SEQUENTIAL CONVERGENCE VIA GALOIS CORRESPONDENCES

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Abstract. Topological sequential spaces are the fixed points of a Galois correspondence between collections of open sets and sequential convergence structures. The same procedure can be followed replacing open sets by other topological concepts, such as closure operators or (ultra)filter convergences. The fixed points of these other Galois correspondences are not topological spaces in general, but they can be embedded into the larger topological classes of pretopological, pseudotopological and convergence spaces.

In this paper, we characterize the sequential convergences which are fixed points of these correspondences as well as their restrictions to topological spaces.

1. Introduction

The notion of convergence of a sequence is fundamental in Topology and it was present from the beginning of its development. In his first attempt to axiomatize topological spaces, M. Fréchet [Fre06] used convergence of sequences to define what he called $\mathcal{L}$-spaces (see Section 3). This axiomatic turned out to be too weak and led to several undesirable properties. Some of the problems could be fixed by P. Urysohn [Ury26] adding a new axiom to thus defining $\mathcal{L}^*$-spaces.

Meanwhile, F. Hausdorff [Hau14] succeeded in giving a satisfactory form to the definition of a topological space using open sets as a basic concept, and then for many years sequential convergence did not play a central role in topology.

In the sixties, sequential convergence became again a subject of interest for many topologists. Among them, J. Kisyński [Kis60] proved that the $\mathcal{L}^*$-spaces are exactly the sequential convergences of topological spaces in which the sequences have unique limits, and S. P. Franklin [Fra65] arrived at the notion of sequential space. Using Franklin's terminology, $\mathcal{L}^*$-spaces are the sequentially Hausdorff sequential spaces. However, if the uniqueness of the limit is dropped, sequential convergences satisfying the remaining axioms of $\mathcal{L}^*$-spaces, (1), (2) and (3) in Section 3, form a larger class than the one of sequential spaces (Example 14). It was only in 1985 that V. Koutnik [Kou85] gave a characterization of sequential spaces in terms of convergence of sequences.

Sequential spaces are the topological spaces for which the open sets are determined by sequential convergence. Although topological spaces are equivalently defined by closure operators, the class of topological spaces for which the closure operator is determined by convergence of sequences is strictly contained in the class of sequential spaces (see, e.g., [Eng89, p.54]). Such a space is called *Fréchet space*.

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In [Čec66], E. Čech uses pretopological closure operators to formulate topological concepts. In that context, we can say that a *presequential space* is a pretopological closure space for which the closure can be described in terms of convergence of sequences. In this paper we prove that the classes of sequential and presequential spaces are better seen as fixed points of Galois correspondences between sequential convergences and collections of open sets or closure operators, respectively. These correspondences are described in Section 2.

As it is well known, sequences are not sufficient to characterize topological spaces, to remedy this the concepts of nets [MS22] and filters [Car37] were introduced. These notions were also used to describe larger classes of spaces, such as pseudotopological or convergence spaces. Using appropriate Galois correspondences, we define pseudosequential spaces (Definition 20) and convergence sequential spaces (Definition 2). In [GH07], the convergence relations of sequential spaces were characterized using axioms that, at the same time, include the ones of Fréchet/Urysohn and relate to the axioms for filter convergence of topological spaces (see [Kow54]). Following that work, we characterize other sequential classes, as well as their restrictions to the topological spaces, which are contained in but are not equal to sequential spaces. We notice that the topological presequential spaces are the Fréchet spaces, and then they are properly contained in the sequential spaces. The sequential spaces can be embedded into presequential spaces via the sequential convergence, which is not the restriction of the embedding from topological into pretopological spaces.

2. The Galois Correspondences

For a fixed set X and any subset A of X, SA denotes the set of all sequences in X which are eventually in A. In particular SX is the set of all sequences in X.

Every topology in a set, or more generally any collection of subsets of a set X, $\tau \subseteq PX$, gives rise to a sequential convergence structure $a \subseteq SX \times X$. One the other hand it is possible to define a topology from a given sequential convergence relation. Since we think of $a \subseteq SX \times X$ as a convergence relation, we often write $(x_n)_{n\in\mathbb{N}} \to x$ instead of $((x_n)_{n\in\mathbb{N}}, x) \in a$.

It is easy to see that the relation between the collections of open sets and the sequential convergences defines a pair of order-reversing maps

$$PPX \underset{\mathcal{O}}{\overset{\Sigma}{\rightleftarrows}} P(SX \times X) \qquad \text{(A)}$$

$$\mathcal{O}(a) := \{A \subseteq X \,|\, (x_n)_n \to x \in A \;\Rightarrow (x_n)_n \in SA\},$$

$$\Sigma(\tau) := \{((x_n)_n, x) \,|\, \forall A \in \tau \; [x \in A \Rightarrow (x_n) \in SA]\}.$$

Since $\tau \subseteq \mathcal{O}\Sigma(\tau)$ and $a \subseteq \Sigma\mathcal{O}(a)$, this pair is a *Galois correspondence*. The fixed points of this correspondence are precisely the the sequential topologies on X, i.e τ is a sequential topology if and only if $\tau = \mathcal{O}\Sigma(\tau)$.

Using other ways of defining topological spaces, such as closure operators or filter convergences, we obtain similar Galois correspondences. Unlike in the case of the open sets, in order to have nice Galois correspondences one has to impose some restrictions.

We first define a pair of order-preserving maps between "closure operators" in X ($k : PX \to PX$) and sequential convergences. The definition of the closure induced by the sequential convergence is just the natural one. To better understand the other direction, one should have in mind that $(x_n)_n \to x$ if and only x is an accumulation point of every subsequence of $(x_n)_n$.

For simplicity, we write $(y_n)_{n\in\mathbb{N}} \preccurlyeq (x_n)_{n\in\mathbb{N}}$ to indicate that $(y_n)_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$.

We have a pair of order-preserving maps

$$PX^{PX} \underset{\sigma}{\overset{\Sigma}{\rightleftarrows}} P(SX \times X) \qquad \text{(B)}$$

$$\sigma(a)(A) := \{x \in X \mid \exists (x_n) \in SA \ \ x_n \to x\},$$

$$\Sigma(k) := \{((x_n)_n, x) \mid [\exists (y_n) \preccurlyeq (x_n) \ \ (y_n) \in SA] \Rightarrow x \in k(A)\},$$

which is not a Galois correspondence in general. However, it will become one if we consider only monotone sequential convergence relations, i.e. if $(y_n)_{n\in\mathbb{N}} \preccurlyeq (x_n)_{n\in\mathbb{N}}$ and $(x_n)_n$ converges to x, then $(y_n)_n$ also converges to x. Another way to obtain a Galois correspondence is to replace σ by γ, where γ is induced by the contravariant correspondence between convergences of sequences and filters of neighborhoods, hence

$$\gamma(a)(A) = \{x \in X \mid \exists (x_n) \in SA \ \ \exists (y_n) \not\succcurlyeq (x_n) \ \ (y_n) \to x).$$

We point out that, if we replace σ by γ, the fixed points of this relation are not modified, i.e $a = \Sigma\sigma(a)$ if and only $a = \Sigma\gamma(a)$.

Definition 1. Let X be a set and $k \in PX^{PX}$. The pair (X, k) is a *presequential space* if k is extensive ($A \subseteq k(A)$) and $k = \sigma\Sigma(k)$.

This definition coincides with the one of sequential space in [Čec66], where topology is done in the context of pretopological closure operators. Recall that k is a pretopological closure in X if it is grounded ($k(\emptyset) = \emptyset$), extensive and additive ($k(A \cup B = k(A) \cup k(B)$).

Clearly, the presequential spaces are precisely the fixed points of the correspondence (B) which are pretopologies. As we pointed out in the introduction, the topological presequential spaces are the *Fréchet spaces*, and not the sequential spaces. Another way to look at this, is to say that the embedding $i : \mathsf{Top} \hookrightarrow \mathsf{PrTop}$ does not give rise to an embedding from the class of sequential spaces, Seq, into the class of presequential spaces, PrSeq, since $i(\mathsf{Seq}) \nsubseteq \mathsf{PrSeq}$. However, Seq can be fully embedded in PrSeq via the *sequential closure*, i.e., $\sigma \cdot \Sigma : \mathsf{Seq} \hookrightarrow \mathsf{PrSeq}$ is an embedding. Consequently, the function $\Sigma \cdot \sigma$ is an inclusion, which is not a surprise since, for a topological space, the convergence of sequences is equivalently defined via open sets or closure operators.

Let us notice that there is a clear difference between the sequential convergence and the (ultra)filter convergence. If one considers similar correspondences to (A) and (B) for the (ultra)filter case (see [HT06]), then fixed points for the first correspondence are the topological spaces and all of them are fixed points of the second correspondence.

The way of defining filter convergence from sequential convergence and vice-versa seems more easy than in the previous cases. One can say that a sequence converges if the filter induced by it converges and, in the other direction, that only the filters induced by convergent sequences converges.

This construction will lead to a Galois correspondence between filter and sequential convergence relations. A sequential convergence relation is a fixed point of this correspondence if and only if when two sequences induce the same filter they have the same limit set.

Although the construction is very natural, it has some problems. The first one is that the fixed points from the filter side are very unnatural, since no filter not induced by a sequence converges. So, our option is to define the Galois correspondence in a way imposing the monotonicity.

To make it precise, we denote by FX the set of all filters in X and by $\mathcal{F}(x_n)$ the filter induced by the sequence $(x_n)_{n\in\mathbb{N}}$. As before, for $b \in P(FX \times X)$, we write sometimes $\mathcal{F} \to x$ meaning that $(\mathcal{F}, x) \in b$. One easily verifies that the pair of order-preserving maps

$$\begin{tikzcd} P(FX \times X) \arrow[r, shift left, "\Sigma"] & P(SX \times X) \arrow[l, shift left, "\mathcal{C}"] \end{tikzcd} \tag{C}$$

$$\mathcal{C}(a) := \{(\mathcal{F}, x) \,|\, \exists (x_n) \to x \;\; \mathcal{F}(x_n) \subseteq \mathcal{F}\},$$
$$\Sigma(b) := \{((x_n)_n, x) \,|\, \forall \mathcal{F} \supseteq \mathcal{F}(x_n) \;\; \mathcal{F} \to x\}.$$

forms a Galois correspondence.

Remark. In order to justify our option for this Galois correspondence, we notice that if we replace sequences by countable filter-basis, then the (pre)topological convergence relations for which $b = \mathcal{C}\Sigma(b)$ will be the ones corresponding to first countable (pre)topologies. In this context, the relation between sequences and countable filter-basis is natural, since sequential and presequential spaces can also be defined with countable filter-basis.

Among the fixed points of this correspondence, we are interested in the ones which are convergence spaces. A *convergence space* (e.g. [Dol96]) is just a pair (X, b), where X is a set, $b \in P(FX \times X)$ is monotone and reflexive, i.e. the principal ultrafilter $\dot{x}$ converges to x. We denote this class by Conv.

Definition 2. Let X be a set and $b \in P(FX \times X)$. The pair (X, b) is a *sequential convergence space* if b is reflexive and $b = \mathcal{C}\Sigma(b)$. The class of all sequential convergence spaces is denoted by CoSeq.

As $\sigma \cdot \Sigma : \mathsf{Seq} \hookrightarrow \mathsf{PrSeq}$ in the previous case, also $\mathcal{C} \cdot \Sigma : \mathsf{PrSeq} \hookrightarrow \mathsf{CoSeq}$ defines an embedding. As before, this inclusion is not a restriction of the usual inclusion from PrTop into Conv, and so the pretopological sequential convergence spaces are a proper subclass of the presequential spaces.

Our goal is now to characterize the sequential convergence relations, which are fixed points of the three Galois correspondences, as well as their restrictions to Conv, PrTop and Top. In the last section, we will also discuss what should be a pseudosequential space.

$$\begin{tikzcd}
\mathsf{Conv} \arrow[r, shift left, "\Sigma"] & P(SX \times X) \arrow[l, shift left, "\mathcal{C}"] \arrow[dl] \arrow[ddl, "\sigma"] \arrow[dddl, "\mathcal{O}"] \\
\mathsf{PsTop} \arrow[u, hook] & \\
\mathsf{PrTop} \arrow[u, hook] & \\
\mathsf{Top} \arrow[u, hook] &
\end{tikzcd}$$

3. $\mathcal{L}$-SPACES

Almost from the beginning of the development of Topology, there was a search for the axiomatization of spaces in terms of convergent sequences. In 1906, M. Fréchet [Fre06] defined the notion of *$\mathcal{L}$-space*. A sequential convergence relation (X, a) is called an *$\mathcal{L}$-space* if

(0) each $(x_n)_{n\in\mathbb{N}}$ has at most one convergence point,
(1) $\dot{x} \to x$,
(2) $(x_n)_n \to x \implies [\forall (y_n)_n \preccurlyeq (x_n)_n \ \ (y_n)_n \to x]$,

for all $x \in X$ and $(x_n)_{n\in\mathbb{N}} \in SX$. Similarly to the filter case, $\dot{x}$ denotes the constant sequence $(x)_n$.

These axioms are very weak, and therefore cause some problems. For instance in an $\mathcal{L}$-space $(x_{2n})_n \to x$ and $(x_{2n+1})_n \to x$ does not imply $(x_n)_n \to x$. Later appeared another problem, in an $\mathcal{L}$-space there is no guarantee that two sequences inducing the same filter have the same limit. These kinds of problems were later solved by P. Urysohn [Ury26] (see also [Fre18]) by adding a fourth axiom thus defining the class of $\mathcal{L}^*$-spaces

(3) $[\forall (y_n)_n \preccurlyeq (x_n)_n \ \exists (z_n)_n \preccurlyeq (y_n)_n \ \ (z_n)_n \to x] \implies (x_n)_n \to x.$

This new axiom, together with (2), is too strong to characterize the fixed points of the Galois correspondence (C), but the right condition is obtain by making a small modification in (3).

Remark. The convergence relations satisfying (1), (2) and (3) were studied by P. T. Johnstone [Joh79] under the name *subsequential spaces*. There it is shown that the category of subsequential spaces and continuous maps is the minimal quasitopos extension of the category of sequential spaces.

Lemma 3. *Given two sequences (x_n) and (w_n), $\mathcal{F}(w_n) \subseteq \mathcal{F}(x_n)$ if and only if*
$$\forall (y_n)_n \preccurlyeq (x_n)_n \ \exists (z_n)_n \preccurlyeq (y_n)_n \ \ (z_n)_n \preccurlyeq (w_n)_n \,.$$

In other words the filter induced by (x_n) contains the filter induced by (w_n) if any subsequence of (x_n) has a common subsequence with (w_n).

Proof. Suppose $\mathcal{F}(w_n) \subseteq \mathcal{F}(x_n)$. Let (y_n) be a subsequence of (x_n). Since $\mathcal{F}(x_n) \subseteq \mathcal{F}(y_n)$, $\mathcal{F}(w_n) \subseteq \mathcal{F}(y_n)$.

Define $A_k := \{w_n \,|\, n \geq k\}$ and $B_k := \{y_n \,|\, n \geq k\}$. $\mathcal{F}(w_n) \subseteq \mathcal{F}(y_n)$ means that $\forall m \, \exists k \ B_k \subseteq A_m \ \Leftrightarrow \ \forall m \, \exists k \ x \in B_k \Rightarrow x \in A_m \ \Leftrightarrow \ \forall m \, \exists k \ \forall l \geq k \, \exists p \geq m \ y_l = w_p$.

We are in conditions to define two increasing functions $\varphi, \psi : \mathbb{N} \to \mathbb{N}$,

$$\begin{aligned}
\varphi(1) &:= \min\{k \in \mathbb{N} \,|\, \exists p \ y_k = w_p\}; \\
\psi(1) &:= \min\{p \in \mathbb{N} \,|\, w_p = y_{\varphi(1)}\}; \\
\varphi(n+1) &:= \min\{l > \varphi(n) \,|\, \exists p > \psi(n) \ y_l = w_p\}; \\
\psi(n+1) &:= \min\{p > \psi(n) \,|\, w_p = y_{\varphi(n+1)}\}.
\end{aligned}$$

The fact that $\mathcal{F}(w_n) \subseteq \mathcal{F}(y_n)$ implies that the functions φ and ψ are well-defined. For every $n \in \mathbb{N}$, $y_{\varphi(n)} = w_{\psi(n)}$, which means that $(y_{\varphi(n)})_n$ is a subsequence of (y_n) and (w_n).

Define now the sets A_k as before and the sets $B_k := \{x_n \,|\, n \geq k\}$. Suppose $\mathcal{F}(w_n) \nsubseteq \mathcal{F}(x_n)$. This means that $\exists m \, \forall k \ B_k \nsubseteq A_m$. Define $m := \min\{m \in \mathbb{N} \,|\, \forall k \ B_k \nsubseteq A_m\}$. Then $\forall k \, \exists l \geq k \, \forall p \geq m \ x_l \neq w_p$.

The increasing function $\kappa : \mathbb{N} \to \mathbb{N}$ with

$$\begin{aligned} \kappa(1) &:= \min\{l \in \mathbb{N} \mid x_l \notin A_m\}; \\ \kappa(n+1) &:= \min\{l > \kappa(n) \mid x_l \notin A_m\}; \end{aligned}$$

defines a subsequence $(x_{\kappa(n)})_n$ of $(x_n)_n$ such that $\{x_{\kappa(n)} \mid n \in \mathbb{N}\} \cap \{w_n \mid n \geq m\} = \emptyset$. Finally, this implies that $\forall (z_n)_n \preccurlyeq (y_n)_n \;\; (z_n)_n \not\preccurlyeq (w_n)_n$. □

Theorem 4. *Let X be a set and $a \in P(SX \times X)$. Then $a = \Sigma\mathcal{C}(a)$ if and only if a satisfies*

(3-) $[(w_n)_n \to x$ and $\forall (y_n)_n \preccurlyeq (x_n)_n \; \exists (z_n)_n \preccurlyeq (y_n)_n \;\; (z_n)_n \preccurlyeq (w_n)_n] \implies (x_n)_n \to x$.

Proof. By the definition of the correspondence, $a = \Sigma\mathcal{C}(a)$ if and only if

$$\exists (w_n)_n \to x \;\; \mathcal{F}(w_n) \subseteq \mathcal{F}(x_n) \implies (x_n)_n \to x \; .$$

From the Lemma 3, the proof is straightforward. □

Corollary 5. *Let a be in $P(SX \times X)$. Then $a = \mathcal{C}(b)$ for some b such that $(X, b) \in$* CoSeq *if and only if a satisfies (1) and (3-).*

In other words, (1) and (3-) characterize the sequential convergence spaces.

We will show now that in this characterization, one can not replace condition (3-) by (3) exhibiting a convergence space which satisfies (3-), also (1) and (2), but not (3).

To simplify, in all examples below, we denote any sequence eventually in $\{x\}$ by $\dot{x}$. Hence, when defining $\dot{x} \to y$, we actually require that every sequence eventually in $\{x\}$ to converge to y.

Example 6. Define $X = \{0, 1\}$ and a convergence in X such that

(a) $\dot{x} \to x$, for $x = 0, 1$;

(b) $\dot{0} \to 1$.

Then (1), (2) and (3-) are fulfilled, but not (3) since the sequence $(x_n)_n$ with $x_{2k} = 0$ and $x_{2k+1} = 1$ does not converge to 1.

Remarks. 1. The Axiom (3-) is formally very similar to (3) and, in the presence of (2), it is implied by it. If we replace the implication by an equivalence in the formulation of (3), then (3) implies (3-), which implies (2).

2. The inclusion of the Axiom (0) in the definition of Fréchet's $\mathcal{L}$-space was natural at the time, if we remember that Hausdorff's separation axiom was originally part of the definition of a topological space.

4. Presequential spaces

We will now characterize the sequential convergences of a presequential space, or almost equivalently, we will characterize the relations $a \in P(SX \times X)$ such that $a = \Sigma\sigma(a)$ (correspondence B in the introduction).

Theorem 7. *Let a be in $P(SX \times X)$. Then $a = \Sigma\sigma(a)$ if and only if it satisfies (2), (3) and*

(4-) $(\dot{x}_n)_n \to \dot{x} \Rightarrow (x_n)_n \to x$.

When we write $(\dot{x}_n)_n \to \dot{x}$, we mean that each of the sequences $\dot{x}_n$ converges to x. More generally, the relation $a \in P(SX \times X)$ induces a relation $Sa \in P(S^2X \times SX)$ between sequences of sequences and sequences. We say that $(x^n_m)_{m,n\in\mathbb{N}} \xrightarrow{m} (y_n)_{n\in\mathbb{N}}$ if for every $n \in \mathbb{N}$, $(x^n_m)_{m\in\mathbb{N}} \to y_n$.

The axiom is called (4-) because it is just a particular case of the Axiom (4) (see Section 5).

Proof. Without losing of generality, we may assume (2). Then $a = \Sigma\sigma(a)$ if and only if

$$[\forall (y_n)_n \preccurlyeq (x_n)_n \ \exists (s_n)_n \in S(\{y_n \,|\, n \in \mathbb{N}\}) \ \ (s_n)_n \to x] \implies (x_n)_n \to x.$$

Suppose this condition holds. The Axiom (3) is just the particular case where $(s_n)_n$ is chosen to be a subsequence of $(y_n)_n$. To prove (4-), let $(x_n)_n$ be a sequence such that $\forall n \in \mathbb{N}\ \dot{x_n} \to x$. If $(y_n)_n \preccurlyeq (x_n)_n$, then $\dot{y_1} \to x$ and we conclude that $(x_n)_n \to x$.

Suppose now that the Axioms (3) and (4-) are satisfied. Let $(y_n)_n$ be a subsequence of $(x_n)_n$ such that there is $(s_n)_n \in S(\{y_n \,|\, n \in \mathbb{N}\})$ with $(s_n)_n \to x$. If $\mathcal{F}(s_n) \supseteq \mathcal{F}(y_n)$, then by Lemma 3 there is $(z_n)_n \preccurlyeq (y_n)_n$ such that $(z_n)_n \preccurlyeq (s_n)_n$ and then by $(z_n)_n \to x$ by (2). If $\mathcal{F}(s_n) \not\supseteq \mathcal{F}(y_n)$, then $(s_n)_n$ only has a finite number of terms. So, at least one the constant sequences $\dot{s_n}$ converges to x and s_n is also in the sequence (y_n). Repeating the process, one finds a subsequence $(z_n)_n$ of $(y_n)_n$ for which $\dot{z_n} \to x$ for any n. Axiom (4-) implies that $(z_n)_n \to x$ and Axiom (3) implies that $(x_n)_n \to x$. □

Corollary 8. *(*[Kou85]*) Let a be in $P(SX \times X)$. Then $a = \Sigma(k)$ for some k such that $(X, k) \in$* PrSeq *if and only if it satisfies (1), (2), (3) and (4-).*

This corollary says that the Axioms (1), (2), (3) and (4-) characterize the presequential spaces.

Remark. This result corresponds exactly to the ultrafilter case, where the fixed points of the respective Galois correspondence are characterized as those which satisfy Axioms like (1) and (4-), since (2) and (3) are trivial for ultrafilters (see [Hof05]).

Corollary 9. *(*[Čec66, Theorem 35 B.6]*) A sequential convergence relation is a convergence of a pretopological T_1-space if and only if satisfies (1), (2), (3) and the constant sequences have unique limits.*

If a pretopological space is T_1, then every sequence has at most one limit. The condition (4-) is always fulfilled whenever the constant sequences have unique limits.

We shall now describe an example of a sequential convergence relation satisfying axioms (1), (2) and (3), but not (4-).

Example 10. Define $X = \mathbb{N} \cup \{\infty\}$ and a convergence in X such that

(a) $\dot{x} \to x$, for all $x \in X$;
(b) $(x_n)_n \to \infty$ if $\{x_n \mid n \in \mathbb{N}\}$ is finite.

Then (1), (2) and (3) are fulfilled. We have $\dot{n} \to \infty$ for all $n \in \mathbb{N}$ but not $(x_n)_n \to \infty$, which contradicts (4-).

As we noticed at the end of Section 2, the pretopological sequential convergence spaces are a proper subcategory of the presequential spaces. A presequential space is convergence sequential if the induced filter convergence is pretopological, i.e. if each member of a family of filters converges to x, then so does the intersection of the family (see [HLCS91]).

Proposition 11. *Let a be in $P(SX \times X)$. Then $a = \Sigma(k)$ for some k such that $(X, k) \in$ $\mathsf{CoSeq} \cap \mathsf{PrTop}$ if and only if it satisfies (1), (2), (3-) and*

(PR) $\forall x \in X\ \ \exists (x_n)_n \to x\ \ \forall (y_n)_n \to x\ \ \exists (z_n)_n \preccurlyeq (y_n)_n\ \ (z_n)_n \preccurlyeq (x_n)_n$.

Proof. The Axiom (PR) must guarantee that the induced convergence space is pretopological, i.e. for any point $x \in X$ there is $\mathcal{F} \to x$ such that if $\mathcal{G} \to x$, then $\mathcal{F} \subseteq \mathcal{G}$. The filter $\mathcal{F}$ has to be of the form $\mathcal{F}(x_n)$ for some sequence (x_n), and the Lemma 3 implies that (PR) just says that $\forall x \in X\ \ \exists (x_n)_n \to x\ \ \forall (y_n)_n \to x\ \ \mathcal{F}(x_n) \subseteq \mathcal{F}(y_n)$. □

Corollary 12. *If $a \in P(SX \times X)$ satisfies (3-) and (PR), then it also satisfies (3) and (4-).*

The result is immediate from the previous proposition. Condition (4-) can be proven from (PR) and (2).

Remark. Condition (PR) implies that there are at most a countable number of "disjoint" sequences converging to the same point.

If one considers the definition of pretopological spaces by means of filters of neighborhoods, a CoSeq presequential space is just a space X where for every $x \in X$, there is a sequence $(x_n)_n$ such that $\mathcal{F}(x_n)$ is the neighborhood filter at x. It is now clear that every CoSeq space is first countable. The reverse is not true, since an indiscrete topological space is first countable (and hence presequential) but in general it does not have a countable neighborhood for any of its points. There are many other known examples of presequential spaces which do not belong to CoSeq. For instance any Fréchet space which is not first countable does so.

5. Sequential spaces

The problem of characterizing the fixed points of the Galois correspondence (A), $a = \Sigma\mathcal{O}(a)$, that is the sequential convergences of a sequential space is very old. The axioms (1), (2) and (3) were formalized when searching for a good characterization of sequential spaces. In case of unique convergence points, J. Kisyński [Kis60] proved that these three simple axioms characterize the sequentially Hausdorff sequential spaces, but this does not extend to the general case. As we have seen, these axioms are not even enough to characterize presequential spaces. A characterization of sequential spaces in terms of convergence of sequences was given by V. Koutnik [Kou85]. We will include the characterization made in [GH07] because it has a very natural relation with other results in this paper. At the same time, the approach we had in [GH07] is more related to the "(ultra)filter case" (see [Kow54]).

Before discussing the next theorem, we need to introduce some concepts. For a set X and an ordinal α, we define the set $S^\alpha X$ by putting $S^{\alpha+1}X = S(S^\alpha X)$ and $S^\lambda = \operatorname{colim}_{\alpha<\lambda} S^\alpha$ if λ is a limit ordinal. The colimit is taken considering the natural embedding from $S^\alpha X$ to $S^\beta X$ if $\alpha < \beta$. Let $\mathfrak{x} = (\mathfrak{x}_n)_n \in S^{\alpha+1}X$ and $(y_n)_n \in SX$. We write $\mathfrak{x} \preccurlyeq (y_n)_n$ if $\mathfrak{x}_k \preccurlyeq (y_n)_n$ for every $k \in \mathbb{N}$. In the limit step the colimit is taken as before. Finally $\mathfrak{x} \to x$ if there is $(z_k)_k \in SX$ such that for every $k \in \mathbb{N}$, $\mathfrak{x}_k \to z_k$. For details see [GH07].

Theorem 13. *([GH07]) Let a be in $P(SX \times X)$. Then $a = \Sigma\mathcal{O}(a)$ if and only if it satisfies (1), (2),*

(3') $[(\forall (y_n)_n \preccurlyeq (x_n)_n\ \exists \mathfrak{z} \preccurlyeq (y_n)_n\ \ \mathfrak{z} \to x] \Rightarrow (x_n)_n \to x$ *and*

(4) $(\dot{x}_n)_{n\in\mathbb{N}} \to (y_n)_{n\in\mathbb{N}} \to x\ \Rightarrow\ (x_n)_{n\in\mathbb{N}} \to x$.

The new Axiom (3') is a generalization of (3). We point out that this is similar to the construction of a topological closure from a pretopological one. The Axiom (4-) is the particular case of (4) where $y_n = x$ for all $n \in \mathbb{N}$.

For an example of a convergence relation for which the axioms (1), (2), (3) and (4) are valid but not (3'), see [GH07]. A much easier example is a relation satisfying (1), (2), (3') and (4-) but not (4).

Example 14. Let $X = \{1, 2, 3\}$ and define $(x_n)_{n\in\mathbb{N}} \to x :\Longleftrightarrow (x_n)_{n\in\mathbb{N}} \in S(\{x, x+1\} \cap X)$. Then (1), (2), (3') and (4-) are fulfilled. We have $\dot{3} \to 2$ and $\dot{2} \to 1$ but not $\dot{3} \to 1$, which shows that (4) is not valid for this convergence.

One of the most studied classes of sequential spaces is the class of Fréchet spaces. As already noticed, a Fréchet space is a presequential space (X, k) with k idempotent. In other words, a Fréchet space is a topological presequential space. It is clear that every Fréchet space is a sequential space. There are many examples of sequential spaces which are not Fréchet spaces.

Proposition 15. *(*[GH07]*) Let a be in $P(SX \times X)$. Then a is a convergence of a Fréchet space if and only if it satisfies (1), (2), (3), (4-) and*

(5) $[(x^n_m)_{n,m} \xrightarrow{m} (y_n)_n \xrightarrow{n} x] \implies \exists (x_k)_k \in S(\{X^n_m \mid m, n \in \mathbb{N}\})\ \ (x_k)_k \to x)$.

The Axiom (5) makes the closure operator $\sigma(a)$ idempotent, and then the presequential space is topological (see, e.g., [Eng89, p.64]).

Corollary 16. *If $a \in P(SX \times X)$ satisfies (2), (3), (4-) and (5), then a also satisfies and (3') and (4).*

The result is immediate because any Fréchet space is a sequential space. It is not a hard work to prove that (1) is not necessary. Although, all the conditions (2), (3), (4-) and (5) cannot be avoidable to prove both (3') and (4). Maybe the most surprising is that (4-) must be used in the prove of (3'). We shall see an example of that.

Example 17. Define $X = \mathbb{N} \cup \{a, b\}$ with $a, b \notin \mathbb{N}$ and a convergence in X such that

(a) $\dot{x} \to x$, for all $x \in X$;
(b) $\dot{x} \to b$, for all $x \in X$;
(c) $(x_n)_n \to a$ if $\{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$

The axioms (2), (3) and (5) (also (1)) are satisfied. Every subsequence of $(n)_n$ converges in two steps to b $\dot{(x_n)_n} \to \dot{a} \to b$, then by (3') should also converge to b, which is not true. In fact, this is the failure of the first step of the iteration in (3'). For details, see condition (3.2) in [GH07].

For the sake completeness, we also characterize the convergence sequential topological spaces.

Proposition 18. *Let a be in $P(SX \times X)$. Then $a = \Sigma(\tau)$ for some τ such that $(X, \tau) \in$ $\mathsf{Seq} \cap \mathsf{CoSeq}$ if and only if it satisfies (1), (2), (3-), (PR) and (5).*

A very easy example of a non-topological presequential space is the following one. This example also shows that $\mathsf{CoSeq} \cap \mathsf{PrTop} \not\subseteq \mathsf{Top}$.

Example 19. Define $X = \{0, 1, 2\}$ and a pretopological closure k such that

(a) $k(\{0\} = \{0,1\}$;
(b) $k(\{1\} = \{1,2\}$;
(c) $k(\{2\} = \{2\}$.

The pretopological space (X,k) belongs to CoSeq, but it is not topological.

6. Pseudosequential spaces

The class PsTop of pseudotopological spaces is easily defined in terms of ultrafilter convergence relations. A pseudotopology on X is a reflexive relation between the set UX of all ultrafilters in X and X, i.e. the fixed ultrafilter $\dot{x}$ converges to x.

At this point we want to know what should be a pseudosequential space. We will use a Galois correspondence, between sequential convergence relations and ultrafilter convergence relations, to answer to this question.

As for filters, we say that an ultrafilter converges if it is finer than the filter induced by a convergent sequence. On the other hand, we define the convergence of sequences in the same way as the convergence of filters is defined from an ultrafilter convergence. A sequence converges if every ultrafilter finer than the sequence converges.

In the case of filters, the fixed points of this correspondence are characterized by conditions like (2) and (3). We could expect that such nice conditions would also work for sequences. However, the class of sequential convergence relations satisfying (2) and (3) is a proper subclass of the class of fixed points of the correspondence

$$P(UX\times X) \underset{\mathcal{C}'}{\overset{\Sigma}{\rightleftarrows}} P(SX\times X) \qquad \text{(D)}$$

$$\mathcal{C}'(a) := \{(\mathcal{U},x)\,|\,\exists (x_n)\to x \;\; \mathcal{F}(x_n)\subseteq\mathcal{U}\},$$
$$\Sigma(b) := \{((x_n)_n,x)\,|\,\forall \mathcal{U}\supseteq\mathcal{F}(x_n)\;\; \mathcal{U}\to x\}.$$

Definition 20. Let X be a set and $b\in P(UX\times X)$. The pair (X,b) is a *pseudosequential space* if b is reflexive and $b=\mathcal{C}'\Sigma(b)$. We denote by PsSeq the class of all pseudosequential spaces.

Before characterizing the sequential convergence relations corresponding to the pseudotopological spaces, we need the following Lemma.

Lemma 21. *If (x_n) and (y_n) are two sequences in X and $\mathcal{U}$ is an ultrafilter in X such that $\mathcal{F}(x_n)\cup\mathcal{F}(y_n)\subseteq\mathcal{U}$, then there is a sequence (z_n) with $(z_n)\preccurlyeq(x_n)$ and $(z_n)\preccurlyeq(y_n)$.*

Proof. Let (x_n) and (y_n) be sequences and let $\mathcal{U}$ be such that $\mathcal{F}(x_n)\cup\mathcal{F}(y_n)\subseteq\mathcal{U}$. If $\mathcal{U}=\dot{x}$ is fixed, then the constant sequence $\dot{x}$ must be a subsequence of (x_n) and (y_n).

Suppose now that $\mathcal{U}$ is a free ultrafilter, which implies that every set in $\mathcal{U}$ is infinite. In particular the set $A=\{x_n\,|\,n\in\mathbb{N}\}\cap\{y_n\,|\,n\in\mathbb{N}\}\in\mathcal{U}$ is infinite. We now define two increasing functions $\varphi,\psi:\mathbb{N}\to\mathbb{N}$,

$$\begin{array}{rcl}\varphi(1) &:=& \min\{n\in\mathbb{N}\,|\,x_n\in A\};\\ \psi(1) &:=& \min\{m\in\mathbb{N}\,|\,y_m=x_{\varphi(1)}\};\\ \varphi(k) &:=& \min\{n>\varphi(k-1)\,|\,\exists m>\psi(k-1)\; x_n=y_m\};\\ \psi(k) &:=& \min\{m>\psi(k-1)\,|\,y_m=x_{\varphi(k)}\}.\end{array}$$

Since A is infinite, the functions φ and ψ are well-defined. The sequence $(x_{\varphi(k)})_k=(y_{\psi(k)})_k$ is simultaneously a subsequence of (x_n) and (y_n). □

Theorem 22. *Let a be in $P(SX \times X)$. Then $a = \Sigma\mathcal{C}'(a)$ if and only if it satisfies*

(PS) *if for every directed family $(\mathfrak{x}_i)_{i\in I}$ of subsequences of $\mathfrak{x}$*
$(\exists\mathfrak{y} \to x)\,(\forall i \in I)(\exists\mathfrak{z}_i \preccurlyeq \mathfrak{y})\;\mathfrak{z}_i \preccurlyeq \mathfrak{x}_i$, *then* $\mathfrak{x} \to x$.

In condition (PS) one considers the (reverse) subsequence order. When we say directed family of subsequences, this means that every two subsequences have a common subsequence. In a similar way, $\mathfrak{z} \preccurlyeq \mathfrak{x} \wedge \mathfrak{y}$ means that $\mathfrak{z}$ is at the same time a subsequence of $\mathfrak{x}$ and of $\mathfrak{y}$.

Proof. A convergence relation is a fixed point of this Galois correspondence if and only if

$$\forall\mathcal{U} \supseteq \mathcal{F}(\mathfrak{x})\;(\exists\mathfrak{y} \to x)\;\;\mathcal{F}(\mathfrak{y}) \subseteq \mathcal{U} \implies \mathfrak{x} \to x.$$

It is enough to prove that the hypothesis of this condition is equivalent to the hypothesis of (PS).

Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}(\mathfrak{x})$. If $\mathcal{U}$ is fixed, the family consisting only of the constant sequence $\dot{x}$ is directed, and so it follows that there is $\mathfrak{y} \to x$ with $\dot{x} \preccurlyeq \mathfrak{y}$. Clearly, $\mathcal{F}(\mathfrak{y}) \subseteq \mathcal{U} = \dot{x}$. Consider now $\mathcal{U}$ a free ultrafilter and $\mathfrak{x}$ an injective sequence, which can be seen as a subsequence of the original one. Let $(\mathfrak{x}_i)_{i\in I}$ be the family of the subsequences of $\mathfrak{x}$ such that $\mathcal{F}(\mathfrak{x}_i) \subseteq \mathcal{U}$. By the Lemma 21, every two of the sequences $\mathfrak{x}_i$ have a common subsequence. In this particular case, the common subsequence can be taken to be the "intersection" of the sequences and then its filter is also contained in $\mathcal{U}$. So, $(\mathfrak{x}_i)_{i\in I}$ is a directed family. Then there is $\mathfrak{y} \to x$ such that for every $\mathfrak{x}_i$ there is $\mathfrak{z}_i \preccurlyeq \mathfrak{y} \wedge \mathfrak{x}_i$. Every set $A \in \mathcal{U}$ contains a countable number of terms of $\mathfrak{x}$ and then induces a subsequence of $\mathfrak{x}$ compatible with $\mathcal{U}$. This implies that every A intersects the set of terms of $\mathfrak{y}$ and hence $\mathcal{F}(\mathfrak{y}) \subseteq \mathcal{U}$.

Conversely, let $(\mathfrak{x}_i)_{i\in I}$ be a directed family of subsequences of $\mathfrak{x}$. The union of the induced filters $\cup_{i\in I}\mathcal{F}(\mathfrak{x}_i)$ is a filter, since the family is directed. There is an ultrafilter $\mathcal{U} \supseteq \cup_{i\in I}\mathcal{F}(\mathfrak{x}_i)$ and there is $\mathfrak{y} \to x$ such that $\mathcal{F}(\mathfrak{y}) \subseteq \mathcal{U}$. By the Lemma 21, for every $i \in I$ there is $\mathfrak{z}_i$ such that $\mathfrak{z}_i \preccurlyeq \mathfrak{x}_i \wedge \mathfrak{y}$. □

Corollary 23. *Let a be in $P(SX \times X)$. Then $a = \mathcal{C}'(b)$ for some b such that $(X, b) \in$* PsSeq *if and only if a satisfies (1) and (PS).*

It is now easy to see that if a sequential convergence relation satisfies (1), (2) and (3), then it is a convergence relation of a pseudosequential space.

Proposition 24. *Let a be in $P(SX \times X)$. If a satisfies (2) and (3), then it satisfies the condition (PS).*

Proof. Suppose that conditions (2) and (3) are satisfied and the hypothesis of condition (PS) holds for a sequence (x_n). Let (y_n) be a subsequence of (x_n). The set consisting only of the sequence (y_n) is directed. So, there is a sequence $(w_n) \to x$ having a sequence $(z_n) \preccurlyeq (y_n) \wedge (w_n)$. By (2), (z_n) converges to x. We proved that every subsequence of (x_n) has a subsequence converging to x, which by (3) means that $(x_n) \to x$ □

The implication of the last proposition is not proper. We will describe a convergence relation satisfying (PS), and also (1) and (2), but not (3).

Example 25. Define $X = \mathbb{N} \cup \{\infty\}$ and let $\mathcal{X}$ be a free ultrafilter in $\mathbb{N}$. Define also a convergence in X such that

(a) $\dot{x} \to x$, for all $x \in X$;

(b) $(x_n)_n \to \infty$ if $\mathcal{F}(x_n) \nsubseteq \mathcal{X}$.

It is easy to see that this convergence is a fixed point of the correspondence (D), i.e. satisfies (PS). Let $(x_n)_n$ be a subsequence of $(n)_n$. We have that either $\mathcal{F}((x_{2n})_n)$ or $\mathcal{F}((x_{2n+1})_n)$ is not contained in $\mathcal{X}$ and then $(x_n)_n$ has a subsequence converging to ∞. If (3) was satisfied, then $(n)_n$ would also converge to ∞, which is not true since its induced filter is the cofinite filter and the cofinite filter is contained in every free ultrafilter.

Note that the pseudosequential space defined above is also a convergence sequential space.

If a sequential convergence relation satisfies (PS), then it also satisfies (3-). The reverse does not hold, as Example 6 shows.

As a last example, for the topological space of the reals we will investigate to which of the sequential classes it belongs.

Example 26. Consider $\mathbb{R}$ with the usual topology. $\mathbb{R}$ is a first countable space and therefore it is also Fréchet and sequential. Moreover, it is not pseudosequential and so neither convergence sequential. To prove this last statement consider the filter base $\mathcal{B} = \{(-\epsilon, \epsilon) \mid \epsilon > 0\} \cup \{\mathbb{R} \setminus A \mid A \text{ is countable}\}$. Any ultrafilter containing $\mathcal{B}$ converges to 0, but it does not contain a filter induced by a sequence.

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