent morphisms are pullback-stable, (2) $\implies$ (3) follows and (3) $\implies$ (4) is obvious. □

Let us call *$\star\star$-quotient morphism* every V-functor $f : (X, a) \to (Y, b)$ such that

$$\forall y_0, y_1, y_2 \in Y, \; u \ll b(y_0, y_1), \; v \ll b(y_1, y_2)$$
$$\exists x_i \in f^{-1}(y_i) \, (i = 0, 1, 2) \; : \; u \le a(x_0, x_1), \; v \le a(x_1, x_2).$$

Clearly, every $\star\star$-quotient morphism is $\star$-quotient, and, if the quantale V is cancellable, the converse holds.

Proposition 5.3. *If V is cancellable, then the following conditions are equivalent, for a V-functor $f : (X, a) \to (Y, b)$.*

(i) f is effective for descent.
(ii) f is a $\star$-quotient morphism.
(iii) f is a $\star\star$-quotient morphism.

Proof. Every effective descent morphism is a $\star$-quotient morphism, and, when V is cancellable, every $\star$-quotient morphism is a $\star\star$-quotient morphism.

To prove the remaining implication we show that $\star\star$-quotient morphisms are pullback-stable in V-**Gph**, and use Theorem 5.2. Given a pullback diagram

$$\begin{array}{ccc} (X \times_Y Z, d) & \xrightarrow{\pi_2} & (Z, c) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Y, b) \end{array}$$

with f a $\star\star$-quotient morphism, let $z_0, z_1, z_2 \in Z$. If $u \ll c(z_0, z_1)$ and $v \ll c(z_1, z_2)$, then $u \ll b(g(z_0), g(z_1))$ and $v \ll b(g(z_1), g(z_2))$. Since f is a $\star\star$-quotient morphism, for $i = 0, 1, 2$ there exist x_i in X such that $f(x_i) = g(z_i)$, $u \le a(x_0, x_1)$ and $v \le a(x_1, x_2)$. Therefore $(x_i, z_i) \in X \times_Y Z$ for $i = 0, 1, 2$, $u \le a(x_0, x_1) \wedge c(z_0, z_1) = d((x_0, z_0), (x_1, z_1))$ by definition of the pullback structure d, and, analogously, $v \le d((x_1, z_1), (x_2, z_2))$. □

Below we present characterizations of effective descent morphisms in the categories under study. In particular our examples show that in V-**Cat**, in general, $\star$-quotient $\not\Rightarrow$ effective descent $\not\Rightarrow$ $\star\star$-quotient. We do not know whether the conditions $(1) - (3)$ of Theorem 5.2 are equivalent.

Examples 5.4. (1) From Proposition 5.3 it follows immediately:

Theorem ([4]). *A morphism f in $[0,\infty]$-**Cat** or in 2-**Cat** is effective for descent if and only if f is $\star$-quotient if and only if f is $\star\star$-quotient.*

(2) We consider now categories enriched in $[0,1]$ equipped with a continuous tensor.

Lemma. *In $[0,1]_\oplus$-**Gph** and in $[0,1]_*$-**Gph**, the class of $\star$-quotient morphisms is pullback stable.*

Proof. Since the quantale $([0,1],*,1)$ is isomorphic to $([0,\infty],+,0)$, the assertion about $[0,1]_*$-**Gph** is just a translation from [4, Proposition 6.3]. Consider now a pullback diagram

$$\begin{tikzcd} (X\times_Y Z,d) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (Z,c) \arrow[d, "g"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

in $[0,1]_\oplus$-**Gph** where f is a $\star$-quotient morphism. Let $z_0,z_1,z_2\in Z$ and put $y_i=g(z_i)$ $(i=0,1,2)$ and $u_i=c(z_i,z_{i+1})$, $v_i=b(y_i,y_{i+1})$ $(i=0,1)$. If $u_0+u_1\le 1$, then clearly

$$0=v_0\oplus v_1=\bigvee_{\pi_2(w_i)=z_i} d(w_0,w_1)\oplus d(w_1,w_2).$$

Assume now $u_0+u_1>1$. Then $v_0+v_1>1$, that is, $v_0+v_1>0$. Since f is $\star$-quotient,

$$v_0\oplus v_1\le\bigvee_{f(x_i)=y_i} a(x_0,x_1)\oplus a(x_1,x_2).$$

Hence, for every $\varepsilon>0$ there exist $x_i\in f^{-1}(y_i)$ $(i=0,1,2)$ with

$$a(x_0,x_1)\oplus a(x_1,x_2)\ge (v_0\oplus v_1)-\varepsilon,$$

or equivalently, $a(x_0,x_1)+a(x_1,x_2)+\varepsilon\ge v_0+v_1$. Since $v_0':=a(x_0,x_1)\le v_0$ and $v_1':=a(x_1,x_2)\le v_1$, we conclude $v_0'+\varepsilon\ge v_0$ and $v_1'+\varepsilon\ge v_1$ and therefore

$$d((x_0,z_0),(x_1,z_1))=v_0'\wedge u_0\ge (v_0-\varepsilon)\wedge u_0\ge u_0-\varepsilon;$$

similarly, $d((x_1,z_1),(x_2,z_2))\ge u_1-\varepsilon$. □

Based on the lemma above as well as on Theorem 1.2 (3), we show now that $\star$-quotient morphisms are pullback stable in $[0,1]_\otimes$-**Gph**, where $\otimes$ is any continuous quantale structure on $[0,1]$ with neutral element 1. In order to do so, we first observe that, for every homomorphism of quantales $\varphi: V\to W$, the corresponding change-of-base functor

$$B_\varphi : V\text{-}\mathbf{Gph}\longrightarrow W\text{-}\mathbf{Gph}$$
$$(f:(X,a)\to(Y,b))\longmapsto(f:(X,\varphi a)\to(Y,\varphi b))$$

preserves $\star$-quotient morphisms. If, moreover, φ preserves finite infima, then $B_\varphi: V$-**Gph** $\to$ W-**Gph** preserves pullbacks.

Proposition. *Let $\otimes$ be a continuous quantale structure on $[0,1]$ with neutral element 1. Then the class of $\star$-quotient morphisms is pullback stable in $[0,1]_\otimes$-**Gph**.*

Proof. As above, consider a pullback diagram

$$\begin{tikzcd} (X\times_Y Z,d) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (Z,c) \arrow[d, "g"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

in $[0,1]_\otimes$-**Gph** where f is a $\star$-quotient morphism, let $z_0, z_1, z_2 \in Z$ and put $y_i = g(z_i)$ $(i = 0,1,2)$ and $u_i = c(z_i, z_{i+1})$, $v_i = b(y_i, y_{i+1})$ $(i = 0,1)$. We wish to show that

$$u_0 \otimes u_1 \leq \bigvee_{\pi_2(w_i)=z_i} d(w_0, w_1) \otimes d(w_1, w_2).$$

- If $u_0 \otimes u_1 = 0$, then the assertion follows trivially.
- If $u_0 \otimes u_1 = 1$, then also $u_0 = u_1 = v_0 = v_1 = v_0 \otimes v_1 = 1$, and the assertion follows.
- Assume now that $u_0 \otimes u_1$ is idempotent. Then $[0, u_0 \otimes u_1]$ (with the restriction of $\otimes$) is isomorphic to $[0,1]$ with an appropriate tensor. Denote this isomorphism by $\varphi : [0, u_0 \otimes u_1] \to [0,1]$, and extend φ to a homomorphism of quantales $\varphi : [0,1] \to [0,1]$ by putting $\varphi(u) = 1$ for $u > u_0 \otimes u_1$. Clearly, $\varphi : [0,1] \to [0,1]$ preserves finite infima. Then $B_\varphi f$ is a $\star$-quotient morphism; hence, by the previous case,

$$\varphi\left(\bigvee_{\pi_2(w_i)=z_i} d(w_0, w_1) \otimes d(w_1, w_2)\right) = 1.$$

  Therefore $u_0 \otimes u_1 \leq \bigvee_{\pi_2(w_i)=z_i} d(w_0, w_1) \otimes d(w_1, w_2)$.
- If $u_0 \otimes u_1$ is not idempotent, then there are idempotents $u, u' \in [0,1]$ with $u < u_0 \otimes u_1 < u'$ and an isomorphism of quantales $\varphi : [u, u'] \to [0,1]$ where $[0,1]$ is equipped with either the tensor $\oplus$ or the multiplication. We extend φ to $\varphi : [0,1] \to [0,1]$ by putting $\varphi(v) = 0$ for $v < u$ and $\varphi(v) = 1$ for $v > u'$. Then $B_\varphi(\pi_2)$ is a $\star$-quotient morphism, therefore

$$\varphi\left(\bigvee_{\pi_2(w_i)=z_i} d(w_0, w_1) \otimes d(w_1, w_2)\right) = \varphi(u_0 \otimes u_1).$$

  Since $u < u_0 \otimes u_1 < u'$, we conclude that

$$\bigvee_{\pi_2(w_i)=z_i} d(w_0, w_1) \otimes d(w_1, w_2) = u_0 \otimes u_1. \qquad \square$$

Theorem. *Let $\otimes$ be a continuous quantale structure on $[0,1]$ with neutral element 1. Then a $[0,1]_\otimes$-functor is effective for descent in $[0,1]_\otimes$-**Cat** if and only if it is a $\star$-quotient morphism.*

Remark. An effective descent $[0,1]_\otimes$-functor does not need to be $\star\star$-quotient morphism. Consider, for instance, the $[0,1]_\oplus$-functor $f : (X, a) \to (Y, b)$ depicted below:

$$\boxed{\begin{tikzcd} x_0 \arrow[r, "\frac{1}{2}"] & x_1 & \\ & x'_1 \arrow[r, "\frac{1}{2}"] & x'_2 \end{tikzcd}} \xrightarrow{\quad f \quad} \boxed{\begin{tikzcd} y_0 \arrow[r, "\frac{1}{2}"] & y_1 \arrow[r, "\frac{1}{2}"] & y_0 \end{tikzcd}}$$

Then f is a $\star$-quotient morphism (since $\frac{1}{2} \oplus \frac{1}{2} = 0$) but not a $\star\star$-quotient morphism.

(3) We turn now our attention to Δ-categories.

Proposition. *The following conditions are equivalent to be effective for descent, for a morphism $f : X \to Y$ in Δ-**Cat**.*

*(i) f is a pullback-stable $\star$-quotient map in Δ-**Gph**.*

(ii) For each $y_0 \xrightarrow{\psi_0} y_1 \xrightarrow{\psi_1} y_2$ in Y, $\alpha_0, \alpha_1 \in [0,\infty]$, $\varepsilon > 0$, there exist $x_0 \xrightarrow{\chi_0} x_1 \xrightarrow{\chi_1} x_2$ in X with $f(x_i) = y_i$, $i = 0,1,2$, and $\chi_0(\alpha_0) + \varepsilon \geq \psi_0(\alpha_0)$, $\chi_1(\alpha_1) + \varepsilon \geq \psi_1(\alpha_1)$.

*(iii) f is a $\star\star$-quotient morphism in Δ-**Cat**.*

Proof. (i) $\implies$ (ii): Let $f : X \to Y$ be a pullback-stable $\star$-quotient morphism in Δ-**Gph**. Consider the coproduct Z of

$$\{\ y_0 \xrightarrow{\varphi_{\alpha_0,\psi_0(\alpha_0)}} y_1 \xrightarrow{\varphi_{\alpha_1,\psi_1(\alpha_1)}} y_2\ \},$$

for all $y_0 \xrightarrow{\psi_0} y_1 \xrightarrow{\psi_1} y_2$ in Y and $\alpha_0, \alpha_1 \in [0,\infty]$, and form the pullback of $f : X \to Y$ along $g : Z \to Y$ (with $g(y_i) = y_i$).

For each $\alpha_0, \alpha_1 \in [0,\infty]$, $y_0 \xrightarrow{\psi_0} y_1 \xrightarrow{\psi_1} y_2$ in Y and $x_0 \xrightarrow{\chi_0} x_1 \xrightarrow{\chi_1} x_2$ in X with $f(x_i) = y_i$, $i = 0,1,2$, we denote by

$$(x_0,y_0) \xrightarrow{\chi_{0,\alpha_0,\psi_0}} (x_1,y_1) \xrightarrow{\chi_{1,\alpha_1,\psi_1}} (x_2,y_2)$$

the corresponding chain in the pullback $X \times_Y Z$, that is, $\chi_{i,\alpha_i,\psi_i} = \chi_i \wedge \varphi_{\alpha_i,\psi_i(\alpha_i)}$.

Let us fix now $y_0 \xrightarrow{\psi_0} y_1 \xrightarrow{\psi_1} y_2$ in Y, $\alpha_0, \alpha_1 \in [0,\infty]$ and $\varepsilon > 0$. By definition of Δ, there exist $\beta_0, \beta_1 \in [0,\infty]$ such that, for $i = 0,1$, $\alpha_i - \frac{\varepsilon}{2} < \beta_i < \alpha_i$ and $\psi_i(\beta_i) + \frac{\varepsilon}{2} \geq \psi_i(\alpha_i)$. Since, by assumption, $\pi_2 : X \times_Y Z \to Z$ is a $\star$-quotient map,

$$\begin{aligned}\psi_0(\beta_0)\ \psi_1(\beta_1) &= \varphi_{\beta_0+\beta_1,\psi_0(\beta_0)\psi_1(\beta_1)}(\alpha_0+\alpha_1)\\ &= \bigvee_{\alpha_0'+\alpha_1'=\alpha_0+\alpha_1} \chi_{0,\beta_0,\psi_0}(\alpha_0')\ \chi_{1,\beta_1,\psi_1(\beta_1)}(\alpha_1').\end{aligned}$$

Therefore, for $\delta = \min\{\frac{\varepsilon\psi_0(\beta_0)}{2}, \frac{\varepsilon\psi_1(\beta_1)}{2}\} > 0$ (if any of these values is 0 then (ii) is trivially satisfied), there exist $\alpha_0' > \beta_0, \alpha_1' > \beta_1$ such that $\alpha_0'+\alpha_1' = \alpha_0+\alpha_1$ and $\chi_{0,\beta_0,\psi_0}(\alpha_0')\ \chi_{1,\beta_1,\psi_1}(\alpha_1')+\delta > \psi_0(\beta_0)\ \psi_1(\beta_1)$. Hence,

$$\chi_0(\alpha_0')\ \psi_1(\beta_1) + \delta \geq \chi_{0,\beta_0,\psi_0}(\alpha_0')\ \chi_{1,\beta_1,\psi_1}(\alpha_1') + \delta > \psi_0(\beta_0)\ \psi_1(\beta_1),$$

and then $\chi_0(\alpha_0') + \frac{\varepsilon}{2} \geq \chi_0(\alpha_0') + \frac{\delta}{\psi_1(\beta_1)} > \psi_0(\beta_0)$, and so

$$\chi_0(\alpha_0) + \varepsilon \geq \chi_0(\alpha_0') + \varepsilon > \psi_0(\beta_0) + \frac{\varepsilon}{2} \geq \psi_0(\alpha_0),$$

as claimed. The condition for $\chi_1(\alpha_1)$ is shown analogously.

(ii) $\implies$ (iii): That f is a $\star\star$-quotient morphism follows from (ii) considering $\alpha_0 = \alpha_1$ a general element of $[0,\infty]$.

(iii) $\implies$ (i) is shown in the proof of Proposition 5.3. $\square$

Remark. We do not know whether every $\star$-quotient morphism is effective for descent in Δ-**Cat**.

(4) Finally, we consider the quantale $V = PM$.

Theorem. *If M is a non-trivial monoid, (i.e. it has at least two elements), then:*
*(a) A PM-functor is effective for descent in PM-***Cat** *if and only if it is a $\star\star$-quotient morphism.*
(b) There are $\star$-quotient morphisms which are not effective for descent.

Proof. (a) First observe that $f : (X,a) \to (Y,b)$ is a $\star\star$-quotient morphism in PM-**Cat** if and only if

$$\begin{gathered}\forall y_0,y_1,y_2 \in Y\ \forall m_0 \in b(y_0,y_1), m_1 \in b(y_1,y_2)\ \exists x_i \in f^{-1}(y_i) :\\ m_0 \in a(x_0,x_1), m_1 \in a(x_1,x_2).\end{gathered}$$

If this condition does not hold, that is, given $y_0, y_1, y_2 \in Y$, $m_0 \in b(y_0,y_1)$, $m_1 \in b(y_1,y_2)$ such that, for all $x_0, x_1, x_2 \in X$ with $f(x_i) = y_i$, either $m_0 \notin a(x_0,x_1)$ or $m_1 \notin a(x_1,x_2)$, then it is easy to check that the pullback $\pi_2 : (X \times_Y Z, d) \to (Z,c)$ of f along $g : (Z,c) \to (Y,b)$, where $Z = \{y_0,y_1,y_2\}$, $c(y_0,y_1) = \{m_0\}$, $c(y_1,y_2) = \{m_1\}$, $c(y_0,y_2) = \{m_0m_1\}$, and g is the inclusion, is not a $\star$-quotient morphism: for all x_0, x_1, x_2 with $f(x_i) = y_i$, $i = 0,1,2$, either $d((x_0,z_0),(x_1,z_1)) = \varnothing$ or $d((x_1,z_1),(x_2,z_2)) = \varnothing$, hence f is not effective for descent.

(b) If M has at least 3 distinct elements, e, m, n, consider the following pullback:

$$
\begin{tikzcd}
\boxed{\begin{array}{r} (x_0, z_0) \xrightarrow{\{m\}} (x_1, z_1) \xrightarrow{\varnothing} (x_2, z_2) \\ (x_1', z_1) \xrightarrow{\{n\}} (x_2', z_2) \end{array}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \boxed{y_0 \xrightarrow{\{mn^k,\, k\in\mathbb{N}_0\}} y_1 \xrightarrow{\{e,n\}} y_2} \arrow[d, "g"] \\
\boxed{\begin{array}{r} x_0 \xrightarrow{\{mn^k,\, k\in\mathbb{N}_0\}} x_1 \xrightarrow{\{e\}} x_2 \\ x_1' \xrightarrow{\{e,n\}} x_2' \end{array}} \arrow[r, "f"] & \boxed{y_0 \xrightarrow{\{mn^k,\, k\in\mathbb{N}_0\}} y_1 \xrightarrow{\{e,n\}} y_2}
\end{tikzcd}
$$

Then f is a $\star$-quotient morphism but its pullback π_2 is not, therefore f is not effective for descent.

If $M = \{e, a\}$, we distinguish two cases. When $a^2 = a$, we consider the pullback

$$
\begin{tikzcd}
\boxed{\begin{array}{r} (x_0, z_0) \xrightarrow{\{a\}} (x_1, z_1) \xrightarrow{\varnothing} (x_2, z_2) \\ (x_1', z_1) \xrightarrow{\{a\}} (x_2', z_2) \end{array}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \boxed{z_0 \xrightarrow{\{a\}} z_1 \xrightarrow{\{a\}} z_2} \arrow[d, "g"] \\
\boxed{\begin{array}{r} x_0 \xrightarrow{\{a\}} x_1 \xrightarrow{\{e\}} x_2 \\ x_1' \xrightarrow{\{e,a\}} x_2' \end{array}} \arrow[r, "f"] & \boxed{y_0 \xrightarrow{\{a\}} y_1 \xrightarrow{\{e,a\}} y_2}
\end{tikzcd}
$$

When $a^2 = e$ we consider the pullback:

$$
\begin{tikzcd}
\boxed{\begin{array}{r} (x_0, z_0) \xrightarrow{\{a\}} (x_1, z_1) \xrightarrow{\varnothing} (x_2, z_2) \\ (x_1', z_1) \xrightarrow{\{a\}} (x_2', z_2) \end{array}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \boxed{z_0 \xrightarrow{\{a\}} z_1 \xrightarrow{\{a\}} z_2} \arrow[d, "g"] \\
\boxed{\begin{array}{r} x_0 \xrightarrow{\{e,a\}} x_1 \xrightarrow{\{e\}} x_2 \\ x_1' \xrightarrow{\{e,a\}} x_2' \end{array}} \arrow[r, "f"] & \boxed{y_0 \xrightarrow{\{e,a\}} y_1 \xrightarrow{\{e,a\}} y_2}
\end{tikzcd}
$$

In both cases f is a $\star$-quotient morphism but π_2 is not. $\square$

Remark 5.5. As for exponentiability, one may want to work on descent in symmetric or separated V-categories, or in $\mathcal{F}$-finitary ones. However, for these restrictions, the results obtained above remain unchanged, as we explain next.

Indeed, the techniques used in [4] to deduce sufficient conditions for effective descent in V-**Cat**, embedding this category in V-**Gph**, can also be used for $V\text{-}\mathbf{Cat}_{\mathrm{sym}}$, $V\text{-}\mathbf{Cat}_{\mathrm{sep}}$ and $V\text{-}\mathbf{Cat}_{\mathrm{sym,sep}}$ since they are also full subcategories of V-**Gph** closed under pullbacks (for details see [21, 4]). Moreover, it is easy to check that, *for a final surjection $f : (X, a) \to (Y, b)$ between V-graphs, if*

(X, a) *is symmetric (separated), then so is* (Y, b). Therefore, all the results on descent stated above have corresponding formulations for these special cases.

The same happens in the case of $\mathcal{F}$-finitary V-categories, given a $\otimes$-filter $\mathcal{F}$. We have already observed, in Lemma 4.6, that $\mathcal{F}$-finitary V-categories descend along surjective V-functors, and, once again, $V\text{-}\mathbf{Cat}_{\mathcal{F}}$ is closed under pullbacks as a full subcategory of $V\text{-}\mathbf{Gph}$.

In particular we can conclude that a morphism of classical (probabilistic) metric spaces is effective for descent precisely if it is so in the category of generalised (probabilistic) metric spaces.

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