APPROXIMATION IN QUANTALE-ENRICHED CATEGORIES

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Abstract. Our work is a foundational study of the notion of approximation in $\mathcal{Q}$-categories and in $(\mathbb{U}, \mathcal{Q})$-categories, for a quantale $\mathcal{Q}$ and the ultrafilter monad $\mathbb{U}$. We introduce auxiliary, approximating and Scott-continuous distributors, the way-below distributor, and continuity of $\mathcal{Q}$- and $(\mathbb{U}, \mathcal{Q})$-categories. We fully characterize continuous $\mathcal{Q}$-categories (resp. $(\mathbb{U}, \mathcal{Q})$-categories) among all cocomplete $\mathcal{Q}$-categories (resp. $(\mathbb{U}, \mathcal{Q})$-categories) in the same ways as continuous domains are characterized among all dcpos. By varying the choice of the quantale $\mathcal{Q}$ and the notion of ideals, and by further allowing the ultrafilter monad to act on the quantale, we obtain a flexible theory of continuity that applies to partial orders and to metric and topological spaces. We demonstrate on examples that our theory unifies some major approaches to quantitative domain theory.

1. Introduction

Quantitative domain theory. The contrast between the needs of denotational semantics and the modelling power that domain theory can offer became well visible when in the early eighties de Bakker and Zucker [deBZ82] presented a quantitative model of concurrent processes based on metric spaces. Their work was later further generalized by America and Rutten [AR89] who considered a general problem of solving recursive domain equations in the category of metric spaces. Since that time much effort has been spent on unification of domain-theoretic and metric approaches to denotational semantics, which in practice meant a search for a class of mathematical structures that can serve as (quantitative) domains of computation. As an early example, Smyth proposed a framework based on quasi-metrics and quasi-uniformities [Smy88]. Both of these quantitative structures differ from their "classical" counterparts by discarding symmetry. However, in Smyth's opinion, in order to accommodate semantic domains used in computer science, a further reformulation of basic definitions involving limits and completeness was necessary. Consequently, he suggested bicomplete totally bounded quasi-uniform spaces [Smy91] as quantitative domains, in his next paper [Smy94] reworked the definition of completeness (that is named Smyth-completeness since then), and introduced so called topological quasi-uniform spaces, in which the quasi-uniform structure is linked to an auxiliary topology by some additional axioms. Smyth's insight immediately inspired further studies in this direction [Sün93, Kün93, Sün95, Sün97].

Other important structures that unify partial orders and metric spaces are *$\mathcal{Q}$-continuity spaces* introduced by Kopperman [Kop88]. The idea was to use a non-symmetric distance that takes values in a set $\mathcal{Q}$ with a rich order structure. Flagg [Fl97] suggested that $\mathcal{Q}$ should be a *value quantale*, that is, a completely distributive unital quantale in which the set of elements that are approximated by the unit is a filter. Soon both authors published a joint paper [FK97] summarizing their research.

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Since F.W. Lawvere's famous 1973 paper [Law73] it is well-known that both ordered sets and metric spaces can be viewed as $\mathcal{Q}$-enriched categories in the sense of Eilenberg and Kelly [EK66, Kel82]: the former ones for the quantale $\mathbf{2}$, the latter ones for the quantale $[0, \infty]$. Clearly Kopperman and Flagg's $\mathcal{Q}$-continuity spaces are exactly categories enriched in a value quantale $\mathcal{Q}$.

Smyth's and Lawvere's ideas have been combined together in a series of papers by the Amsterdam research group at CWI [BvBR96, Rut98, BvBR98] that showed, among other things, how to construct the (sequential Yoneda) completion and powerdomains for $[0, \infty]$-categories. Their work has been complemented by Künzi and Schellekens in [KS02] (they proposed the netwise version of the Yoneda completion). A completion by flat modules for generalized metric spaces (resp. completion by type 1 filters) was further discussed by Vickers [Vic05] (resp. by Schmitt [Sch06]). Independently, Flagg, Sünderhauf and Wagner [FS02, FSW96] studied ideal completion of $\mathcal{Q}$-continuity spaces and they in effect demonstrated that for $\mathcal{Q} = [0, \infty]$ their results phrased in terms of ideals (called FSW-ideals here) agree with results of the CWI group phrased in terms of Cauchy nets. They also gave a representation theory for *algebraic* $\mathcal{Q}$-continuity spaces.

Furthermore, in [Wag94] and later in [Wag97], Wagner proposed a framework for solving recursive domain equations in certain complete $\mathcal{Q}$-categories, thereby unifying original attempts of Scott [GHK+03, SP82] and de Bakker and Zucker [deBZ82] that in the eighties seemed to be fundamentally different. Since then these ideas of domain-theoretic origin have been successfully applied in semantics. Most notably, solving recursive equations over metric spaces proved to be one of the fundamental tools in semantics of concurrency, see e.g. [vBr01, vBW05, vBMOW05, vBHMW07].

Our motivation and related work. A central part of domain theory revolves around a notion of *approximation*, which provides a mathematical content to the idea that infinite objects are given in some coherent way as limits of their finite approximations. This leads to considering, not arbitrary complete partial orders, but the continuous ones. Our work is to be thought of as a foundational study of approximation in $\mathcal{Q}$-categories, that generalizes the domain-theoretic notion. Our exposition is categorical but kept close to the domain-theoretic language [AJ94, GHK+03]. Consequently, we speak about auxiliary, approximating and Scott-continuous $\mathcal{Q}$-distributors, about the way-below $\mathcal{Q}$-distributor, and we introduce continuous $\mathcal{Q}$-categories. The generalization from domain theory to $\mathcal{Q}$-categories that we propose proceeds on various levels, as we shall explain below, comparing our paper to related work in the area.

Relative continuity. There is no canonical choice for $\mathcal{Q}$-categorical counterparts of even the most fundamental notions of domain theory. For instance, as we saw above, order ideals can be generalized to several non-equivalent concepts on the $\mathcal{Q}$-level (e.g. forward Cauchy nets, flat modules, FSW-ideals) which nevertheless yield the same definitions in both metric and order-theoretic cases [FSW96, BvBR98, Vic05]. Consequently, one obtains different notions of (co)completeness for $\mathcal{Q}$-categories based on a specific choice of ideals. The starting point of our paper is the conviction that one is not obliged to make this choice right at the beginning, and we study cocompleteness and continuity of $\mathcal{Q}$-categories relative to an abstract class of ideals J subject to suitable axioms. Accordingly, we speak about J-cocompleteness and J-continuity. Although there are many papers in the literature dealing with relative cocompleteness [AK88, KS05, CH09b, LZ07], we are not aware of any systematic study of relative continuity in $\mathcal{Q}$-categories. We therefore introduce a concept of a J-continuous $\mathcal{Q}$-category and develop its basic characterisations. For appropriate choices of $\mathcal{Q}$ and J we recover many of the well-known classical structures: continuous domains, completely distributive complete lattices, Cauchy-complete metric spaces but there remain many more settings where the meaning of J-continuity is still to be explored.

Continuous categories. The difference between continuous categories of Johnstone and Joyal [Joh82, Kos86, ALR03] and our J-continuous $\mathcal{Q}$-categories is that the former are **Set**-based and their continuity is not relative to the choice of ideals. On the other hand, our Theorem 4.15(i) confirms that in essence we introduce continuity in the same way — by the requirement that the left adjoint to the Yoneda embedding itself has a left adjoint.

Other relevant literature. In [Stu07] Stubbe considers totally continuous cocomplete $\mathcal{Q}$-categories enriched over a quantaloid $\mathcal{Q}$. On the one hand, a significant part of the results from [Stu07] can be recovered from our paper as soon as we fix J to be the class of all $\mathcal{Q}$-distributors. On the other hand, Stubbe shows that it is possible to work in the more general context of quantaloids, rather than that of quantales.

$(\mathbb{U}, \mathcal{Q})$-categories. In the last part of our paper we propose a further substantial generalization of continuous domains by considering so called $(\mathbb{U}, \mathcal{Q})$-categories, where the ultrafilter monad $\mathbb{U}$ is allowed to act on the quantale $\mathcal{Q}$. The most important for us is the example of $(\mathbb{U}, \mathbf{2})$-categories as the category of $(\mathbb{U}, \mathbf{2})$-categories is isomorphic to the category of all topological spaces. As a consequence, our theory applies uniformly to partial orders, metric spaces and topologies. We believe that discovering fundamental links between the three types of structure will deepen our understanding of each of them separately. In Section 5.7 we introduce J-continuous $(\mathbb{U}, \mathcal{Q})$-categories and show that defining approximation — while still possible 'locally' — becomes difficult globally, which is of course a price paid for such a generous generality. We close the paper by giving a full characterization of J-continuous $(\mathbb{U}, \mathcal{Q})$-categories among all $(\mathbb{U}, \mathcal{Q})$-categories in the same ways as continuous domains are characterized among all dcpos. We also remark that further work [Hof10] showed that suitably defined categories of J-continuous $(\mathbb{U}, \mathcal{Q})$-categories are dually equivalent to certain categories of $(\mathbb{U}, \mathcal{Q})$-categories. However the meaning of approximation in general topological spaces is yet to be explored.

It is worth mentioning that the ultrafilter monad $\mathbb{U}$ is made compatible with the quantale structure $\mathcal{Q}$ by the convergence structure of a compact topology on $\mathcal{Q}$. Under some natural assumptions this topology happens to be the Lawson topology, and this observation simplifies the presentation of our results.

2. Preliminaries

2.1. **Quantales.** A $\mathcal{Q} = (Q, \leqslant, \otimes, \mathbf{1})$ is a completely distributive commutative unital quantale (in short: a quantale) such that the unit element $\mathbf{1}$ is greatest with respect to the order on Q. We also assume that $\bot \neq \mathbf{1}$. Recall that complete distributivity of the complete lattice $(Q, \leqslant)$ amounts to the fact that for all $b \in Q$ we have $b = \bigvee\{a \in Q \mid a \prec b\}$, where $a \prec b$ whenever $\forall S \subseteq Q\ (b \leqslant \bigvee S \Rightarrow (\exists s \in S\ a \leqslant s))$.

Examples of quantales include: the two element lattice $\mathbf{2} = (\{\bot, \mathbf{1}\}, \leqslant, \wedge, \mathbf{1})$; the unit interval $[0, 1]$ in the order opposite to the natural one, with truncated addition as tensor; the extended real half line $[0, \infty]$ in the order opposite to the natural one, with addition as tensor. In general, every frame with infimum as tensor is a quantale.

2.2. **$\mathcal{Q}$-categories.** A *$\mathcal{Q}$-category* is a set X with a map (called *the structure of* X) $X\colon X \times X \to Q$ satisfying $\mathbf{1} \leqslant X(x, x)$ (reflexivity), and $X(x, y) \otimes X(y, z) \leqslant X(x, z)$ (transitivity), for all $x, y, z \in X$. A *$\mathcal{Q}$-functor* $f\colon X \to Y$ is a function that satisfies $X(x, y) \leqslant Y(fx, fy)$ for all $x, y \in X$. The resulting category $\mathcal{Q}$-**Cat** is isomorphic to the category **Ord** of (pre)ordered sets if $\mathcal{Q} = \mathbf{2}$, to the category **Met** of generalized metric spaces [Law73] if $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$ (metrics bounded by 1 in the latter case). Furthermore, $\mathcal{Q}$ with its internal hom becomes a $\mathcal{Q}$-category. Moreover, any $\mathcal{Q}$-category has its dual X^{op} defined as $X^{op}(x, y) = X(y, x)$ for all $x, y \in X$.

The category $\mathcal{Q}$-**Cat** admits a tensor product $X \otimes Y((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$, and internal hom: $Y^X(f, g) = \bigwedge_{x \in X} Y(fx, gx)$. The internal hom describes the pointwise order if $\mathcal{Q} = \mathbf{2}$, and the non-symmetrized sup-metric if $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$. Since tensor is left

adjoint to internal hom, every $\mathcal{Q}$-functor $g\colon X\otimes Y\to Z$ has its *exponential mate* $\ulcorner g\urcorner\colon Y\to Z^X$. For example, the structure of X is always a $\mathcal{Q}$-functor of type $X^{op}\otimes X\to Q$, and its exponential mate $\mathsf{y}_X\colon X\to Q^{X^{op}}$, $x\mapsto X(-,x)$ is a $\mathcal{Q}$-functor called the *Yoneda embedding*. The Yoneda Lemma then states that for any $\phi\in\widehat{X}$ (where $\widehat{X}:=Q^{X^{op}}$) we have $\phi x=\widehat{X}(\mathsf{y}_X x,\phi)$.

2.3. **$\mathcal{Q}$-distributors.** A $\mathcal{Q}$-functor of type $X^{op}\otimes Y\to Q$ is called a *$\mathcal{Q}$-distributor*. Examples:

— The structure of any $\mathcal{Q}$-category X is a $\mathcal{Q}$-distributor.

— Any two $\mathcal{Q}$-distributors $\phi\colon X^{op}\otimes Y\to Q$ and $\psi\colon Y^{op}\otimes Z\to Q$ can be composed to give a $\mathcal{Q}$-distributor of type $X^{op}\otimes Z\to Q$:

$$(\psi\cdot\phi)(x,z):=\bigvee_{y\in Y}(\phi(x,y)\otimes\psi(y,z)).$$

Therefore we think of $\phi\colon X^{op}\otimes Y\to Q$ as an arrow $\phi\colon X\multimap Y$, which, by the above, can be composed with $\psi\colon Y\multimap Z$ to give $\psi\cdot\phi\colon X\multimap Z$. Note also that $Y\cdot\phi=\phi=\phi\cdot X$.

— Using $X^{\mathrm{op}}\otimes -\dashv(-)^{X^{\mathrm{op}}}$, a $\mathcal{Q}$-distributor $\phi : X\multimap Y$ can be also seen as a $\mathcal{Q}$-functor $\ulcorner\phi\urcorner : Y\to\widehat{X}$.

— Any function $f\colon X\to Y$ gives rise to two $\mathcal{Q}$-distributors, namely $f_*\colon X\multimap Y$, $f_*(x,y)=Y(fx,y)$ and $f^*\colon Y\multimap X$, $f^*(y,x)=Y(y,fx)$. Note that the Yoneda Lemma states that $(\mathsf{y}_X)_*(x,\phi)=\phi(x)$.

— For any $\phi\colon X\multimap Y$ and $\psi\colon Z\multimap Y$ we define *lifting of ψ along ϕ* to be the following $\mathcal{Q}$-distributor:

$$(\phi\multimapdot\psi)(z,x)=\bigwedge_{y\in Y}Q(\phi(x,y),\psi(z,y)).$$

We further observe that for any element $x\colon 1\to X$ (1 is the one-element $\mathcal{Q}$-category that should not be confused with the unit of the quantale), the distributor $x^*\colon X\multimap 1$ is in fact the same as the $\mathcal{Q}$-functor $\mathsf{y}_X x:=X(-,x)\in\widehat{X}$.

In **Ord**, distributors of type $X\multimap 1$ are precisely (characteristic maps of) lower sets, and distributors of type $1\multimap X$ are upper sets of the poset X.

On the other hand, in **Met**, any Cauchy sequence $(x_n)_{n\in\omega}$ induces a distributor $\phi\colon 1\multimap X$ via $\phi(x)=\lim_{n\to\infty}X(x_n,x)$, and a distributor $\psi\colon X\multimap 1$ via $\psi(x)=\lim_{n\to\infty}X(x,x_n)$. Observe that $\psi\cdot\phi\leqslant 0$ and $\phi\cdot\psi\geqslant X$ in the pointwise order. Conversely, any pair of distributors that satisfies the above equations comes from some Cauchy sequence on X.

More generally, we will say that $\mathcal{Q}$-distributors $\phi\colon Z\multimap X$, $\psi\colon X\multimap Z$ are adjoint whenever $\phi\cdot\psi\leqslant X$ and $\psi\cdot\phi\geqslant Z$. In this case we say that ϕ is a left adjoint to ψ and ψ is a right adjoint to ϕ.

3. J-cocomplete $\mathcal{Q}$-categories

Suppose that for each $\mathcal{Q}$-category X there is given a collection JX of $\mathcal{Q}$-distributors of type $X\multimap 1$ (called thereafter *J-ideals*) such that JX contains $x^*\in JX$, for every $x\in X$, and such that for every $\phi\in JX$ and every $\mathcal{Q}$-functor $f : X\to Y$ one has $\phi\cdot f^*\in JY$. The first condition on JX tells us in fact that the Yoneda embedding $\mathsf{y}_X\colon X\to\widehat{X}$ corestricts to JX. We now define X to be *J-cocomplete* if $\mathsf{y}_X\colon X\to JX$ has a left adjoint in $\mathcal{Q}$-**Cat**. That is, there must exist a $\mathcal{Q}$-functor $\mathcal{S}_X\colon JX\to X$ such that for all $\phi\in JX$ and all $x\in X$:

$$X(\mathcal{S}_X\phi,x)=\widehat{X}(\phi,\mathsf{y}_X x). \tag{1}$$

The element $\mathcal{S}_X\phi\in X$ is called the *supremum* of ϕ.

A $\mathcal{Q}$-functor $f\colon X\to Y$ is *J-cocontinuous* if, for every $\phi\in JX$ which has a supremum $\sup_X(\phi)$ in X, also $\underline{f}(\phi):=\phi\cdot f^*\in JY$ has a supremum in Y and, moreover, $f(\mathcal{S}_X\phi)=\mathcal{S}_Y\underline{f}(\phi)$.

If $JX = \widehat{X}$, then evidently $\widehat{X}$ itself is cocomplete (meaning: $\widehat{X}$-cocomplete), the supremum of $\psi\colon \widehat{X} \multimap 1$ is the $\mathcal{Q}$-distributor $X \multimap 1$ defined by

$$x \mapsto \bigvee_{\phi \in \widehat{X}} \psi(\phi) \otimes \phi(x),$$

which can be written very compactly in the language of $\mathcal{Q}$-distributors: $\mathcal{S}_{\widehat{X}}(\psi) = \psi \cdot (\mathsf{y}_X)_*$. For example, if $\mathcal{Q} = \mathbf{2}$, then $\widehat{X}$ is a poset of lower subsets of the poset X ordered by inclusion, ψ is a lower set of lower sets of X, and the supremum of ψ is nothing else but $\bigcup \psi$.

Unfortunately, in general JX is not itself J-cocomplete since the composite $\psi \cdot \mathsf{y}_*$ may not be an element of JX, for $\psi \in JJX$. On the other hand, closure under certain composition provides exactly what is needed:

Theorem 3.1. *The following are equivalent:*

(1) *for all $\psi \in JJX$, $\psi \cdot \mathsf{y}^J_* \in JX$, where $\mathsf{y}^J : X \to JX$ denotes the corestriction to JX of the Yoneda embedding $\mathsf{y}\colon X \to \widehat{X}$.*
(2) *JX is J-cocomplete and the inclusion functor $i : JX \hookrightarrow \widehat{X}$ preserves J-suprema.*
(3) *J-**Dist** $:= \{\phi\colon X \multimap Y \mid \forall y \in Y \quad y^* \cdot \phi \in JX\}$ is closed under composition.*

Proof. Assume first (1), and let $\psi \in JJX$. As for $\widehat{X}$, the supremum of $\psi : JX \multimap 1$ is given by $\psi \cdot \mathsf{y}^J_*$. Furthermore, from $\mathsf{y} = i \cdot \mathsf{y}^J$ one obtains $\mathsf{y}^J_* = i^* \cdot \mathsf{y}_*$, hence $\psi \cdot i^* \cdot \mathsf{y}_* = \psi \cdot \mathsf{y}^J_*$, which proves (2). Assume now (2). We argue here as in [Stu10, Proposition 4.2]. For $\mathcal{Q}$-distributors $\phi : X \multimap Y$ and $\psi : Y \multimap Z$ and $z \in Z$, $\mathcal{S}_{\widehat{X}}(z^* \cdot \psi \cdot \ulcorner\phi\urcorner^*) = z^* \cdot \psi \cdot \phi \in \widehat{X}$. If both ϕ and ψ belong to J-**Dist**, then $\ulcorner\phi\urcorner$ sends y to $y^* \cdot \phi$ and hence corestricts to JX. Therefore, by hypothesis, $z^* \cdot \psi \cdot \phi \in JX$ for every $z \in Z$, which proves $\psi \cdot \phi \in J$-**Dist**. To see (3)$\Rightarrow$(1), just observe that $\phi^* \cdot \mathsf{y}^J_* = \phi$, for all $\phi \in JX$. $\square$

One calls a choice JX of $\mathcal{Q}$-distributors *saturated* if it satisfies any of the equivalent conditions above.

Convention 3.2. In the rest of our paper we consider only saturated collections JX of $\mathcal{Q}$-distributors.

Relative cocompleteness allows for a unified presentation of seemingly unrelated notions of order- and metric completeness:

Example 3.3. For $\mathcal{Q} = \mathbf{2}$, we consider all $\mathcal{Q}$-distributors of type $X \multimap 1$ corresponding to order-ideals in X (i.e. directed and lower subsets of X), and write $J = \mathbf{Idl}$. Then X is **Idl**-cocomplete if and only if X is a directed-complete. Moreover, **Idl**-cocontinuous **2**-functors are precisely Scott-continuous maps between posets.

Example 3.4. For $\mathcal{Q} = [0, \infty]$ we consider all $\mathcal{Q}$-distributors of type $X \multimap 1$ corresponding to ideals in X in the sense of [BvBR98], and write $J = \mathbf{FC}$. These ideals in turn correspond to equivalence classes of forward Cauchy sequences on X. Hence, X is **FC**-cocomplete if and only if each forward Cauchy sequence on X converges if and only if X is sequentially Yoneda complete. The **FC**-cocontinuous $[0, \infty]$-functors are precisely those non-expansive maps that preserve limits of forward Cauchy sequences.

Example 3.5. For any $\mathcal{Q}$ we can choose J to consist of all right adjoint $\mathcal{Q}$-distributors (i.e. $\mathcal{Q}$-distributors that have left adjoints). Recall from [Law73] that, for $\mathcal{Q} = [0, \infty]$ and for $\mathcal{Q} = [0, 1]$, a right adjoint $\mathcal{Q}$-distributor $X \multimap 1$ corresponds to an equivalence class of Cauchy sequences on X. A generalized metric space X is J-cocomplete if and only if each Cauchy sequence on X converges.

Example 3.6. For a completely distributive quantale $\mathcal{Q}$ and any $\mathcal{Q}$-category X, a $\mathcal{Q}$-distributor $\psi\colon X \multimap 1$ is a **FSW**-ideal if: (a) $\bigvee_{z \in X} \psi z = \mathbf{1}$, and (b) for all $e_1, e_2, d \prec \mathbf{1}$, for all $x_1, x_2 \in X$, whenever $e_1 \prec \psi x_1$ and $e_2 \prec \psi x_2$, then there exists $z \in X$ such that $d \prec \psi z$, $e_1 \prec X(x_1, z)$

and $e_2 \prec X(x_2, z)$. Now for $\mathcal{Q} = [0, \infty]$ **FSW**-ideals on X are in a bijective correspondence with equivalence classes of forward Cauchy nets on X [FSW96]; for $\mathcal{Q} = \mathbf{2}$, **FSW**-ideals are characteristic maps of order-ideals on X. Therefore this example unifies Examples 3.3, 3.4. The **FSW**-cocontinuous **2**-functors are precisely Scott-continuous maps between posets, and the **FSW**-cocontinuous $[0, \infty]$-functors are precisely the non-expansive maps that preserve limits of forward Cauchy nets.

Further examples are mentioned in [Sch06, CH09b, ZF05].

4. J-continuous $\mathcal{Q}$-categories

We now come to the main subject of this paper and introduce J-continuous $\mathcal{Q}$-categories that provide generalization for many structures that play a major role in theoretical computer science, e.g. continuous domains, complete metric spaces, or completely distributive complete lattices.

Let $J_S X$ be a subset of JX consisting of these J-ideals that have suprema, i.e. $\phi \in J_S X$ if and only if there exists $\mathcal{S}_X \phi \in X$ such that the equation (1) holds for all $x \in X$. Observe that this enables us to consider, for any $\mathcal{Q}$-category X, the $\mathcal{Q}$-functor $\mathcal{S}_X \colon J_S X \to X$. Moreover, we note that by the Yoneda lemma, for any $x \in X$, $\mathcal{S}\mathsf{y}_X x = x$ and hence $\mathsf{y}_X \colon X \to JX$ further corestricts to $\mathsf{y}_X \colon X \to J_S X$.

Convention 4.1. In what follows we will drop the indices in $\mathcal{S}_X$ and y_X if the context allows us to do so.

Definition 4.2. A $\mathcal{Q}$-category X is *J-continuous* if the supremum $\mathcal{S} \colon J_S X \to X$ has a left adjoint.

Note that any $\mathcal{Q}$-functor of type $X \to J_S X$ corresponds to a certain $\mathcal{Q}$-distributor $X \multimap X$ belonging to J. Hence, X is J-continuous if and only if there exists a $\mathcal{Q}$-distributor $\Downarrow \colon X \multimap X$ necessarily in J so that, moreover, $\ulcorner \Downarrow \urcorner$ is of type $X \to J_S X$ and is left adjoint to $\mathcal{S} \colon J_S X \to X$.

Let us locate $\Downarrow$ among other $\mathcal{Q}$-distributors of the same type. Firstly, for any $\mathcal{Q}$-distributor $v \colon X \multimap X$ one has:

$$\begin{aligned} \forall \psi \in J_S X \quad (\ulcorner v \urcorner \cdot \mathcal{S}(\psi) \leqslant \psi) \quad &\text{iff} \quad \forall \psi \in J_S X \ \forall x \in X \quad (v(x, \mathcal{S}\psi) \leqslant \psi x) \\ &\text{iff} \quad \forall \psi \in J_S X \ \forall x \in X \quad ((\mathcal{S}^* \cdot v)(x, \psi) \leqslant \mathsf{y}_*(x, \psi)) \\ &\text{iff} \quad \mathcal{S}^* \cdot v \leqslant \mathsf{y}_*. \end{aligned}$$

In particular, $\mathcal{S}^* \cdot \Downarrow \leqslant \mathsf{y}_*$, and $\Downarrow \colon X \multimap X$ is the largest such $\mathcal{Q}$-distributor since, for every $\mathcal{Q}$-distributor $v : X \multimap X$,

$$\begin{aligned} \mathcal{S}^* \cdot v \leqslant \mathsf{y}_* \quad &\text{implies} \quad \forall x \in X \quad ((\ulcorner v \urcorner \cdot \mathcal{S})(\ulcorner \Downarrow \urcorner x) \leqslant \ulcorner \Downarrow \urcorner x) \\ &\text{implies} \quad \forall x \in X \quad \ulcorner v \urcorner x \leqslant \ulcorner \Downarrow \urcorner x \\ &\text{implies} \quad v \leqslant \Downarrow . \end{aligned}$$

We have identified $\Downarrow \colon X \multimap X$ as the lifting $\Downarrow = \mathcal{S}^* \multimapdot \mathsf{y}_*$ of $\mathsf{y}_* : X \multimap J_S X$ along $\mathcal{S}^* \colon X \multimap J_S X$. Of course, this lifting exists in any $\mathcal{Q}$-category and can be studied in its own right. In Section 4.3 we will do so, and give conditions which guarantee that it provides a left adjoint to $\mathcal{S} \colon J_S X \to X$.

Turning to the classical case $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$, the distributor $\Downarrow$ is given by the way-below relation. Therefore, we will call the distributor $\Downarrow \colon X \multimap X$ the *way-below $\mathcal{Q}$-distributor* on X. In the case of metric spaces, as a consequence of symmetry, $\Downarrow \colon X \multimap X$ is the same as the structure $X \colon X \multimap X$.

As it is well-known, the way-below relation on a continuous dcpo is the smallest approximating auxiliary relation. In what follows, we aim for a similar characterisation of the way-below $\mathcal{Q}$-distributor in the general case.

Definition 4.3. Define a $\mathcal{Q}$-distributor $v \colon X \multimap X$ to be:

— *auxiliary*, if $v \leqslant X$.
— *interpolative*, if $v \leqslant v \cdot v$;
— *approximating*, if $v \in J$ and $X \bullet\!\!- v = X$.

Furthermore, a $\mathcal{Q}$-distributor $v\colon X \multimap Y$ is:

— *J-cocontinuous*, if $\mathcal{S}^* \cdot v = \mathsf{y}_* \cdot v$.

The nomenclature of the above definition matches precisely the nomenclature used in domain theory in the case $\mathcal{Q} = \mathbf{2}$, see e.g. Section I-1 of [GHK+03]. Thus for example approximating $\mathbf{2}$-distributors are approximating relations, and **Idl**-cocontinuous $\mathbf{2}$-distributors of type $X \multimap 1$ are precisely the (characteristic maps) of Scott-open subsets of X.

4.1. **Approximating $\mathcal{Q}$-distributors.** Since approximating $\mathcal{Q}$-distributors naturally generalize approximating relations on posets, they enjoy analogous properties:

Lemma 4.4. *Every approximating $\mathcal{Q}$-distributor $v\colon X \multimap X$ is auxiliary. If $v, w\colon X \multimap X$ are approximating, then so is $w \cdot v$.*

Proof. If v is approximating, then $v = X \cdot v = (X \bullet\!\!- v) \cdot v \leqslant X$. Let now $v, w\colon X \multimap X$ be approximating $\mathcal{Q}$-distributors. By hypothesis on J, $w \cdot v \in J$. Furthermore, $X \bullet\!\!- (w \cdot v) = (X \bullet\!\!- v) \bullet\!\!- w = X$. □

Lemma 4.5. *A $\mathcal{Q}$-distributor $v\colon X \multimap X$ is approximating if and only if its exponential mate $\ulcorner v \urcorner$ is of type $X \to J_S X$ and $\mathcal{S} \ulcorner v \urcorner = 1_X$.*

Proof. By definition, $v\colon X \multimap X$ is approximating if and only if $\ulcorner v \urcorner$ is of type $X \to JX$ and, for each $x \in X$, $x_* = X \bullet\!\!- (x^* \cdot v)$. This in turn is equivalent to $\ulcorner v \urcorner x \in J_S X$ and $(\mathcal{S} \cdot \ulcorner v \urcorner)(x) = x$, for each $x \in X$. □

Lemma 4.6. *Any approximating J-cocontinuous $\mathcal{Q}$-distributor is interpolative.*

Proof. From $\mathcal{S}^* \cdot v = \mathsf{y}_* \cdot v$ we deduce $v = \ulcorner v \urcorner^* \cdot \mathcal{S}^* \cdot v = \ulcorner v \urcorner^* \cdot \mathsf{y}_* \cdot v = v \cdot v$. □

4.2. **J-cocontinuous $\mathcal{Q}$-distributors.**

Proposition 4.7. *A $v\colon X \multimap Y$ is J-cocontinuous if and only if $\ulcorner v \urcorner\colon Y \to \widehat{X}$ is J-cocontinuous.*

Proof. It is routine to check that for any $\psi \in J_S Y$, $\mathcal{S}_X(\underline{\ulcorner v \urcorner}(\psi)) = \psi \cdot v$. Hence the $\mathcal{Q}$-functor $\ulcorner v \urcorner$ is J-cocontinuous if and only if $\ulcorner v \urcorner(\mathcal{S}_Y \psi) = \psi \cdot v$. We have, for $x \in X$: $\ulcorner v \urcorner(\mathcal{S}_Y \psi)(x) = v(x, \mathcal{S}_Y \psi) = (\mathcal{S}_Y^* \cdot v)(x, \psi) = ((\mathsf{y}_Y)_* \cdot v)(x, \psi) = (\ulcorner \psi \urcorner^* \cdot (\mathsf{y}_X)_* \cdot v)(x) = (\psi \cdot v)(x)$, as required. □

Corollary 4.8. *If $v\colon Y \multimap Z$ is J-cocontinuous, then $v \cdot w\colon X \multimap Z$ is J-cocontinuous, for any $w\colon X \multimap Y$.*

Corollary 4.9. *A $\mathcal{Q}$-distributor $v\colon X \multimap Y$ is J-cocontinuous if and only if $v \cdot x_*\colon X \multimap Y$ is J-cocontinuous for all $x \in X$.*

Proof. From $\mathcal{S}^* \cdot v \cdot x_* = \mathsf{y}_* \cdot v \cdot x_*$ for all $x \in X$ we deduce $\mathcal{S}^* \cdot v = \mathsf{y}_* \cdot v$. □

4.3. **The way-below $\mathcal{Q}$-distributor.** Recall from the beginning of Section 4, that we define the *way-below* $\mathcal{Q}$-distributor $\Downarrow\colon X \multimap X$ to be the largest v such that $\mathcal{S}^* \cdot v \leqslant \mathsf{y}_*$, that is, $\Downarrow := \mathcal{S}^* \multimapdotbothA \mathsf{y}_*$.

$$\begin{tikzcd} J_S X & X \arrow[l, "\mathsf{y}_*"', "\circ"] \arrow[dl, dotted, "\circ"', "\Downarrow"] \\ X \arrow[u, "\mathcal{S}^*", "\circ"'] & \end{tikzcd}$$

As in the poset case, the way-below $\mathcal{Q}$-distributor is not, in general, approximating; however, it is smaller than any approximating $\mathcal{Q}$-distributor:

Lemma 4.10. *If $v\colon X \multimap X$ is approximating, then $\Downarrow \leqslant v$. Hence, the way-below $\mathcal{Q}$-distributor is auxiliary.*

Proof. Since $\ulcorner v \urcorner^* \cdot \mathsf{y}_* \leqslant v$, we have $\mathsf{y}_* \leqslant \ulcorner v \urcorner^* \multimap v$. Hence $\Downarrow = \mathcal{S}^* \multimap \mathsf{y}_* \leqslant \mathcal{S}^* \multimap (\ulcorner v \urcorner^* \multimap v) = \ulcorner v \urcorner^* \cdot \mathcal{S}^* \multimap v = X \multimap v = v$. □

Corollary 4.11. *If $\Downarrow$ is approximating, then $\Downarrow$ is interpolative.*

Proof. If $\Downarrow$ is approximating, then so is $\Downarrow \cdot \Downarrow$, and therefore $\Downarrow \leqslant \Downarrow \cdot \Downarrow$. □

Lemma 4.12. *Any auxiliary J-cocontinuous $v\colon X \multimapinv X$ satisfies $v \leqslant \Downarrow$.*

Proof. $\mathcal{S}^* \cdot v \leqslant \mathsf{y}_* \cdot v \leqslant \mathsf{y}_* \cdot X = \mathsf{y}_*$. Therefore $v \leqslant \mathcal{S}^* \multimap \mathsf{y}_* = \Downarrow$. □

Lemma 4.13. *If $v\colon X \multimapinv X$ is interpolative and $v \leqslant \Downarrow$, then v is J-cocontinuous.*

Proof. $v \leqslant \mathcal{S}^* \multimap \mathsf{y}_*$ if and only if $\mathcal{S}^* \cdot v \leqslant \mathsf{y}_*$, which yields $\mathcal{S}^* \cdot v \leqslant \mathcal{S}^* \cdot v \cdot v \leqslant \mathsf{y}_* \cdot v$. □

Also from [Stu07] we have:

Lemma 4.14. *Let $\alpha\colon X \to J_S X$ be a J-cocontinuous $\mathcal{Q}$-functor with $\mathcal{S}\alpha \cong 1$. Then $\alpha \dashv \mathcal{S}$.*

We gather the most important consequences of the above considerations here:

Theorem 4.15. *Let X be a $\mathcal{Q}$-category and let $v\colon X \multimapinv X \in J$. The following are equivalent:*

(i) *$\ulcorner v \urcorner$ is of type $X \to J_S X$ and $\ulcorner v \urcorner \dashv \mathcal{S}$,*
(ii) *v is approximating and $v = \Downarrow$,*
(iii) *v is approximating and J-cocontinuous,*
(iv) *v is approximating and $\ulcorner v \urcorner\colon X \to J_S X$ is J-cocontinuous,*
(v) *for all $x \in X$ and $\phi \in J_S X$ we have $\widehat{X}(\ulcorner v \urcorner(x), \phi) = X(x, \mathcal{S}\phi)$.*

Proof. The implication (i)⇒(ii) we have already discussed at the beginning of this section. To see (ii)⇒(iii), assume that $\Downarrow$ is approximating. Then $\Downarrow$ is interpolative and therefore J-cocontinuous. Assume now (iii). Then $\ulcorner v \urcorner\colon X \to \widehat{X}$ is J-cocontinuous. Therefore also $\ulcorner v \urcorner\colon X \to J_S X$, which shows that (iii)⇒(iv). Lemma 4.5 and Lemma 4.14 imply immediately (iv)⇒(i). Clearly, (i) implies (v). Finally, assume (v). Then

$$(X \multimapinv v)(x, y) = \widehat{X}(\ulcorner v \urcorner(x), \ulcorner X \urcorner(y)) = \widehat{X}(\ulcorner v \urcorner(x), \mathsf{y}(y)) = X(x, \mathcal{S}\mathsf{y}(y)) = X(x, y),$$

which proves that v is approximating. Hence, $\ulcorner v \urcorner$ is of type $X \to J_S X$ and indeed left adjoint to $\mathcal{S}$. □

The following theorem provides a full characterization of J-continuity of $\mathcal{Q}$-categories:

Theorem 4.16. *The following are equivalent, for a $\mathcal{Q}$-category X.*

(i) *X is J-continuous,*
(ii) *The way-below $\mathcal{Q}$-distributor $\Downarrow\colon X \multimapinv X$ is approximating,*
(iii) *There exists a J-cocontinuous approximating $\mathcal{Q}$-distributor $v\colon X \multimapinv X$.*

Proof. By equivalence of (v), (ii) and (iv), respectively, in Theorem 4.15.

Examples:

— **FSW**-continuous **FSW**-cocomplete **2**-categories are precisely continuous domains. Indeed, let $(X, \leqslant)$ be a poset. By examining Example 3.6 we gather that a lower set $\psi \subseteq X$ is an **FSW**-ideal if and only if (a) there exists $z \in X$ such that $z \in \psi$ (b) if $x_1, x_2 \in \psi$, then there exists $z \in \psi$ such that $x_1, x_2 \leqslant z$. Therefore **FSW**-ideals are precisely the order ideals of X. Now, the Yoneda embedding y_X is nothing else that the lower closure $\downarrow\colon X \to JX$, and thus the adjunction $\mathcal{S}_X \dashv \downarrow$ amounts to the fact that each order ideal has a supremum $\mathcal{S}_X = \bigvee$; therefore X is **FSW**-cocomplete if and only if it is a directed-complete partial order. Finally, the adjunction $\ulcorner \Downarrow \urcorner \dashv \mathcal{S}_X$ (compare Definition 4.2 and Theorem 4.15(v)) amounts to the equivalence: $\{z \in X \mid z \ll x\} \subseteq \psi \iff x \leqslant \bigvee \psi$, for any $x \in X$ and any order ideal $\psi \subseteq X$, where $\ll$ is the way-below relation on X, which, in turn is the same as saying that $x = \bigvee\{z \in X \mid z \ll x\}$ and the supremum is directed.

Therefore X is **FSW**-continuous if and only if it is continuous in the sense of domain theory.

— cocontinuous cocomplete **2**-categories are completely distributive complete lattices; the way-below distributor becomes the 'totally-below' relation $\prec$ associated with complete distributivity of the underlying lattice (see Section 2.1. above for the definition of $\prec$ and observe that we could have equivalently defined $\prec$ via the adjunction $\ulcorner \Downarrow \urcorner \dashv \mathcal{S}$, where $\mathcal{S} = \bigvee$ and $\ulcorner \Downarrow \urcorner\, x = \{y \mid y \prec x\}$).

— complete metric spaces are **FSW**-continuous $[0,\infty]$-categories. Indeed, by the discussion in Example 3.6 we know that **FSW**-ideals correspond to forward Cauchy nets. However, by symmetry of distance, forward Cauchy nets are in fact Cauchy nets. Moreover, by symmetry again, a metric space X is **FSW**-continuous if and only if it is **FSW**-cocomplete. Since **FSW**-cocompleteness amounts to the fact that every Cauchy net has a limit, we conclude that **FSW**-continuous metric spaces are precisely the complete metric spaces.

5. J-continuous $(\mathbb{U},\mathcal{Q})$-categories

Besides metric spaces, also other geometric objects such as topological and approach spaces can be viewed as generalized ordered sets. The topological case is very elegantly expressed in [Bar70] where topological spaces are presented as sets X equipped with a relation $\mathfrak{x} \to x$ between ultrafilters and points, subject to the *reflexivity* and the *transitivity* condition

$$\dot{x} \to x, \qquad\qquad (\Upsilon \to \sigma \;\&\; \sigma \to x) \;\Rightarrow\; m_X(\Upsilon) \to x, \tag{2}$$

for all $x \in X$, $\sigma \in UX$ and $\Upsilon \in UUX$. Here $e_X(x) = \dot{x}$ is the principal ultrafilter induced by x and

$$m_X(\Upsilon) = \{A \subseteq X \mid A^{\#} \in \Upsilon\} \qquad\qquad (A^{\#} = \{\sigma \in UX \mid A \in \sigma\})$$

is the filtered sum of the filters in Υ. Furthermore, approach spaces [Low97] are to topological spaces what metric spaces are to ordered sets: one trades the quantale **2** for $[0,\infty]$. Hence, an approach space can be presented as a pair (X,a) consisting of a set X and a $[0,\infty]$-relation $a : UX \nrightarrow X$ satisfying

$$0 \geq a(\dot{x},x) \qquad \text{and} \qquad Ua(\Upsilon,\sigma) + a(\sigma,x) \geq a(m_X(\Upsilon),x), \tag{3}$$

and a mapping $f : X \to Y$ between approach spaces $X = (X,a)$ and $Y = (Y,b)$ is a *contraction* whenever $a(\sigma,x) \geq b(Uf(\sigma), f(x))$ for all $\sigma \in UX$ and $x \in X$. In the sequel **App** denotes the category of approach spaces and contraction maps. It is now a little step to admit that the domain $\mathfrak{x}$ of $\mathfrak{x} \to x$ in X is an element of a set TX other then the set UX of all ultrafilters of X. Eventually, we arrive at the notion of a $(\mathbb{T},\mathcal{Q})$-category, for a Set-monad $\mathbb{T} = (T,e,m)$ and quantale $\mathcal{Q}$, as introduced in [CH03, CT03, CHT04]. However, to keep our presentation simple, in this paper we decided to limit our choice of monad to $\mathbb{U}$ (the identity monad case already implicitly discussed in preceeding sections) but we hasten to remark that the majority of the results that follow can be restated and proved in the general setting.

5.1. **The ultrafilter monad.** The ultrafilter monad $\mathbb{U} = (U,e,m)$ consists of:

— a functor $U\colon \mathbf{Set} \to \mathbf{Set}$ that to each set X assigns the set of all ultrafilters on X, and to each map $f\colon X \to Y$ assigns a map $Uf\colon UX \to UY$ given by $Uf(\sigma) := \{B \subseteq Y \mid f^{-1}[B] \in \sigma\}$;

— the unit e, which is a natural transformation from the identity functor on **Set** to U given componentwise by: $e_X\colon X \to UX$, $e_X(x) := \dot{x} = \{A \subseteq X \mid x \in A\}$;

— the multiplication m, which is a natural transformation of type $UU \to U$. Its component $m_X\colon UUX \to UX$ assigns to each ultrafilter of ultrafilters Υ these subsets A of X for which $A^{\#} = \{\sigma \in UX \mid A \in \sigma\}$ belongs to Υ.

5.2. **The Lawson topology on $\mathcal{Q}$.** Note that the transitivity axiom in both (2) and (3) above involves the application of U to a relation $r : X \longrightarrow\!\!\!\!\!\!\!+ Y$: for a $\mathbf{2}$-relation one puts

$$\sigma\,(Ur)\,\nu \quad \text{if} \quad \forall A \in \sigma, B \in \nu\, \exists x \in A, y \in B\,.\, x\, r\, y,$$

and for a $[0,\infty]$-relation

$$Ur(\sigma,\nu) = \sup_{A\in\sigma, B\in\nu} \inf_{x\in A, y\in B} r(x,y),$$

where $\sigma \in UX$ and $\nu \in UY$. These examples suggest that, for a general quantale $\mathcal{Q}$, one defines

$$Ur(\sigma,\nu) = \bigwedge_{A\in\sigma, B\in\nu} \bigvee_{x\in A, y\in B} r(x,y).$$

in order to obtain an extension of U to $\mathcal{Q}$-relations. It is interesting to observe that this formula can be rewriten as

$$(\sigma,\nu) \mapsto \bigvee\{\xi \cdot Ur(\omega) \;\Big|\; \omega \in U(X\times Y), U\pi_1(\omega) = \sigma, U\pi_2(\omega) = \nu\}$$

where $\xi : U\mathcal{Q} \to \mathcal{Q}, \sigma \mapsto \bigwedge_{A\in\sigma} \bigvee A$ is the (convergence of the) Lawson topology on $\mathcal{Q}$ [GHK+03, Thm.III-3.17]. Being the convergence of a compact Hausdorff topology on $\mathcal{Q}$, the diagrams

$$\begin{tikzcd} Q \arrow[r, "e_Q"] \arrow[dr, "1_Q"'] & UQ \arrow[d, "\xi"] \\ & Q \end{tikzcd} \qquad \begin{tikzcd} UUQ \arrow[r, "U\xi"] \arrow[d, "m_Q"'] & UQ \arrow[d, "\xi"] \\ UQ \arrow[r, "\xi"'] & Q \end{tikzcd}$$

commute. Furthermore, in order to guarantee functoriality and other good properties of the above extension of U to $\mathcal{Q}$-**Rel** we assume that

— $(\mathcal{Q}, \mathbf{1}, \otimes)$ is a monoid in the category of compact Hausdorff spaces, i.e. the diagrams

$$\begin{tikzcd} U1 \arrow[r, "U\mathbf{1}"] \arrow[d, "!"'] & UQ \arrow[d, "\xi"] \\ 1 \arrow[r, "\mathbf{1}"'] & Q \end{tikzcd} \qquad \begin{tikzcd} U(Q\times Q) \arrow[r, "U(\otimes)"] \arrow[d, "\langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle"'] & UQ \arrow[d, "\xi"] \\ Q\times Q \arrow[r, "\otimes"'] & Q \end{tikzcd}$$

commute, and

— we also require the following technical property: whenever for $f\colon X \to Y$, $\phi\colon X \to Q$, $\psi\colon Y \to Q$ we have $\psi(y) \leqslant \bigvee_{\{x|fx=y\}} \phi(x)$, then $\xi(U(\psi)(\sigma)) \leqslant \bigvee_{\{\nu|U(f)(\nu)=\sigma\}} \xi(U(\phi)(\nu))$ holds.

In conclusion, the triple $(\mathbb{U}, \mathcal{Q}, \xi)$ is a strict topological theory in the sense of [Hof07]. In the following subsections we summarise the main aspects of the theory of $(\mathbb{U}, \mathcal{Q})$-categories, referring to [CT03, Hof07, CH09a, Hof11, CH09b] for further details. We remark that many notions and results do not differ dramatically from the $\mathcal{Q}$-case, with the notable exception of the dual category and, consequently, the Yoneda lemma (see Proposition 5.3, Lemma 5.4 and Corollary 5.5 below). Our main contribution here is the introduction and study of continuity (see Subsection 5.7), which has to face yet another problem: the lifting of distributors is not always available in the $(\mathbb{U}, \mathcal{Q})$-case. Therefore we cannot use freely the way-below distributor $\Downarrow$, however, we prove that local versions of $\Downarrow$ do exist and can often be used instead.

Remark 5.1. By the Fundamental Theorem of Compact Semilattices (VI-3.4 of [GHK+03]), the only compact Hausdorff topology on $\mathcal{Q}$ making $\wedge$ continuous is the Lawson topology. Therefore, if the tensor on $\mathcal{Q}$ is given by infimum, the Lawson topology is the only compact Hausdorff topology on $\mathcal{Q}$ turning $(\mathbb{U}, \mathcal{Q}, \xi)$ into a strict topological theory.

5.3. $(\mathbb{U},\mathcal{Q})$**-relations.** A $\mathcal{Q}$-relation of the form $\alpha\colon UX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ we call $(\mathbb{U},\mathcal{Q})$-relation from X to Y, and write $\alpha\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$. For $(\mathbb{U},\mathcal{Q})$-relations $\alpha\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ and $\beta\colon Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$ we define the Kleisli convolution $\beta\circ\alpha\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$ as $\beta\circ\alpha = \beta\cdot U\alpha\cdot m_X^{op}$. Kleisli convolution is associative and has the $(\mathbb{U},\mathcal{Q})$-relation $e_X^{op}\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ as a lax identity: $a\circ e_X^{op} = a$ and $e_Y^{op}\circ a \geqslant a$ for any $a\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$. We call $a\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ unitary if $e_Y^{op}\circ a = a$. Furthermore, for a $(\mathbb{U},\mathcal{Q})$-relation $\alpha\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$, the composition function $(-)\circ\alpha$ still has a right adjoint $(-) \mathrel{\circ\!\!-} \alpha$ (we define $\gamma \mathrel{\circ\!\!-} \alpha\colon = \gamma \mathrel{\bullet\!\!-} (U(\alpha)\cdot m_X^{op})$) but $\alpha\circ(-)$ in general does not.

5.4. $(\mathbb{U},\mathcal{Q})$**-categories.** An $(\mathbb{U},\mathcal{Q})$-category is a pair consisting of a set X and an $(\mathbb{U},\mathcal{Q})$-endorelation $X(-,-)\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ such that $e_X^{op}\leqslant X$ and $X\circ X\leqslant X$. Expressed elementwise, these conditions become

$$\mathbf{1}\leqslant X(e_X(x),x) \qquad\text{and}\qquad UX(\Upsilon,\upsilon)\otimes X(\upsilon,x)\leqslant X(m_X(\Upsilon),x)$$

for all $\Upsilon\in UUX$, $\upsilon\in UX$ and $x\in X$. A function $f\colon X\to Y$ between $(\mathbb{U},\mathcal{Q})$-categories is a $(\mathbb{U},\mathcal{Q})$-functor if $f\cdot X\leqslant Y\cdot Tf$, which in pointwise notation reads as $X(\upsilon,x)\leqslant Y(Uf(\upsilon),f(x))$ for all $\upsilon\in UX$, $x\in X$. If we have above even equality, we call $f\colon X\to Y$ fully faithful. The resulting category of $(\mathbb{U},\mathcal{Q})$-categories and $(\mathbb{U},\mathcal{Q})$-functors we denote as $(\mathbb{U},\mathcal{Q})$-**Cat**. The quantale $\mathcal{Q}$ becomes a $(\mathbb{U},\mathcal{Q})$-category $\mathcal{Q}=(\mathcal{Q},\hom_\xi)$, where $\hom_\xi\colon U\mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}$, $(\sigma,v)\mapsto\hom(\xi(\sigma),v)$. By $|X|$ we denote the $(\mathbb{U},\mathcal{Q})$-category (UX,m_X). There is also a free $(\mathbb{U},\mathcal{Q})$-category on a set X given by (X,e_X^{op}). We have a canonical forgetful functor $\mathrm{S}\colon(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}\to\mathcal{Q}\text{-}\mathbf{Cat}$ sending a $(\mathbb{U},\mathcal{Q})$-category X to its underlying $\mathcal{Q}$-category $\mathrm{S}X=(X,X\cdot e_X)$. Furthermore, S has a left adjoint $\mathrm{A}\colon\mathcal{Q}\text{-}\mathbf{Cat}\to(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}$ defined by $\mathrm{A}X=(X,e_X^{op}\cdot UX)$, for each $\mathcal{Q}$-category X.

Example 5.2. For $\mathcal{Q}=\mathbf{2}$, a $(\mathbb{U},\mathcal{Q})$-category is a topological space presented via its ultrafilter convergence structure, and a function $f:X\to Y$ between topological spaces is continuous if and only if it is $(\mathbb{U},\mathcal{Q})$-functor (see [Bar70]). The functor $\mathrm{S}\colon\mathbf{Top}\to\mathbf{Ord}$ sends a topological space X to the ordered set X where $x\leqslant y :\iff \dot{x}\to y \iff y\in\overline{\{x\}}$, and its left adjoint $\mathrm{A}\colon\mathbf{Ord}\to\mathbf{Top}$ takes an ordered set to its Alexandroff space. Note that we consider X here with the dual of the specialization order. The quantale $\mathbf{2}$ becomes the Sierpiński space with $\{1\}$ closed, and $|X|$ is the Čech-Stone compactification of the discrete space X.

There is yet another functor connecting $(\mathbb{U},\mathcal{Q})$-categories with $\mathcal{Q}$-categories, namely $\mathrm{M}\colon(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}\to\mathcal{Q}\text{-}\mathbf{Cat}$ which sends a $(\mathbb{U},\mathcal{Q})$-category X to the $\mathcal{Q}$-category $(UX,UX\cdot m_X^{op})$. These functors are all needed to define the dual of a $(\mathbb{U},\mathcal{Q})$-category X, namely $X^{\mathrm{op}}:=\mathrm{A}((\mathrm{M}X)^{\mathrm{op}})$.

As studied in [Hof07] the tensor product of $\mathcal{Q}$ can be transported to $(\mathbb{U},\mathcal{Q})$-**Cat** by putting $X\otimes Y:=X\times Y$ with structure $(X\otimes Y)(\sigma,(x,y))=X(\upsilon,x)\otimes Y(\nu,y)$, where $\sigma\in U(X\times Y)$, $x\in X$, $y\in Y$, $\upsilon=U\pi_1(\sigma)$ and $\nu=U\pi_2(\sigma)$. The $(\mathbb{U},\mathcal{Q})$-category $E=(1,\mathbf{1})$ is a $\otimes$-neutral object, where 1 is a singleton set and $\mathbf{1}\colon U1\times1\to\mathcal{Q}$ the constant relation with value $\mathbf{1}\in\mathcal{Q}$. In general, this constructions does not result in a closed structure on $(\mathbb{U},\mathcal{Q})$-**Cat**; however we have that: $|X|\otimes(-)\colon(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}\to(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}$ has a right adjoint $(-)^{|X|}\colon(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}\to(\mathbb{U},\mathcal{Q})\text{-}\mathbf{Cat}$.

5.5. $(\mathbb{U},\mathcal{Q})$**-distributors.** Let X and Y be $(\mathbb{U},\mathcal{Q})$-categories and $\phi\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ be a $(\mathbb{U},\mathcal{Q})$-relation. We call ϕ a $(\mathbb{U},\mathcal{Q})$-distributor, and write $\phi\colon X \relbar\joinrel\circ\joinrel\rightharpoonup Y$, if $\phi\circ X=\phi$ and $Y\circ\phi=\phi$. Kleisli convolution is associative, and it follows that $\psi\circ\phi$ is a $(\mathbb{U},\mathcal{Q})$-distributor if $\psi\colon Y \relbar\joinrel\circ\joinrel\rightharpoonup Z$ and $\phi\colon X \relbar\joinrel\circ\joinrel\rightharpoonup Y$ are so. Furthermore, we have $X(-,-)\colon X \relbar\joinrel\circ\joinrel\rightharpoonup X$ for each $(\mathbb{U},\mathcal{Q})$-category X, and, by definition, $X(-,-)$ is the identity $(\mathbb{U},\mathcal{Q})$-distributor on X for the Kleisli convolution. In other words, $(\mathbb{U},\mathcal{Q})$-categories and $(\mathbb{U},\mathcal{Q})$-distributors form a category, denoted as $(\mathbb{U},\mathcal{Q})$-**Dist**, with Kleisli convolution as compositional structure. In fact, $(\mathbb{U},\mathcal{Q})$-**Dist** is an ordered category with the structure on hom-sets inherited from $(\mathbb{U},\mathcal{Q})$-**Rel**. Finally, a $(\mathbb{U},\mathcal{Q})$-relation $\phi\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ is unitary precisely if ϕ is a $(\mathbb{U},\mathcal{Q})$-distributor $\phi\colon(X,e_X^{op}) \relbar\joinrel\circ\joinrel\rightharpoonup (Y,e_Y^{op})$ between the corresponding discrete $(\mathbb{U},\mathcal{Q})$-categories.

A $(\mathbb{U},\mathcal{Q})$-functor $f\colon X\to Y$ induces $(\mathbb{U},\mathcal{Q})$-distributors $f_*\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ and $f^*\colon Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ by putting $f_*=Y\cdot Uf$ and $f^*=f^{op}\cdot Y$ respectively. Hence, for $\sigma\in UX$, $\nu\in UY$, $x\in X$ and $y\in Y$, we have $f_*(\sigma,y)=b(Uf(\sigma),y)$ and $f^*(\nu,x)=b(\nu,f(x))$. These $(\mathbb{U},\mathcal{Q})$-distributors

form an adjunction $f_* \dashv f^*$ in $(\mathbb{U},\mathcal{Q})$-**Dist**. Moreover, given a $(\mathbb{U},\mathcal{Q})$-functor $g\colon Y \to Z$, $g_* \circ f_* = (g \cdot f)_*$ and $f^* \circ g^* = (g \cdot f)^*$, plus $(1_X)_* = (1_X)^* = X$.

We will often need the following crucial property.

Proposition 5.3 ([CH09a])**.** *For an $(\mathbb{U},\mathcal{Q})$-relation $\psi\colon X \nrightarrow Y$, the following are equivalent:*

(i) *$\psi\colon X \multimap Y$ is a $(\mathbb{U},\mathcal{Q})$-distributor;*
(ii) *Both $\psi\colon |X| \otimes Y \to \mathcal{Q}$ and $\psi\colon X^{\mathrm{op}} \otimes Y \to \mathcal{Q}$ are $(\mathbb{U},\mathcal{Q})$-functors.*

Therefore, each $(\mathbb{U},\mathcal{Q})$-distributor $\phi\colon X \multimap Y$ defines a $(\mathbb{U},\mathcal{Q})$-functor $\ulcorner\phi\urcorner\colon Y \to \mathcal{Q}^{|X|}$ which factors through the embedding $\widehat{X} \hookrightarrow \mathcal{Q}^{|X|}$, where $\widehat{X} = \{\psi \in \mathcal{Q}^{|X|} \mid \psi\colon X \multimap 1\}$ and 1 denotes the $(\mathbb{U},\mathcal{Q})$-category $(1, e_1^{op})$.

$$\begin{tikzcd} Y \arrow[r, "\ulcorner\phi\urcorner"] \arrow[dr, "\ulcorner\phi\urcorner"'] & \mathcal{Q}^{|X|} \\ & \widehat{X} \arrow[u, hook] \end{tikzcd}$$

In particular, for each $(\mathbb{U},\mathcal{Q})$-category X we have $X(-,-)\colon X \multimap X$, and therefore obtain the Yoneda $(\mathbb{U},\mathcal{Q})$-functor $\mathsf{y} = \ulcorner X \urcorner\colon X \to \widehat{X}$. The following result is crucial to transport $\mathcal{Q}$-categorical ideas into the $(\mathbb{U},\mathcal{Q})$-setting.

Lemma 5.4 ([Hof11])**.** *Let $\psi\colon X \multimap Z$ and $\phi\colon X \multimap Y$ be $(\mathbb{U},\mathcal{Q})$-distributors. Then, for all $\zeta \in UZ$ and $y \in Y$, $\mathcal{Q}^{|X|}(U\ulcorner\psi\urcorner(\zeta), \ulcorner\phi\urcorner(y)) = (\phi \circ\!\!- \psi)(\zeta, y)$.*

Corollary 5.5. *For each $\phi \in \widehat{X}$ and each $\sigma \in UX$, $\phi(\sigma) = \mathcal{Q}^{|X|}(U\mathsf{y}(\sigma), \phi)$, that is, $(\mathsf{y})_*\colon X \multimap \widehat{X}$ is given by the evaluation map* $\mathrm{ev}\colon UX \times \widehat{X} \to \mathcal{Q}$. *As a consequence, $\mathsf{y}\colon X \to \widehat{X}$ is fully faithful.*

Example 5.6. We consider the quantale $\mathcal{Q} = \mathbf{2}$. In Example 5.2 we have already seen that this case captures precisely topological spaces and continuous maps. It is shown in [HT08] that a distributor $X \multimap 1$ corresponds to a (possibly improper) filter on the lattice of open subsets of X, and the "presheaf space" $\widehat{X}$ is homeomorphic to the space $F_0(X)$ of all such filters, where the sets

$$\{F \in F_0(X) \mid A \in F\} \qquad\qquad (A \subseteq X \text{ open})$$

form a basis for the topology on $F_0(X)$ (see also [Esc97]). Note that $F \leqslant G$ if and only if $F \supseteq G$ in the underlying ordered set $\mathrm{S}(F_0(X))$. The Yoneda embedding $\mathsf{y} : X \to F_0(X)$ sends each point x to the filter $\mathcal{N}(x)$ of all open neighbourhoods of x.

5.6. ***J*-cocomplete $(\mathbb{U},\mathcal{Q})$-categories.** As in the case of $\mathcal{Q}$-categories, we consider cocompleteness and continuity with respect to chosen distributors. To do so, let J-**Dist** be a subcategory of $(\mathbb{U},\mathcal{Q})$-**Dist** such that, for every $(\mathbb{U},\mathcal{Q})$-functor f, $f^* \in J$ and, for all $\phi\colon X \multimap Y \in (\mathbb{U},\mathcal{Q})$-**Dist**,

$$(\forall y \in Y \quad y^* \circ \phi \in J) \Rightarrow \phi \in J.$$

We write JX for the full subcategory of $\hat{X}$ defined by all J-distributors of type $X \multimap 1$. A $(\mathbb{U},\mathcal{Q})$-category X is J-cocomplete if $\mathsf{y} : X \to JX$ has a left adjoint $\mathcal{S} : JX \to X$ in the ordered category $(\mathbb{U},\mathcal{Q})$-**Cat**. By definition, $\mathcal{S} : JX \to X$ is a $(\mathbb{U},\mathcal{Q})$-functor such that, for all $x \in X$ and $\Upsilon \in U(JX)$,

$$X(U\mathcal{S}(\Upsilon), x) = \hat{X}(\Upsilon, \mathsf{y}(x)).$$

It is worthwhile to mention that any left inverse $(\mathbb{U},\mathcal{Q})$-functor $\mathcal{S} : JX \to X$ of y_X is actually a left adjoint. However, we should also mention that the situation slightly differs here from the $\mathcal{Q}$-case. As before, the map $\mathcal{S}$ gives for each $\psi \in JX$ a supremum, i.e. $x \in X$ with $x_* = 1_X \circ\!\!- \psi$. But it is *not true* that X is J-cocomplete if each $\psi \in JX$ has a supremum x in X since the induced map $\mathcal{S} : JX \to X, \psi \mapsto x$ is in general only a $\mathcal{Q}$-functor.

Example 5.7. We consider the quantale $\mathcal{Q} = \mathbf{2}$, that is, topological spaces and continuous maps, and the the absolute case $J = (\mathbb{U}, \mathcal{Q})$-**Dist**. A topological space X is cocomplete if and only if $\mathsf{y} : X \to F_0(X)$ has a left adjoint $\mathcal{S} : F_0(X) \to X$ in **Top**, which is equivalent to $\mathrm{S}(X)^{\mathrm{op}}$ (the dual of the underlying ordered set) being a continuous lattice.
Let X be a complete ordered set. We define a sub-basis $\mathcal{B}$ for a topology on X as follows: $A \in \mathcal{B}$ whenever A is down-closed and, for any $B \subseteq X$, $\bigwedge B \in A$ implies $B \cap A \neq \varnothing$. One easily verifies that the underlying order of the induced topology is just the order we started with; moreover, X is the only neighbourhood of the top-element of X. Hence, each filter ψ (of opens) converges and has indeed a smallest convergence point. To see this, let B be the set of all convergence points of ψ, and put $y = \bigwedge B$. Let $A \in \mathcal{B}$ with $y \in A$. Then there is some $x \in B \cap A$ and $A \in \psi$ since ψ converges to x. Consequently, y is the smallest convergence point of ψ. Therefore each distributor $\psi : X \multimapinv 1$ has a supremum in X but X cannot be a cocomplete topological space if the dual of X is not a continuous lattice.

Hence, a $(\mathbb{U}, \mathcal{Q})$-category X is J-cocomplete if and only if each $\psi : X \multimapinv 1$ in J-**Dist** has "continuously" a supremum. We remark *en passant* that, if one allows distributors in J-**Dist** with arbitrary codomain, then again one has that X is J-cocomplete if and only if each $\psi : X \multimapinv Y$ in J-**Dist** has a supremum in X (see [Hof11, CH09b]). This is one of the reasons why we prefer to define relative cocompletness with respect to a category J-**Dist** of distributors rather then a choice of presheafs $X \multimapinv 1$, for each X.

5.7. **J-continuous $(\mathbb{U}, \mathcal{Q})$-categories.** We come now to our main purpose in this section and introduce J-continuous $(\mathbb{U}, \mathcal{Q})$-categories. Due to the difficulties described in the previous subsection, we cannot introduce $J_S(X)$ as in Section 4 and therefore define J-continuity only for J-cocomplete $(\mathbb{U}, \mathcal{Q})$-categories.

Definition 5.8. A J-cocomplete $(\mathbb{U}, \mathcal{Q})$-category X is called *J-continuous* if the $(\mathbb{U}, \mathcal{Q})$-functor $\mathcal{S} : JX \to X$ has a left adjoint in $(\mathbb{U}, \mathcal{Q})$-**Cat**.

As in the $\mathcal{Q}$-case, such a left adjoint $(\mathbb{U}, \mathcal{Q})$-functor $X \to JX$ corresponds to a $(\mathbb{U}, \mathcal{Q})$-distributor $\Downarrow\colon X \multimapinv X$ which necessarily belongs to J and, moreover, must be the lifting $\Downarrow = \mathcal{S}^* \multimap \mathsf{y}_*$ of $\mathsf{y}_* : X \multimapsto JX$ along $\mathcal{S}^* : X \multimapsto JX$. However, an immediate problem in generalizing the way-below relation to an $(\mathbb{U}, \mathcal{Q})$-distributor in an analogous way to the $\mathcal{Q}$-distributor case stems from the fact that in general the lifting $\multimap$ between $(\mathbb{U}, \mathcal{Q})$-distributors does not exist. We deal first with this problem.

Lemma 5.9. *Let $\psi\colon UY \nrightarrow X$ and $\phi\colon Z \nrightarrow X$ be $\mathcal{Q}$-relations, and let $\phi \multimapdot \psi\colon UY \nrightarrow Z$ the lifting of ψ along ϕ in $\mathcal{Q}$-**Rel**.*

$$\begin{tikzcd} X & UY \arrow[l, "\psi"'] \arrow[dl, dotted, "\phi \multimapdot \psi"] \\ Z \arrow[u, "\phi"] & \end{tikzcd}$$

If ψ is a unitary $(\mathbb{U}, \mathcal{Q})$-relation $\psi\colon Y \nrightarrow X$, then so is $\phi \multimapdot \psi\colon Y \nrightarrow Z$.

Proof. We have to show that

$$\phi \multimapdot \psi\colon |Y| \otimes Z_D \to \mathcal{Q},\ (\nu, z) = \bigwedge_{x \in X} \mathcal{Q}(\phi(z, x), \psi(\nu, x))$$

is a $(\mathbb{U}, \mathcal{Q})$-functor, where Z_D denotes the free $(\mathbb{U}, \mathcal{Q})$-category $Z_D = (Z, e_Z^{op})$ on the set Z. Since $\bigwedge : \mathcal{Q}^{X_D} \to \mathcal{Q}$ is a $(\mathbb{U}, \mathcal{Q})$-functor, it is enough to show functoriality of

$$|Y| \otimes Z_D \otimes X_D \to \mathcal{Q},\ (\nu, z, x) = \mathcal{Q}(\phi(z, x), \psi(\nu, x)).$$

But this function can be expressed as a composite of $(\mathbb{U}, \mathcal{Q})$-functors

$$|Y| \otimes Z_D \otimes X_D \to Z_D \otimes X_D \otimes |Y| \otimes X_D \xrightarrow{\phi \otimes \psi} \mathcal{Q}_D \otimes \mathcal{Q} \to \mathcal{Q}.$$

Note that we use here symmetry of the tensor product $\otimes$ and functoriality of $\Delta_X\colon X_D \to X_D \otimes X_D$. $\square$

Lemma 5.10. *Let $\phi\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ and $\psi\colon Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$ be $(\mathbb{U},\mathcal{Q})$-relations. Furthermore, assume that ϕ is unitary and Y finite. Then $\psi \circ \phi = \psi \cdot e_Y \cdot \phi$.*

Proof. Just observe that

$$\psi \circ \phi = \psi \cdot U\phi \cdot m_X^{op} = \psi \cdot e_Y \cdot e_Y^{op} \cdot U\phi \cdot m_X^{op} = \psi \cdot e_Y \cdot \phi. \qquad \square$$

Lemma 5.11. *For all $(\mathbb{U},\mathcal{Q})$-distributors $\phi\colon Y \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ and $\psi\colon 1 \mathrel{-\!\!\circ\!\!\rightharpoonup} X$, ϕ has a lifting along an ψ in $(\mathbb{U},\mathcal{Q})$-**Dist** which is given by $\psi \multimap \phi = \psi \cdot e_1 \mathrel{-\!\!\bullet} \phi$.*

$$\begin{tikzcd}
X & Y \arrow[l, "\phi"', "\circ" marking] \arrow[dl, dotted, "\circ" marking, "\psi\multimap\phi"] \\
1 \arrow[u, "\psi", "\circ" marking] &
\end{tikzcd}$$

Proof. Let $\gamma\colon Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ be an unitary $(\mathbb{U},\mathcal{Q})$-relation. Then $\psi \circ \gamma \leqslant \phi$ if and only if $\psi \cdot e_1 \cdot \gamma \leqslant \phi$ if and only if $\gamma \leqslant \psi \cdot e_1 \mathrel{-\!\!\bullet} \phi$. $\square$

By analogy with $\mathcal{Q}$-distributors, define $v\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ to be

- *auxiliary*, if $v \leqslant X$;
- *approximating*, if: $v \in J$, and $X \multimapinv v = X$;
- *interpolative*, if $v \leqslant v \circ v$.

We call a $(\mathbb{U},\mathcal{Q})$-distributor $v : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$

- *J-cocontinuous* if $\mathcal{S}^* \circ v = \mathsf{y}_* \circ v$.

Any approximating $(\mathbb{U},\mathcal{Q})$-distributor is auxiliary, and any approximating J-cocontinuous $(\mathbb{U},\mathcal{Q})$-distributor is interpolative. Furthermore, the composition of approximating $(\mathbb{U},\mathcal{Q})$-distributors is again approximating (compare with Lemmata 4.4 and 4.6).

With the same proof as for Proposition 4.7 one verifies that $v\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ is J-cocontinuous if and only if the $(\mathbb{U},\mathcal{Q})$-functor $\ulcorner v \urcorner\colon Y \to \widehat{X}$ is J-cocontinuous.

We also define the way-below $(\mathbb{U},\mathcal{Q})$-distributor $\Downarrow\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ as the lifting of $\mathsf{y}_*\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} JX$ along $\mathcal{S}^*\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} JX$, whenever it exists. Since we do not have in general the way-below distributor 'globally', we define its 'local' version at $x \in X$ to be the lifting of y_* along $\mathcal{S}^* \circ x_*$,

$$\begin{tikzcd}
JX & X \arrow[l, "\mathsf{y}_*"', "\circ" marking] \arrow[dl, dotted, "\circ" marking, "\Downarrow_x := (\mathcal{S}^*\circ x_*)\multimap \mathsf{y}_*"] \\
1 \arrow[u, "\mathcal{S}^*\circ x_*", "\circ" marking] &
\end{tikzcd}$$

which does exist for each $(\mathbb{U},\mathcal{Q})$-category X and each $x \in X$. Of course, if $\Downarrow$ exists on X, then $\Downarrow_x = x^* \circ \Downarrow$ for each $x \in X$.

Lemma 5.12. *For every $(\mathbb{U},\mathcal{Q})$-category X, the map $\Downarrow_- : X \to \widehat{X}$, $x \mapsto \Downarrow_x$ is a $\mathcal{Q}$-functor.*

Proof. For any $x, y \in X$, we have to show that

$$X(x,y) \leq \Downarrow_y \multimapinv \Downarrow_x .$$

First note that $X(x,y) = y^* \circ x_*$. Now,

$$y^* \circ x_* \leqslant (\mathcal{S}^* \circ y_* \multimap \mathsf{y}_*) \multimapinv (\mathcal{S}^* \circ x_* \multimap \mathsf{y}_*)$$

if and only if

$$y^* \circ x_* \circ (\mathcal{S}^* \circ x_* \multimap \mathsf{y}_*) \leqslant \mathcal{S}^* \circ y_* \multimap \mathsf{y}_*,$$

which in turn is equivalent to

$$\mathcal{S}^* \circ y_* \circ y^* \circ x_* \circ (\mathcal{S}^* \circ x_* \multimap \mathsf{y}_*) \leqslant \mathsf{y}_*;$$

and this is indeed true since $y_* \circ y^* \leqslant X$. $\square$

So far we are not able to prove or disprove that $\Downarrow_-$ is a $(\mathbb{U},\mathcal{Q})$-functor. Of course, $\Downarrow_-$ is a $(\mathbb{U},\mathcal{Q})$-functor if X is J-continuous, since in this case $\Downarrow_- = \ulcorner \Downarrow \urcorner$.

Proposition 5.13. *A J-cocomplete $(\mathbb{U},\mathcal{Q})$-category X is J-continuous if and only if $\Downarrow_-$ is a $(\mathbb{U},\mathcal{Q})$-functor and, for each $x \in X$, $\Downarrow_x \in JX$ and $X \multimapinv \Downarrow_x = x_*$.*

Proof. Clearly, the conditions are necessary. Assume now that $\Downarrow_-$ is a $(\mathbb{U},\mathcal{Q})$-functor and $\Downarrow_x \in JX$ and $X \multimapinv \Downarrow_x = x_*$ for each $x \in X$. Hence $\Downarrow_-$ is of type $X \to J_S X$ and we have $\mathcal{S}\cdot \Downarrow_x \cong x$. Let now $\psi \in J_S X$. Then

$$\Downarrow_{\mathcal{S}\psi}(\nu) = \bigwedge_{\phi \in J_S X} Q(X(\dot{\mathcal{S}\psi}, \mathcal{S}\phi), \phi(\nu)) \leq Q(X(\dot{\mathcal{S}\psi}, \mathcal{S}\psi), \psi(\nu)) \leq \psi(\nu),$$

hence $\Downarrow_- \cdot \mathcal{S} \leqslant 1_{J_S X}$, and therefore $\Downarrow_- \dashv \mathcal{S}$. □

In general, for a distributor $v : X \multimapsto X$ and $x \in X$, we consider its local version $v_x : X \multimapsto 1$ at x defined as $v_x := x^* \circ v$. Observe that for any two $(\mathbb{U},\mathcal{Q})$-distributors v, w of type $X \multimapsto X$, if $v_x \leqslant w_x$ for all $x \in X$, then $v \leqslant w$, since $v_x \leqslant w_x$ if and only if $v(\nu, x) \leqslant w(\nu, x)$ for all ν, if and only if $v \leqslant w$. Furthermore, we call a $(\mathbb{U},\mathcal{Q})$-distributor $v\colon X \multimapsto X$ with $v \in J$ *approximating at $x \in X$* if v_x satisfies $X \multimapinv v_x = x_*$. The counterparts to Lemma 4.5 and Lemma 4.10 read as follows.

Proposition 5.14. *A $(\mathbb{U},\mathcal{Q})$-distributor $v\colon X \multimapsto X$ is approximating at x, for every $x \in X$, if and only if its mate $\ulcorner v \urcorner$ is of type $X \to JX$ and $\mathcal{S} \cdot \ulcorner v \urcorner \cong 1_X$.*

Lemma 5.15. *If $v\colon X \multimapsto X$ is approximating at $x \in X$, then $\Downarrow_x \leqslant v_x$.*

Proof. $\Downarrow_x(\nu) = \bigwedge_{\phi \in J_S X} Q(X(\dot{x}, \mathcal{S}\phi), \phi(\nu)) \leqslant Q(X(\dot{x}, \mathcal{S}v_x), v_x(\nu)) = v_x(\nu)$. □

Hence, if the way-below distributor exists, it is smaller then any approximating distributor. In particular, $\Downarrow$ is necessarily auxiliary. As in the $\mathcal{Q}$-case we deduce:

Corollary 5.16. *If $\Downarrow$ exists and is approximating, then $\Downarrow$ is interpolative.*

Lemma 5.17. *Let $v\colon X \multimapsto X$ be auxiliary and J-cocontinuous. Then, for each $x \in X$, $v_x \leq \Downarrow_x$. Hence, if the way-below distributor $\Downarrow$ exists, then $v \leq \Downarrow$.*

Lemma 5.18. *Let $v\colon X \multimapsto X$ be interpolative such that $\mathcal{S}^* \circ v \leq \mathsf{y}_*$. Then v is J-cocontinuous.*

Of course, $\mathcal{S}^* \circ v \leq \mathsf{y}_*$ is equivalent to $v \leq \Downarrow$ assuming that the way-below distributor $\Downarrow$ exists.

Lemma 5.19. *Let $\alpha\colon X \to JX$ be a J-cocontinuous $(\mathbb{U},\mathcal{Q})$-functor with $\mathcal{S}\alpha \cong 1$. Then $\alpha \dashv \mathcal{S}$.*

Theorem 5.20. *Let $v\colon X \multimapsto X \in J$. Then the following are equivalent:*

(i) *$\ulcorner v \urcorner$ is of type $X \to JX$ and $\ulcorner v \urcorner \dashv \mathcal{S}$,*
(ii) *v is approximating and provides the lifting of S^* along y_*, i.e. $v = \Downarrow$,*
(iii) *v is approximating and J-cocontinuous,*
(iv) *v is approximating at $x \in X$ for every $x \in X$ and J-cocontinuous,*
(v) *v is approximating at $x \in X$ for every $x \in X$ and $\ulcorner v \urcorner\colon X \to JX$ is J-cocontinuous,*
(vi) *for all $\sigma \in UX$ and $\psi \in JX$ we have $\widehat{X}(U\ulcorner v \urcorner(\sigma), \psi) = X(\sigma, \mathcal{S}(\psi))$.*

Theorem 5.21. *The following are equivalent, for a J-cocomplete $(\mathbb{U},\mathcal{Q})$-category X.*

(i) *X is J-continuous,*
(ii) *The way-below $(\mathbb{U},\mathcal{Q})$-distributor $\Downarrow\colon X \multimapsto X$ exists and is approximating,*
(iii) *There exists a J-cocontinuous approximating $(\mathbb{U},\mathcal{Q})$-distributor $v\colon X \multimapsto X$,*
(iv) *There exists a J-cocontinuous $(\mathbb{U},\mathcal{Q})$-distributor $v\colon X \multimapsto X$ which is approximating at x, for each $x \in X$.*

Example 5.22. We consider the quantale $\mathcal{Q} = \mathbf{2}$, that is, topological spaces and continuous maps. We start with the absolute case $J = (\mathbb{U}, \mathcal{Q})$-**Dist**. A left adjoint $\mathcal{S} : F_0(X) \to X$ of $\mathsf{y} : X \to F_0(X)$ associates to each filter $F \in F_0(X)$ its smallest convergence point with respect to the order in $\mathrm{S}(X)$. Furthermore, the local version $\Downarrow_x$ of the way-below distributor is given by the filter

$$\Downarrow_x = \left\langle \bigcup \{F \in F_0(X) \mid x \leq \mathcal{S}(F)\} \right\rangle$$

generated by $\bigcup\{F \in F_0(X) \mid x \leq \mathcal{S}(F)\}$. A space X is J-continuous if and only if $\Downarrow_-\colon X \to F_0(X)$ is continuous and every $x \in X$ is the smallest convergence point of $\Downarrow_x$. If X is cocomplete, then continuity of $\Downarrow_-\colon X \to F_0(X)$ reduces to Scott-continuity of the monotone map $\Downarrow_-\colon \mathrm{S}(X)^{\mathrm{op}} \to (F_0(X), \subseteq)$ in the usual order-theoretic sense. So far we are not able to give a more elementary topological description of (absolute) continuity in topological spaces, however, we remark that

- each space of the form $F_0(X)$ is cocomplete and J-continuous, and more general, a topological T_0 space X is continuous if and only if it is the filter space of a frame (this will be the topic of a forthcoming paper).
- and therefore every T_0-space can be embedded into a cocomplete and continuous space.

We finish this paper by mentioning two more examples.

For J being the class of all right adjoint distributors, a topological space X is J-cocomplete if and only if it is weakly sober [CH09a], and every topological space is J-continuous.

Further possible choices of J are discussed in [CH09b]. For instance, we may consider the class J of all those $(\mathbb{U}, \mathcal{Q})$-distributors $\phi\colon X \multimap Y$ for which $\phi \circ (-) : \mathsf{Dist}(1, X) \to \mathsf{Dist}(1, Y)$ preserves certain infima. Note that a distributor $1 \multimap X$ corresponds to a continuous map $X \to \mathbf{2}$, which in turn corresponds to a closed subset of X. Hence $\mathsf{Dist}(1, X)$ is isomorphic to the lattice of closed subsets of X. In particular, we can chose $J = \{\phi\colon X \multimap Y \mid \phi \circ (-) \text{ preserves the top element}\}$. Then

$$\phi \in J \iff \forall y \in Y\ \exists \nu \in UX \quad \nu \phi y.$$

Hence, a distributor $\phi\colon X \multimap 1$ belongs to J if and only if it corresponds to a proper filter. Therefore

$$\Downarrow_x = \left\langle \bigcup \{F \in F_0(X) \mid x \leq \mathcal{S}(F) \text{ and } F \text{ is proper}\} \right\rangle,$$

and a continuous map $f : X \to Y$ is J-dense precisely if it is dense in the usual topological sense. Consequently, X is J-cocomplete if and only if X is densely injective. Finally, X is J-continuous if and only if $\Downarrow_-\colon X \to F_0(X)$ is continuous and, for every $x \in X$, the filter $\Downarrow_x$ is proper and x is its smallest convergence point.

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