TOPOLOGICAL THEORIES AND CLOSED OBJECTS

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Abstract. Recent work of several authors shows that many categories of interest to topologists can be represented as categories of lax algebras. In this paper we introduce the concept of a topological theory as a syntactical tool to deal with lax algebras, and show the usefulness of our approach by applying it to the study of function spaces.

Introduction

In 1970 M. Barr [Bar70] observed that, for the canonical extension $\hat{U}$ of the ultrafilter monad $\mathbb{U} = (U, e, m)$ on Set to the bicategory Rel of sets and relations, topological spaces are precisely the lax Eilenberg–Moore algebras $(X, a : UX \longrightarrow\!\!\!\!\!\!\mapsto X)$ for this extension. Here "lax" refers to the fact that we require the convergence relation a to satisfy only the *in*equalities

$$1_X \le a \cdot e_X \qquad \text{and} \qquad a \cdot \hat{U}a \le a \cdot m_X.$$

This presentation of topological spaces turned out to be very useful for the description of properties of topological spaces and continuous maps in the language of ultrafilter convergence, see for instance [CH02, CHT03, CHJ05]. Of particular interest to this work is the characterisation of exponentiable topological spaces [Möb81, Pis99] as being exactly those spaces where the second axiom holds strictly, that is, $a \cdot \hat{U}a = a \cdot m_X$.

In our recent work [CH03, CT03] we obtain more examples of lax Eilenberg–Moore algebras by substituting Rel by a *suitable* bicategory Y which admits a *suitable* extension of a Set-monad $\mathbb{T}$. However, in order to obtain interesting results, it is often desireable to know more about "suitable". A first step in this direction was already done in [CT03] where the authors considered instead of the generic bicategory Y the (more concrete) bicategory V-Mat of V-matrices, for a quantale V. We feel, however, that this specialisation gives "only half of the way" since the basic ingredients – the quantale V and the monad $\mathbb{T}$ – do not determine the extension of $\mathbb{T}$ to V-Mat. In this paper we will take the next step and introduce a further "player" which connects $\mathbb{T}$ and V, and they together provide enough information to develope the theory unambiguously.

The main motivation for our approach comes from our recent study of completeness [CH07] (in the sense of Lawvere [Law73]), where we observed that the canonical extension [CH04] of $\mathbb{T}$ to V-Mat is determined by (and determines) a map $\xi : T\mathsf{V} \to \mathsf{V}$. Guided by this observation, we introduce *topological theories* as triples $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of a Set-monad $\mathbb{T}$, a quantale V and a map $\xi : T\mathsf{V} \to \mathsf{V}$ compatible with the monad and the quantale structure. Now T can be naturally extended to a lax functor T_ξ on V-Mat. Moreover, we give conditions on $\mathscr{T}$ which guarantee special properties of T_ξ. Based on this extension, *models* of a theory $\mathscr{T}$ are defined

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2000 *Mathematics Subject Classification.* 18C15, 18D05, 18D20, 54B30.

Key words and phrases. Lax algebra, V-category, monad, quantale, function space, compactness.

The author acknowledges partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT.

as lax Eilenberg-Moore algebras as indicated above for the ultrafilter monad. We consider as an important feature of our approach the fact that the map $\xi : T\mathsf{V} \to \mathsf{V}$ allows not only for an extension of T, but also to lift the V-category structure $\hom : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ on the quantale V to a $\mathscr{T}$-algebra structure on V. We prove several properties of the $\mathscr{T}$-algebra V. We remark that this common root of the extension of T and of the structure on V is crucial for our arguments.

Finally, we wish to prove that the framework developed so far is useful for the study of special properties of lax algebras and characterise those objects $X = (X, a)$ which admit nice function spaces. By the latter we mean a (canonical) right adjoint to $X \otimes \_$, where we consider the natural lifting of the tensor product of the quantale V to the model category of $\mathscr{T}$. To do so, for a $\mathscr{T}$-algebra Y we equip Y^X with the largest structure which makes $\mathrm{ev} : X \otimes Y^X \to Y$ a lax homomorphism and show that Y^X is a $\mathscr{T}$-algebra for each Y precisely if $a \cdot T_\xi a = a \cdot m_X$. Of course, in the case of topological spaces we recover the characterisation mentioned at the beginning.

1. The category of V-matrices

Throughout this paper we consider a (commutative and unital) *quantale* $\mathsf{V} = (\mathsf{V}, \otimes, k)$. Hence V is a complete lattice, $k \in \mathsf{V}$ and $\otimes : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ is a commutative and associative operation on V such that

$$u \otimes k = u, \qquad\qquad u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} (u \otimes v_i)$$

for all $u, v_i \in \mathsf{V}$ and $i \in I$. Since V is complete, the preservation of suprema by $u \otimes \_ : \mathsf{V} \to \mathsf{V}$ is equivalent to the existence of a right adjoint $\hom(u, \_) : \mathsf{V} \to \mathsf{V}$ to $u \otimes \_$. Therefore we have a map $\hom : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ such that, for all $u, v, w \in \mathsf{V}$,

$$u \otimes v \leq w \iff v \leq \hom(u, w).$$

Commutativity of $\otimes$ implies $u \leq \hom(v, w) \iff v \leq \hom(u, w)$, hence $\hom(\_, w) : \mathsf{V}^{\mathrm{op}} \to \mathsf{V}$ is right adjoint to $\hom(\_, w) : \mathsf{V} \to \mathsf{V}^{\mathrm{op}}$. Obviously, if $k = \bot$ the bottom element of V we have $\mathsf{V} = \{k\}$. Otherwise V is called *non-trivial* and *throughout this article we will always assume* V *to be so.*

Each complete Heyting algebra V is a quantale with $\otimes = \wedge$ and $k = \top$ the top element of V. Of particular interest to us is the two-element Boolean algebra $\mathsf{2} = \{\text{false} \models \text{true}\}$. A rich source of examples provides the real half line $\mathsf{P} = [0, \infty]$ ordered by the "greater or equal" relation $\geq$. With respect to this order, 0 is the top and ∞ is the bottom element, and we have $\bigvee = \inf$ and $\bigwedge = \sup$ in P. Then P is a Heyting algebra (with $\wedge = \max$) denoted by $\mathsf{P}_\wedge$. Another way of viewing P as a quantale goes as follows: we let $x \otimes y = x + y$ (with $x + \infty = \infty + x = \infty$) for all $x, y \in \mathsf{P}$, then $k = 0$ is obviously the neutral element for $\otimes = +$ and we have $\hom(x, y) = \max\{y - x, 0\}$. We denote this quantale by P_+.

The category V-Mat of V-*matrices* [BCSW83, CT03] has sets as objects, and a morphism $r : X \nrightarrow Y$ in V-Mat is a mapping $r : X \times Y \to \mathsf{V}$. Composition of V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is defined as matrix multiplication

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

and the identity arrow $1_X : X \nrightarrow X$ in V-Mat is the V-matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\bot$ of V. The complete order of V

induces a complete order on $\mathsf{V}\text{-}\mathsf{Mat}(X,Y) = \mathsf{V}^{X\times Y}$: for V-matrices $r, r' : X \nrightarrow Y$ we write $r \leq r'$ if $r(x,y) \leq r'(x,y)$ for all $x \in X$ and $y \in Y$. The composition functions preserve suprema in each variable:

$$s \cdot \bigvee_{i\in I} r_i = \bigvee_{i\in I}(s \cdot r_i) \qquad \text{and} \qquad \left(\bigvee_{i\in I} r_i\right) \cdot t = \bigvee_{i\in I}(r_i \cdot t)$$

for matrices $t : X \nrightarrow Y$, $r_i : Y \nrightarrow Z$ $(i \in I)$ and $s : Z \nrightarrow W$. Given a V-matrices $r : X \nrightarrow Y$, we define its *transpose* $r^\circ : Y \nrightarrow X$ by $r^\circ(y,x) = r(x,y)$. We have the laws

$$1_X^\circ = 1_X, \qquad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \qquad r^{\circ\circ} = r,$$

as well as $r^\circ \leq s^\circ$ whenever $r \leq s$. From that we see that $\mathsf{V}\text{-}\mathsf{Mat}$ is selfdual.

The category Set can be naturally embedded[1] into $\mathsf{V}\text{-}\mathsf{Mat}$ by sending a map $f : X \to Y$ to the V-matrix

$$f : X \nrightarrow Y,\ f(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

To keep notation simple, in the sequel we will write $f : X \to Y$ rather then $f : X \nrightarrow Y$ for a V-matrix induced by a map. We remark that the formula for matrix composition becomes considerable easier if one of the V-matrices is a Set-map:

$$s \cdot f(x,z) = s(f(x),z), \qquad g \cdot r(x,z) = \bigvee_{y\in g^{-1}(z)} r(x,y)$$

for maps $f : X \to Y$ and $g : Y \to Z$ and V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. In particular we see that each function $f : X \to Y$ satisfies the inequalities $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, i.e. f is left adjoint to f° and we write $f \dashv f^\circ$.

Examples 1.1. We have $\mathsf{2}\text{-}\mathsf{Mat} \cong \mathsf{Rel}$. Here Rel denotes the category with sets as objects and (binary) relations as morphisms, composition is given by the usual relational composition. For each relation $r : X \nrightarrow Y$ we let G_r denote its graph $G_r \subseteq X \times Y$, then $r = q \cdot p^\circ$ in Rel, where $p : G_r \to X$ and $q : G_r \to Y$ are the projection maps. $\mathsf{P}_+\text{-}\mathsf{Mat}$ is the 2-category whose morphisms $a : X \nrightarrow Y$ are generalised distances $a : X \times Y \to \mathsf{P}_+$ with composition given by

$$b \cdot a(x,z) = \inf\{a(x,y) + b(y,z) \mid y \in Y\};$$

$1_X : X \nrightarrow X$ is the discrete distance sending the diagonal to 0 and all other pairs (x,x') to ∞.

Let V and W be quantales. A *homomorphism of quantales* is a map $\varphi : \mathsf{V} \to \mathsf{W}$

$$\varphi(u) \otimes \varphi(v) = \varphi(u \otimes v), \qquad k_{\mathsf{W}} = \varphi(k_{\mathsf{V}}), \qquad \bigvee_{i\in I} \varphi(v_i) = \varphi(\bigvee_{i\in I} v_i), \tag{1}$$

for all $u, v, v_i \in \mathsf{V}$ and $i \in I$. Every homomorphism of quantales $\varphi : \mathsf{V} \to \mathsf{W}$ induces a 2-functor $\Phi : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{W}\text{-}\mathsf{Mat}$ which is the identity on objects and sends a V-matrix $a : X \times Y \to \mathsf{V}$ to $\varphi a : X \times Y \to \mathsf{W}$. It is routine to check that $1_X = \Phi(1_X)$, $\Phi(s) \cdot \Phi(r) = \Phi(s \cdot r)$ and

[1] Here we use in fact that V is non-trivial.

$\Phi(r) \leq \Phi(r')$ whenever $r \leq r'$. Moreover, Φ commutes with the embedding of Set as well as with transposing matrices, that is, the diagrams

$$\begin{tikzcd} \mathsf{Set} \arrow[r] \arrow[dr] & \mathsf{V\text{-}Mat} \arrow[d, "\Phi"] \\ & \mathsf{W\text{-}Mat} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{V\text{-}Mat}^{\mathrm{op}} \arrow[r, "(\text{-})^\circ"] \arrow[d, "\Phi^{\mathrm{op}}"'] & \mathsf{V\text{-}Mat} \arrow[d, "\Phi"] \\ \mathsf{V\text{-}Mat}^{\mathrm{op}} \arrow[r, "(\text{-})^\circ"'] & \mathsf{W\text{-}Mat} \end{tikzcd}$$

commute.

The notion of a homomorphism of quantales turns out to be to restrictive for our purpose. More useful will be the concept of a *lax homomorphism of quantales* where we require, instead of (1), only

$$\varphi(u) \otimes \varphi(v) \leq \varphi(u \otimes v), \qquad k_{\mathsf{W}} \leq \varphi(k_{\mathsf{V}}), \qquad \bigvee_{i \in I} \varphi(v_i) \leq \varphi(\bigvee_{i \in I} v_i)$$

for all $u, v \in \mathsf{V}$. If $\varphi : \mathsf{V} \to \mathsf{W}$ is a lax homomorphism of quantales, then $\Phi : \mathsf{V\text{-}Mat} \to \mathsf{W\text{-}Mat}$ is only a *lax functor*, that is,

$$1_X \leq \Phi(1_X), \qquad \Phi(s) \cdot \Phi(r) \leq \Phi(s \cdot r), \qquad \Phi(r) \leq \Phi(r') \text{ whenever } r \leq r',$$

for all V-matrices $r, r' : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. However, Φ still commutes with the involution: $\Phi(r^\circ) = \Phi(r)^\circ$. Moreover, we have $\Phi(f) \geq f$ as well as $\Phi(f^\circ) \geq f^\circ$ for each map f with equality provided that $\varphi(\bot_{\mathsf{V}}) = \bot_{\mathsf{W}}$ and $\varphi(k_{\mathsf{V}}) = k_{\mathsf{W}}$. In general, a lax functor $\hat{T} : \mathsf{V\text{-}Mat} \to \mathsf{W\text{-}Mat}$ is a *lax extension* of the Set-functor $T : \mathsf{Set} \to \mathsf{Set}$ if $\hat{T}X = TX$ and $Tf \leq \hat{T}f$ and $(Tf)^\circ \leq \hat{T}(f^\circ)$ for all $f : X \to Y$ in Set. It was already observed in [Sea05] that such an extension necessarily preserves composition of V-matrices with Set-maps:

$$\hat{T}(s \cdot f) = \hat{T}s \cdot \hat{T}f = \hat{T}s \cdot Tf \qquad \text{and} \qquad \hat{T}(g^\circ \cdot r) = \hat{T}g^\circ \cdot \hat{T}r = Tg^\circ \cdot \hat{T}r$$

for V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ and Set-maps $f : X \to Y$ and $g : Z \to Y$. In particular we have $\Phi(s \cdot f) = \Phi(s) \cdot f$ and $\Phi(g^\circ \cdot r) = g^\circ \cdot \Phi(r)$.

Examples 1.2. (a) For each quantale V, the canonical embedding $\theta_{\mathsf{V}} : 2 \rightarrowtail \mathsf{V}$ sending false to $\bot$ and true to k is a (strict) homomorphism of quantales. It induces the embedding $\Theta_{\mathsf{V}} : \mathsf{Rel} \rightarrowtail \mathsf{V\text{-}Mat}$ which interprets a relation $r : X \nrightarrow Y$ as the V-matrix

$$(x, y) \mapsto \begin{cases} k & \text{if } xry, \\ \bot & \text{else.} \end{cases}$$

To simplify notation, in the sequel we will not use Θ_{V} and write simply r for the V-matrix $\Theta_{\mathsf{V}}(r)$.

(b) The embedding $\theta_{\mathsf{P}_+} : 2 \rightarrowtail \mathsf{P}_+, \text{false} \mapsto \infty, \text{true} \mapsto 0$ has a left adjoint

$$\lambda : \mathsf{P}_+ \to 2,\ x \mapsto \begin{cases} \text{false} & \text{if } x = \infty, \\ \text{true} & \text{else,} \end{cases}$$

and a right adjoint

$$\varrho : \mathsf{P}_+ \to 2,\ x \mapsto \begin{cases} \text{true} & \text{if } x = 0, \\ \text{false} & \text{else.} \end{cases}$$

Both λ and ϱ preserve the tensor product and satisfy $\varrho(0) = \text{true} = \lambda(0)$ and $\varrho(\infty) = \text{false} = \lambda(\infty)$. Being a left adjoint, λ preserves suprema, but ϱ does not. Hence λ is a homomorphism of quantales and ϱ is a lax one, and therefore induce a 2-functor $\Lambda : \mathsf{P}_+\text{-}\mathsf{Mat} \to \mathsf{Rel}$ and a lax functor $R : \mathsf{P}_+\text{-}\mathsf{Mat} \to \mathsf{Rel}$ respectively.

In the sequel it will be often convenient to assume that suprema commute with infima in V, that is, $\bigvee : 2^{\mathsf{V}^{\mathrm{op}}} \to \mathsf{V}$ preserves infima. Since $2^{\mathsf{V}^{\mathrm{op}}}$ is complete, preservation of infima is equivalent to the existence of a left adjoint to $\bigvee$. A complete lattice X where $\bigvee : 2^{X^{\mathrm{op}}} \to X$ has a left adjoint is called *constructively completely distributive (ccd)* (see [Woo04] for a nice presentation of this topic). Classically, a complete lattice X is called *completely distributive (cd)* if

$$\forall I\ \forall (A_i)_{i\in I} \in PX^I. \quad \bigvee_{f\in\prod_{i\in I} A_i} \bigwedge_{i\in I} f(i) = \bigwedge_{i\in I} \bigvee A_i.$$

It can be shown (see [Woo04], for instance) that the equivalence (ccd) $\iff$ (cd) requires and implies the axiom of choice. Each powerset PS is constructively completely distributive independently of the axiom of choice, whereby complete distributivity of PS (for $S \neq \varnothing$) depends on choice. Note that a constructively completely distributive lattice is automatically Heyting.

Let us now have a closer look at the left adjoint to $\bigvee : 2^{X^{\mathrm{op}}} \to X$. It singles out, for each $x \in X$, a down-closed subset $A(x)$ with the following property:

$$\forall S \subseteq X \ \ A(x) \subseteq {\downarrow}S \iff x \leq \bigvee S.$$

Choosing $S = A(x)$ and $S = {\downarrow}x$ we see that $x \leq \bigvee A(x)$ and $A(x) \leq {\downarrow}x$ respectively, hence $x = \bigvee A(x)$. For $u, x \in X$ we define $u \lll x$ (u is *totally below* x) if, for every $S \subseteq X$, $x \leq \bigvee S$ entails $u \in {\downarrow}S$. We remark that $u \lll x$ implies $u \leq x$ and $u \lll v \leq x$ implies $u \lll x$. From the properties mentioned so far it follows now that $A(x) = \{u \in X \mid u \lll x\}$. Hence we have seen that

Theorem 1.3. *A complete lattice X is constructively completely distributive if and only if, for each $x \in X$, $x = \bigvee\{u \in X \mid u \lll x\}$.*

The theorem above implies at once that P as well as PS for any set S are constructively completely distributive. In the first case the totally below relation is given by $>$, whereby in the latter $A \lll B$ if and only if "A is an element of B", that is, $A = \{y\}$ for some $y \in B$. The next result shows that, in order to decide if X is constructively completely distributive, it is not necessary to compute $\lll$.

Proposition 1.4. *A complete lattice X is constructively completely distributive if and only if there exists a relation $\sqsubset$ on X such that, for all $u, v, x \in X$,*

(a) $u \sqsubset v \leq x \implies u \sqsubset x$,
(b) $x = \bigvee\{u \in X \mid u \sqsubset x,\ u \text{ is } \sqsubset\text{-atomic}\}$.

Here $u \in X$ is called $\sqsubset$-atomic *if, for each $S \subseteq X$, $u \sqsubset \bigvee S$ implies $u \in {\downarrow}S$.*

Proof. See for instance [CH04, 5.3]. □

Remark 1.5. Of course, we can always choose $\sqsubset = \lll$. In that case each $x \in X$ is $\lll$-atomic. For $X = PS$ another possible choice is $\sqsubset = \subseteq$. Then A is $\subseteq$-atomic if and only if A is a singleton.

2. Monads

Recall that a *monad* $\mathbb{T} = (T, e, m)$ on a category C consists of a functor $T : \mathsf{C} \to \mathsf{C}$ together with natural transformations $e : \mathrm{Id}_{\mathsf{C}} \to T$ (unit) and $m : TT \to T$ (multiplication) such that the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "m_T"] \arrow[d, "Tm"'] & T^2 \arrow[d, "m"] \\ T^2 \arrow[r, "m"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "e_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "m"] & T \arrow[l, "Te"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

commute. If $\mathbb{T}$ and $\mathbb{T}'$ are monads on C, a *monad morphism* $j : \mathbb{T} \to \mathbb{T}'$ is a natural transformation $j : T \to T'$ such that the diagrams

$$\begin{tikzcd} & \mathrm{Id} \arrow[dl, "e"'] \arrow[dr, "e'"] & \\ T \arrow[rr, "j"'] & & T' \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "j^2"] \arrow[d, "m"'] & T'T' \arrow[d, "m'"] \\ T \arrow[r, "j"'] & T' \end{tikzcd}$$

commute, where $j^2 = j_{T'} \cdot Tj = T'j \cdot j_T$.

There are two *trivial monads* on Set, one with $TX = 1$ for every set X and the other with $T\varnothing = \varnothing$ and $TX = 1$ for $X \neq \varnothing$. Any other monad is called *non-trivial*. They are characterised by the following

Lemma 2.1. *Let $\mathbb{T} = (T, e, m)$ be a monad on Set. Then the following assertions are equivalent.*

(a) *$\mathbb{T}$ is non-trivial.*
(b) *e is monic.*
(c) *T is faithful.*

Proof. See [Law63, Man76]. □

Examples 2.2. We list some monads on Set which will be considered in this paper.

(a) The *identity monad* $\mathbb{1} = (\mathrm{Id}, 1, 1)$. Trivially, the identity functor $\mathrm{Id} : \mathsf{Set} \to \mathsf{Set}$ together with the identity transformation $1 : \mathrm{Id} \to \mathrm{Id}$ forms a monad. It is the initial monad, for every monad $\mathbb{T} = (T, e, m)$ the unit e is the unique monad morphism $\mathbb{1} \to \mathbb{T}$. In particular, every non-trivial monad $\mathbb{T}$ has $\mathbb{1}$ as a submonad.
(b) The *word monad* $\mathbb{L} = (L, e, m)$. The word functor $L : \mathsf{Set} \to \mathsf{Set}$ sends each set X to the set LX of all finite words $(x_1, \dots, x_n)$ ($n \in \mathbb{N}$) of elements of X. Note that this includes the empty word $()$. For each function $f : X \to Y$, $Lf : LX \to LY$ sends $(x_1, \dots, x_n)$ to $(f(x_1), \dots, f(x_n))$. The X-component of the natural transformation $e : \mathrm{Id} \to L$ is given by $e_X : X \to LX, x \mapsto (x)$. An element of LLX is a word of words of X, by removing inner brackets we obtain an element of LX. This defines the X-component $m_X : LLX \to LX$ of $m : LL \to L$.
(c) The *(contravariant) double powerset monad* $\mathbb{P}^{-2} = (P^{-2}, e, m)$. Here $P^{-2}X = PPX$ for every set X and $P^{-2}f : P^{-2}X \to P^{-2}Y, \mathcal{A} \mapsto \{B \subseteq Y \mid f^{-1}[B] \in \mathcal{A}\}$ for every map $f : X \to Y$. The natural transformations $e : \mathrm{Id} \to P^{-2}$ and $m : P^{-2}P^{-2} \to P^{-2}$ are

given by

$$e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\} \qquad \text{and} \qquad m_X(\mathfrak{X}) = \{A \subseteq X \mid A^\# \in \mathfrak{X}\},$$

for all sets X, $\mathfrak{X} \in P^{-2}P^{-2}X$ and $x \in X$, where $A^\# = \{\mathfrak{a} \in P^{-2}X \mid A \in \mathfrak{a}\}$. Important submonads of $\mathbb{P}^{-2}$ are

(i) The *filter monad* $\mathbb{F} = (F, e, m)$. The filter functor $F : \mathsf{Set} \to \mathsf{Set}$ is the subfunctor of P^{-2} that assigns to each set X the set FX of all (possibly improper) filters on X. It is now easy to see that the monad structure of $\mathbb{P}^{-2}$ can be restricted to F. We remark that $Ff(\mathfrak{f}) = \Uparrow\{f[A] \mid A \in \mathfrak{f}\}$.

(ii) The *proper filter monad* $\mathbb{F}^+ = (F^+, e, m)$. The proper filter monad is defined as above except that we replace filter by proper filter everywhere.

(iii) The *ultrafilter monad* $\mathbb{U} = (U, e, m)$. The ultrafilter monad is defined as the filter monad except that we replace filter by ultrafilter everywhere.

Example 2.3. Given a quantale V, we define the V-*powerset* functor $P_\mathsf{V} : \mathsf{Set} \to \mathsf{Set}$ by putting $P_\mathsf{V}(X) = \mathsf{V}^X$ and, for $f : X \to Y$ and $\varphi \in \mathsf{V}^X$,

$$P_\mathsf{V}(f)(\varphi)(y) = \bigvee_{x \in f^{-1}(y)} \varphi(x).$$

We may interpret $\varphi \in \mathsf{V}^X$ as a V-matrix $\varphi : 1 \relbar\joinrel\mapstochar\joinrel\rightarrow X$, and then write $P_\mathsf{V}(f)(\varphi) = f \cdot \varphi$. The functor P_V is actually part of a monad (P_V, e, m) where $e_X = \ulcorner \Delta_X \urcorner : X \to \mathsf{V}^X$ and $m_X : P_\mathsf{V}P_\mathsf{V}(X) \to P_\mathsf{V}(X)$ is defined by $m_X(\Phi) = \varepsilon \cdot \Phi$, where the V-matrix $\varepsilon : P_\mathsf{V}X \times X \to \mathsf{V}$ is given by the evaluation map $\varepsilon(\varphi, x) = \varphi(x)$. Elementwise,

$$m_X(\Phi)(x) = \bigvee_{\varphi \in \mathsf{V}^X} \Phi(\varphi) \otimes \varphi(x)$$

for each $\Phi \in P_\mathsf{V}P_\mathsf{V}(X)$ and $x \in X$. Of course, in case $\mathsf{V} = 2$ we obtain the usual powerset monad.

Let $\mathbb{T} = (T, e, m)$ be a monad on C. A $\mathbb{T}$-*algebra* (also called *Eilenberg-Moore algebra*) is a pair (X, α) consisting of a C-object X and a C-morphism $\alpha : TX \to X$ making the diagrams

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, equal, "1_X"'] & TX \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad\qquad \begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "T\alpha"'] & TX \arrow[d, "\alpha"] \\ TX \arrow[r, "\alpha"'] & X \end{tikzcd}$$

commutative. Given $\mathbb{T}$-algebras (X, α) and (Y, β), a C-morphism $f : X \to Y$ is a $\mathbb{T}$-*algebra homomorphism* if the diagram

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TY \arrow[d, "\beta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. The category of $\mathbb{T}$-algebras and $\mathbb{T}$-algebra homomorphisms is denoted by $\mathsf{C}^\mathbb{T}$. Every monad morphism $j : \mathbb{T} \to \mathbb{T}'$ induces a functor

$$\mathsf{C}^j : \mathsf{C}^{\mathbb{T}'} \to \mathsf{C}^\mathbb{T}, (X, \alpha) \mapsto (X, \alpha \cdot j_X),$$

and we have $\mathsf{C}^{j'\cdot j} = \mathsf{C}^{j'} \cdot \mathsf{C}^{j}$. In particular there is a canonical forgetful functor $\mathsf{C}^e : \mathsf{C}^{\mathbb{T}} \to \mathsf{C}$ sending a $\mathbb{T}$-algebra (X, α) to its underlying C-object X.

The notion of a taut monad was independently introduced by A. Möbus [Möb81] and E. Manes [Man02]. We sketch here some important results and refer to [Man02] for further details. Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor which preserves monomorphisms. Note that this property comes for free when T is part of a monad $\mathbb{T} = (T, e, m)$ (see [Man02, 1.10]). Given $\mathfrak{x} \in TX$ and a subset $i : A \hookrightarrow X$, we write $\mathfrak{x} \in TA$ whenever $\mathfrak{x}$ is in the image of Ti, and call A a *support* of $\mathfrak{x}$. Following [Man02], we denote the set of all supports of $\mathfrak{x} \in TX$ by $\mathrm{supp}_X(\mathfrak{x})$.

We address now the question when $\mathrm{supp} = (\mathrm{supp}_X)_X$ is a natural transformation. Let $f : X \to Y$ be a function and $\mathfrak{x} \in TX$. $P^{-2}f \cdot \mathrm{supp}_X(\mathfrak{x}) = \mathrm{supp}_Y \cdot Tf(\mathfrak{x})$ translates to $Tf(\mathfrak{x}) \in TB \iff \mathfrak{x} \in T(f^{-1}[B])$ for every $B \subseteq Y$. This in turn is equivalent to T preserves inverse images. Generally, a functor T is called *taut* [Man02] when it preserves pullbacks of monomorphisms along arbitrary maps. Using the fact that $m : A \to Y$ is a monomorphism if and only if

(2)
$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & Y \end{tikzcd}$$

is a pullback square, we see that a taut functor preserves monomorphisms. From that follows that the composite of taut functors is again taut. With the exception of the double powerset functor, all functors of Examples 2.2 are taut. Every taut functor T preserves in particular intersections $A \cap B$, hence $\mathrm{supp}_X(\mathfrak{x})$ is a filter and $\mathrm{supp} : T \to F$ a natural transformation from T to the filter functor F. Directly from the definition of supp we deduce that the naturality square

$$\begin{tikzcd} TA \arrow[r, "\mathrm{supp}_A"] \arrow[d, tail, "Ti"'] & FA \arrow[d, tail, "Fi"] \\ TX \arrow[r, "\mathrm{supp}_X"'] & FX \end{tikzcd}$$

induced by a monomorphism $i : A \rightarrowtail X$ is a pullback. In general, a natural transformation $j : T \to T'$ with this property is called *taut*. It is easy to see that, for natural transformations $j : T \to T'$ and $j' : T' \to T''$, $j' \cdot j$ is taut provided that j and j' are, and j is taut provided that $j' \cdot j$ and j' are. Every taut natural transformation $j : T \to T'$ reflects tautness, that is, with T' also T is taut. We conclude that *a functor $T : \mathsf{Set} \to \mathsf{Set}$ is taut if and only if there exists a taut natural transformation $T \to F$ to the filter functor F.* Moreover, for each natural transformation $\eta : T \to F$ we have

$$A \in \mathrm{supp}_X(\mathfrak{x}) \iff \mathfrak{x} \in TA \Rightarrow A \in \eta_X(\mathfrak{x})$$

for each set X, $A \subseteq X$ and $\mathfrak{x} \in TX$, where the latter implication is an equivalence provided that η is taut. In other words, supp *is the smallest (w.r.t. the inclusion order of filters) natural transformation and the unique taut natural transformation from a taut functor to the filter functor.* This can be seen as a generalisation of a result of Börger [Bör87] which states that for each finite coproduct preserving functor $T : \mathsf{Set} \to \mathsf{Set}$ there exists a unique natural transformation $\kappa : T \to U$ from T to the ultrafilter functor U. In fact, extensivity of Set

(see [CLW93]) implies that each finite coproduct preserving functor $T : \mathsf{Set} \to \mathsf{Set}$ (and each natural transformation $T \to U$) is taut. Moreover, $\mathrm{supp}_X(\mathfrak{x})$ is an ultrafilter, for each set X and $\mathfrak{x} \in TX$, since T preserves coproducts.

A monad $\mathbb{T} = (T, e, m)$ is called *taut* if T, e and m are taut. Looking again at Examples 2.2, we have that all except the double powerset monad $\mathbb{P}^{-2}$ are taut, in particular the filter monad (F, e, m) is taut. On the other hand, both trivial monads are not taut. Let (T, η, μ) be any monad with a taut functor T, then $e : \mathrm{Id} \to F$ and $m \cdot \mathrm{supp}^2 : T^2 \to F$ are taut and hence

$$e \subseteq \mathrm{supp} \cdot \eta \qquad \text{and} \qquad m \cdot \mathrm{supp}^2 \subseteq \mathrm{supp} \cdot \mu,$$

with equality if and only if η respectively μ are taut. We conclude that *a monad* $\mathbb{T}$ *on* Set *is taut if and only if* $\mathrm{supp} : \mathbb{T} \to \mathbb{F}$ *is a taut monad morphism* ([Man02, Theorem 3.3]). We remark that tautness of η follows from tautness of T ([Man02, Proposition 2.3]) provided that $\mathbb{T}$ is non-trivial.

Let now V be a (ccd)-lattice. We define maps

$$\xi_{\mathsf{V}} : F\mathsf{V} \to \mathsf{V}, \;\; \mathfrak{f} \mapsto \bigvee_{A \in \mathfrak{f}} \bigwedge_{v \in A} v = \bigvee \{v \in \mathsf{V} \mid \uparrow v \in \mathfrak{f}\}$$

and

$$\zeta_{\mathsf{V}} : F\mathsf{V} \to \mathsf{V}, \;\; \mathfrak{f} \mapsto \bigwedge_{A \in \mathfrak{f}} \bigvee_{v \in A} v = \bigwedge \{v \in \mathsf{V} \mid \downarrow v \in \mathfrak{f}\}.$$

It is easy to see that ξ is order preserving and ζ is order reversing, and both are Eilenberg-Moore structures on V. Moreover, for each ultrafilter $\mathfrak{x}$ we have $\xi_{\mathsf{V}}(\mathfrak{x}) = \zeta_{\mathsf{V}}(\mathfrak{x})$. Let now $\mathbb{T} = (T, e, m)$ be a monad with T taut. By composing with $\mathrm{supp}_{\mathsf{V}}$ we obtain maps

$$\xi_{\mathsf{V}} : T\mathsf{V} \to \mathsf{V}, \;\; \mathfrak{x} \mapsto \bigvee \{v \in \mathsf{V} \mid \mathfrak{x} \in T(\uparrow v)\} \tag{3}$$

and

$$\zeta_{\mathsf{V}} : T\mathsf{V} \to \mathsf{V}, \;\; \mathfrak{x} \mapsto \bigwedge \{v \in \mathsf{V} \mid \mathfrak{x} \in T(\downarrow v)\}$$

satisfying

$$\xi_{\mathsf{V}} \cdot e_{\mathsf{V}} \geq 1_{\mathsf{V}} \geq \zeta_{\mathsf{V}} \cdot e_{\mathsf{V}}, \qquad \xi_{\mathsf{V}} \cdot T\xi_{\mathsf{V}} \leq \xi_{\mathsf{V}} \cdot m_{\mathsf{V}}, \qquad \zeta_{\mathsf{V}} \cdot T\zeta_{\mathsf{V}} \geq \zeta_{\mathsf{V}} \cdot m_{\mathsf{V}}. \tag{4}$$

We have even equality provided that e respectively m is taut.

Example 2.4. In [Bar70] Barr devices an extension of a Set-functor $T : \mathsf{Set} \to \mathsf{Set}$ to Rel by putting $\hat{T}(r) = Tq \cdot (Tp)^\circ$, where $r : X \rightarrowtail\!\!\!\!\!\!\rightarrow Y$ is a relation with projection maps $p : G_r \to X$ and $q : G_r \to Y$ (see Example 1.1). We have $\hat{T}(r^\circ) = (\hat{T}r)^\circ$ and $\hat{T}(s \cdot r) \geq \hat{T}s \cdot \hat{T}r$ with equality whenever s is a function. If T is taut, then we have also equality if r is an injective function. This extension can be described in an interesting alternative way using the map $\xi_2 : T2 \to 2$ (where $\xi_2(\mathfrak{x}) = \text{true} \iff \mathfrak{x} \in T(\text{true})$) provided that T is taut. Considering a relation $r : X \rightarrowtail\!\!\!\!\!\!\rightarrow Y$ as a function $r : X \times Y \to 2$, then $\hat{T}r(\mathfrak{x}, \mathfrak{y}) = \text{true}$ if and only if there exists some $\mathfrak{w} \in T(X \times Y)$ with $T\pi_X(\mathfrak{w}) = \mathfrak{x}$, $T\pi_Y(\mathfrak{w}) = \mathfrak{y}$ and $\xi_2 \cdot Tr(\mathfrak{w}) = \text{true}$.

Lemma 2.5. *Let* $T : \mathsf{Set} \to \mathsf{Set}$ *be a taut functor and* $e : \mathrm{Id} \to T$ *be a natural transformation. Let* X *and* Y *be sets and* $\mathfrak{x} \in TX$ *and* $y \in Y$*. Then there exists an element* $\mathfrak{w} \in T(X \times Y)$ *with* $T\pi_X(\mathfrak{w}) = \mathfrak{x}$ *and* $T\pi_Y(\mathfrak{w}) = e_Y(y)$ *if and only if* $T!(\mathfrak{x}) = e_1(\star)$*. Moreover, such* $\mathfrak{w}$*, if exists, is unique and given by* $\mathfrak{w} = T\langle 1_X, y\rangle(\mathfrak{x})$*.*

Here $! : X \to 1$ is the unique map into the terminal set and we identify $y \in Y$ with the map $1 \to Y, \star \mapsto y$.

Proof. If $T!(\mathfrak{x}) = e_1(\star)$, then $\mathfrak{w} = T\langle 1_X, y\rangle(\mathfrak{x})$ clearly satisfies $T\pi_X(\mathfrak{w}) = \mathfrak{x}$ and $T\pi_Y(\mathfrak{w}) = e_Y(y)$. Assume now that there is some $\mathfrak{w} \in T(X \times Y)$ with $T\pi_X(\mathfrak{w}) = \mathfrak{x}$ and $T\pi_Y(\mathfrak{w}) = e_Y(y)$. Since

$$\begin{tikzcd} X \arrow[r, "\langle 1_X, y\rangle"] \arrow[d, "!"'] & X \times Y \arrow[d, "\pi_Y"] \\ 1 \arrow[r, "y"'] & Y \end{tikzcd}$$

is a pullback square, tautness of T guarantees the existence of a unique $\mathfrak{x}' \in TX$ with $T!(\mathfrak{x}') = e_1(\star)$ and $T\langle 1_X, y\rangle(\mathfrak{x}') = \mathfrak{w}$. From $T\pi_X(\mathfrak{w}) = \mathfrak{x}$ we deduce $\mathfrak{x}' = \mathfrak{x}$. □

So far we have only considered preservation of pullbacks of monomorphisms (along arbitrary maps), further study requires also compatibility of T with arbitrary pullbacks. We say that a commutative square

$$\begin{tikzcd} P \arrow[r, "l"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \tag{5}$$

in Set satisfies the *Beck-Chevalley Condition* (BC) if $f^\circ \cdot g = l \cdot h^\circ$ in Rel. Such a diagram we call for short a (BC)-diagram. Note that the inequality $f^\circ \cdot g \geq l \cdot h^\circ$ follows already from the commutativity of (5). Hence the Beck-Chevalley condition requires, for each $x \in X$ and $y \in Y$ with $f(x) = g(y)$, the existence of a (not necessarily unique) $u \in P$ with $l(u) = x$ and $h(u) = y$. This in turn is equivalent to the surjectivity of the canonical map $\mathrm{can} : P \to X \times_Y Z$ into the pullback. From that follows immediately that (5) is a pullback square if and only if it satisfies (BC) and (h, l) are jointly monic. The following facts will be useful in the sequel.

Lemma 2.6. *Consider the commutative diagrams*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "[1]"] & B \arrow[r, "l"] \arrow[d, "h"'] \arrow[dr, phantom, "[2]"] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

in Set. *Then the following statements hold.*

(a) *If [1] and [2] satisfy (BC), then so does the outer square [1,2]).*
(b) *If the outer square [1,2] satisfies (BC) and (h, l) are jointly monic, then [1] satisfies (BC).*

A functor $T : \mathsf{Set} \to \mathsf{Set}$ satisfies the *Beck-Chevalley condition* (BC) if T sends (BC)-diagrams into (BC)-diagrams. Since every Set-functor preserves surjections, it is enough to consider pullback diagrams. Using again the description (2) of monomorphisms via pullbacks, we see that every functor which satisfies (BC) preserves monomorphisms and is therefore taut.

For later use we record the following consequence of the lemma above.

Corollary 2.7. *Assume that $T : \mathsf{Set} \to \mathsf{Set}$ satisfies (BC) and let $f : A \to X$ and $g : B \to Y$. Then*

$$\begin{tikzcd} T(A \times B) \arrow[r, "T(f\times g)"] \arrow[d, "\mathrm{can}"'] & T(X \times Y) \arrow[d, "\mathrm{can}"] \\ TA \times TB \arrow[r, "Tf\times Tg"'] & TX \times TY \end{tikzcd}$$

satisfies (BC). Here the vertical arrows denote the canonical maps into the product.

Remark 2.8. Of course, tautness of T is sufficient if f and g are monomorphisms.

A natural transformation $j : T \to T'$ satisfies the *Beck-Chevalley condition* (BC) if every naturality square satisfies (BC). We say that a monad $\mathbb{T} = (T, e, m)$ satisfies the *Beck-Chevalley condition* (BC) if T and m satisfy (BC). Note that we do *not* require anything about e. Of course, e is still taut if $\mathbb{T}$ is not trivial, and therefore every non-trivial monad satisfying (BC) is automatically taut. All monads of Examples 2.2 satisfy (BC), with the exception of the double powerset monad.

3. Topological theories

We will now join the two notions – quantale and monad – considered so far and introduce the concept of a topological theory. Besides a monad $\mathbb{T} = (T, e, m)$ and a quantale V it consists of a map $\xi : T\mathsf{V} \to \mathsf{V}$ compatible with both $\mathbb{T}$ and V. Using ξ we are able to extend the Set-functor T to V-Mat, and show that this extension has particularly nice properties if $\xi : T\mathsf{V} \to \mathsf{V}$ is "strictly" compatible with $\mathbb{T}$ and V. Finally, models of such a theory are defined as lax Eilenberg–Moore algebras for this extension.

Definition 3.1. A *topological theory* $\mathscr{T}$ is a triple $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of

(a) a monad $\mathbb{T} = (T, e, m)$,
(b) a quantale[2] $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and
(c) a map $\xi : T\mathsf{V} \to \mathsf{V}$

such that

($\mathrm{M_e}$) $1_{\mathsf{V}} \leq \xi \cdot e_{\mathsf{V}}$,
($\mathrm{M_m}$) $\xi \cdot T\xi \leq \xi \cdot m_{\mathsf{V}}$,
($\mathrm{Q}_\otimes$) $\otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \leq \xi \cdot T(\otimes)$,

$$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd}$$

($\mathrm{Q_k}$) $k \leq \xi \cdot Tk(\mathfrak{x})$ for all $\mathfrak{x} \in T1$, and

$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd}$$

[2]Recall that we assume $k \neq \bot$.

($\mathrm{Q}_{\bigvee}$) for all maps $f : X \to Y$ $\varphi : X \to \mathsf{V}$ and $\psi : Y \to \mathsf{V}$ where $\psi(y) \leq \bigvee_{x \in f^{-1}(y)} \varphi(x)$ for all $y \in Y$, we have

$$\xi \cdot T\psi(\mathfrak{y}) \leq \bigvee_{\mathfrak{x} \in Tf^{-1}(\mathfrak{y})} \xi \cdot T\varphi(\mathfrak{x})$$

for all $\mathfrak{y} \in TY$.

We say that $\mathscr{T}$ satisfies ($\mathrm{M_e}$), ($\mathrm{M_m}$), ($\mathrm{Q}_\otimes$) or ($\mathrm{Q_k}$) strictly, and write ($\mathrm{M}_{\mathrm{e}}^{=}$), ($\mathrm{M}_{m}^{=}$), ($\mathrm{Q}_{\otimes}^{=}$) and ($\mathrm{Q}_{k}^{=}$) respectively, if we have equality instead of "$\leq$". In case $\mathscr{T}$ satisfies all four axioms strictly we call $\mathscr{T}$ a *strict topological theory.*

Note that we have automatically ($\mathrm{Q}_{\otimes}^{=}$) in case $\otimes = \wedge$ and ($\mathrm{Q}_{k}^{=}$) if k is terminal in V. Condition ($\mathrm{Q}_{\bigvee}$) can be expressed more elegantly using the functor P_{V} of Example 2.3. First note that, for each set X, ξ induces a map

$$\xi_X : P_{\mathsf{V}}(X) \to P_{\mathsf{V}}T(X), \quad \varphi \mapsto \xi \cdot T\varphi.$$

Now ($\mathrm{Q}_{\bigvee}$) implies that ξ_X is order-preserving, for each set X. Moreover, supposing that ξ_Y preserves the order, it is enough to consider "=" in ($\mathrm{Q}_{\bigvee}$) which in turn means precisely that the diagram

$$\begin{tikzcd} P_{\mathsf{V}}(X) \arrow[r, "P_{\mathsf{V}}(f)"] \arrow[d, "\xi_X"'] & P_{\mathsf{V}}(Y) \arrow[d, "\xi_Y"] \\ P_{\mathsf{V}}T(X) \arrow[r, "P_{\mathsf{V}}T(f)"'] & P_{\mathsf{V}}TY \end{tikzcd}$$

commutes. Hence, considering P_{V} as a functor $P_{\mathsf{V}} : \mathsf{Set} \to \mathsf{Ord}$, ($\mathrm{Q}_{\bigvee}$) is equivalent to

($\mathrm{Q}'_{\bigvee}$) $\quad (\xi_X)_X : P_{\mathsf{V}} \to P_{\mathsf{V}}T$ is a natural transformation.

Our next lemma shows that ξ must be also compatible with the right adjoint hom.

Lemma 3.2. *A topological theory $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ satisfies also*

($\mathrm{Q_{hom}}$) $\quad \xi \cdot T(\mathrm{hom}) \leq \mathrm{hom} \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle$.

$$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\mathrm{hom})"] \arrow[d, "\langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\mathrm{hom}"'] & \mathsf{V} \end{tikzcd}$$

Proof. Let $\mathfrak{w} \in T(\mathsf{V} \times \mathsf{V})$, we have to show that

$$\xi \cdot T\pi_1(\mathfrak{w}) \otimes \xi \cdot T\,\mathrm{hom}(\mathfrak{w}) \leq \xi \cdot T\pi_2(\mathfrak{w}).$$

To this end, consider $\langle \pi_1, \mathrm{hom} \rangle : \mathsf{V} \times \mathsf{V} \to \mathsf{V} \times \mathsf{V}$ and put $\tilde{\mathfrak{w}} = T\langle \pi_1, \mathrm{hom} \rangle(\mathfrak{w})$. Note that

$$\otimes \cdot \langle \pi_1, \mathrm{hom} \rangle \leq \pi_2 \qquad \text{and} \qquad \pi_2 \cdot \langle \pi_1, \mathrm{hom} \rangle = \mathrm{hom}\,.$$

From ($\mathrm{Q}_{\bigvee}$) we deduce

$$\xi \cdot T\otimes(\tilde{\mathfrak{w}}) \leq \xi \cdot T\pi_2(\mathfrak{w}), \qquad \xi \cdot T\pi_2(\tilde{\mathfrak{w}}) = \xi \cdot T\,\mathrm{hom}(\mathfrak{w}), \qquad \xi \cdot T\pi_1(\tilde{\mathfrak{w}}) = \xi \cdot T\pi_1(\mathfrak{w})$$

and ($\mathrm{Q}_{\otimes}$) implies $\xi \cdot T\pi_1(\mathfrak{w}) \otimes \xi \cdot T\,\mathrm{hom}(\mathfrak{w}) \leq \xi \cdot T\pi_2(\mathfrak{w})$. □

The following theorem provides us with some examples of topological theories.

Theorem 3.3. *The following statements hold.*

(a) $(\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$ *is a strict topological theory for each quantale* V.

(b) *Let* $\mathbb{T} = (T, e, m)$ *be a monad where* T *is taut and let* V *be a (ccd)-quantale. Then* $(\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ *is a topological theory where* $\xi_{\mathsf{V}} : T\mathsf{V} \to \mathsf{V}$ *is the canonical map as defined in* (3) *(Section 2).*

(c) $(\mathbb{L}, \mathsf{V}, \xi_{\otimes})$ *is a strict topological theory for each quantale* V*, where*

$$\begin{aligned} \xi_{\otimes} : L\mathsf{V} &\to \mathsf{V}. \\ (v_1, \ldots, v_n) &\mapsto v_1 \otimes \ldots \otimes v_n \\ () &\mapsto k \end{aligned}$$

Proof. (a) Obvious.

(b) Let $\mathbb{T} = (T, e, m)$ be a monad where T is taut and let V be a constructively completely distributive quantale V. It was already observed in (4) that $(\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ satisfies $(\mathrm{M}_{\mathrm{e}}^{=})$ and $(\mathrm{M}_{\mathrm{m}})$. In order to see $(\mathrm{Q}_{\otimes})$, let $\mathfrak{w} \in T(\mathsf{V} \times \mathsf{V})$ and put $\mathfrak{x} = T\pi_1(\mathfrak{w})$ and $\mathfrak{y} = T\pi_2(\mathfrak{w})$. Let $u, v \in \mathsf{V}$ such that $\mathfrak{x} \in T(\uparrow u)$ and $\mathfrak{y} \in T(\uparrow v)$. By Corollary 2.7,

$$\begin{tikzcd} T(\uparrow u \times \uparrow v) \arrow[r, tail] \arrow[d, "\mathrm{can}"'] & T(\mathsf{V} \times \mathsf{V}) \arrow[d, "\mathrm{can}"] \\ T(\uparrow u) \times T(\uparrow v) \arrow[r, tail] & T\mathsf{V} \times T\mathsf{V} \end{tikzcd}$$

is a pullback. Therefore $\mathfrak{w} \in T(\uparrow u \times \uparrow v)$ and consequently $T\otimes(\mathfrak{w}) \in T(\uparrow(u \otimes v))$.

To see $(\mathrm{Q}_{\mathrm{k}})$, just note that $\mathfrak{x} \in Tk$ implies $\mathfrak{x} \in T(\uparrow k)$.

Assume now that we have maps $f : X \to Y$, $\varphi : X \to \mathsf{V}$ and $\psi : Y \to \mathsf{V}$ such that, for each $y \in Y$, $\psi(y) \leq \bigvee_{x \in f^{-1}(y)} \varphi(x)$. Let $\mathfrak{y} \in TY$ and $v \in \mathsf{V}$ with $T\psi(\mathfrak{y}) \in T(\uparrow v)$. Then $\mathfrak{y} \in TY_v$ where Y_v is the pullback

$$\begin{tikzcd} Y_v \arrow[r, tail] \arrow[d, "g"'] & Y \arrow[d, "g"] \\ \uparrow v \arrow[r, tail] & \mathsf{V}. \end{tikzcd}$$

Let $u \lll v$. For each $y \in Y_v$ we have $v \leq g(y) \leq \bigvee_{x \in h^{-1}(y)} f(x)$, hence there exists some $x \in h^{-1}(y)$ with $f(x) \geq u$. Therefore the restriction $h^{-1}[Y_v] \cap X_u \to Y_v$ of h is surjective, where X_u is the pullback

$$\begin{tikzcd} X_u \arrow[r, tail] \arrow[d, "f"'] & X \arrow[d, "f"] \\ \uparrow u \arrow[r, tail] & \mathsf{V}. \end{tikzcd}$$

We conclude that there exists some $\mathfrak{x} \in TX$ with $Tf(\mathfrak{x}) = \mathfrak{y}$ and $\xi \cdot T\varphi(\mathfrak{x}) \geq u$.

(c) Let V be a quantale. Then $\xi_{\otimes}$ is a $\mathbb{L}$-algebra structure since $\otimes$ is associative and k is a unit for $\otimes$. Axiom $(\mathrm{Q}_{\otimes}^{=})$ follows from commutativity of $\otimes$, $(\mathrm{Q}_{k}^{=})$ from $k \otimes k = k$ and $(\mathrm{Q}_{\bigvee})$ from the (componentwise) preservation of $\bigvee$ by $\otimes$. $\square$

In the sequel we will denote the identity theory $(\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$ by $\mathscr{I}_{\mathsf{V}}$, the "word theory" $(\mathbb{L}, \mathsf{V}, \xi_{\otimes})$ by $\mathscr{L}_{\mathsf{V}}^{\otimes}$ and the "ultrafilter theory" $(\mathbb{U}, \mathsf{V}, \xi_{\mathsf{V}})$ by $\mathscr{U}_{\mathsf{V}}$. In general, we write shortly $\mathscr{T}_{\mathsf{V}}$ for a theory of the form $(\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ as in (b).

We are now turning to the semantic side and going to introduce models of a topological theory. This requires in the first place an extension of the Set-functor T to $\mathsf{V}\text{-}\mathsf{Mat}$. Motivated by the observation made in Example 2.4, we define

Definition 3.4. Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory. We extend the Set-functor T to $\mathsf{V}\text{-}\mathsf{Mat}$ by putting $T_\xi X = TX$ for each set X and

$$\begin{aligned} T_\xi r : TX \times TY &\to \mathsf{V} \\ (\mathfrak{x}, \mathfrak{y}) &\mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\} \end{aligned}$$

for each V-matrix $r : X \times Y \to \mathsf{V}$.

Note that so far we do not claim any functoriality properties of T_ξ. Let $\mathrm{can} = \mathrm{can}_{X,Y} : T(X \times Y) \to TX \times TY$ denote the canonical map into the product. Then $T_\xi(\_)$ can be written as the composite

$$P_\mathsf{V}(X \times Y) \xrightarrow{\xi_{X \times Y}} P_\mathsf{V} T(X \times Y) \xrightarrow{P_\mathsf{V}(\mathrm{can})} P_\mathsf{V}(TX \times TY).$$

In order to make use of the description above, we remark that the composition in $\mathsf{V}\text{-}\mathsf{Mat}$ can be written as

$$P_\mathsf{V}(X \times Y) \times P_\mathsf{V}(Y \times Z) \xrightarrow{\tilde{\otimes}} P_\mathsf{V}(X \times Y \times Z) \xrightarrow{P_\mathsf{V}(\pi)} P_\mathsf{V}(X \times Z),$$

where $\tilde{\otimes}(r, s)$ is given by

$$X \times Y \times Z \xrightarrow{1_X \times \Delta_Y \times 1_Z} X \times Y \times Y \times Z \xrightarrow{r \times s} \mathsf{V} \times \mathsf{V} \xrightarrow{\otimes} \mathsf{V}$$

and $\pi = \pi_{X,Y,Z} : X \times Y \times Z \to X \times Z$ is the projection map.

Theorem 3.5. *Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory. Then the following statements hold.*

(a) *For each V-matrix $r : X \,-\!\!\!\!\!\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\; Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$ (and we write $T_\xi r^\circ$).*

(b) *For each function $f : X \to Y$, $Tf \le T_\xi f$ and $Tf^\circ \le T_\xi f^\circ$.*

(c) *For each V-matrix $r : X \,-\!\!\!\!\!\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\; Y$ and functions $f : A \to X$ and $g : Y \to Z$,*

$$T_\xi(g \cdot r) = Tg \cdot T_\xi r \qquad \text{and} \qquad T_\xi(r \cdot f) \le T_\xi r \cdot Tf.$$

In the latter case we have equality provided that T satisfies (BC) or that T is taut and f is injective.

(d) *For all V-matrices $r : X \,-\!\!\!\!\!\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\; Y$ and $s : Y \,-\!\!\!\!\!\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\; Z$, $T_\xi s \cdot T_\xi r \le T_\xi(s \cdot r)$ provided that T satisfies (BC), and $T_\xi s \cdot T_\xi r \ge T_\xi(s \cdot r)$ provided that $(Q^{=}_{\otimes})$ holds.*

(e) *The natural transformations e and m become op-lax, that is, for every V-matrix $r : X \,-\!\!\!\!\!\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\; Y$ we have the inequalities:*

$$e_Y \cdot r \le T_\xi r \cdot e_X, \qquad\qquad m_Y \cdot T_\xi T_\xi r \le T_\xi r \cdot m_X.$$

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & T_\xi X \arrow[d, "T_\xi r", "\shortmid" marking] \\ Y \arrow[r, "e_Y"'] & T_\xi Y \end{tikzcd} \qquad\qquad \begin{tikzcd} T_\xi T_\xi X \arrow[r, "m_X"] \arrow[d, "T_\xi T_\xi r"', "\shortmid" marking] \arrow[dr, phantom, "\le"] & T_\xi X \arrow[d, "T_\xi r", "\shortmid" marking] \\ T_\xi T_\xi Y \arrow[r, "m_Y"'] & T_\xi Y \end{tikzcd}$$

We have even equality provided that e (resp. m) satisfies (BC) and $(M^{=}_{e})$ (resp. $(M^{=}_{m})$).

Proof. (a) Obvious.
(b) Let $f : X \to Y$ be a map. By definition we have that the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1_X, f\rangle"] \arrow[d, "!"'] & X \times Y \arrow[d, "f"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd}$$

commutes. Applying T to the diagram above and combining it with ($\mathrm{Q_k}$) we see that, for each $\mathfrak{x} \in TX$, there is some $\mathfrak{w} \in T(X \times Y)$ with $T\pi_X(\mathfrak{w}) = \mathfrak{x}$, $T\pi_Y(\mathfrak{w}) = Tf(\mathfrak{x})$ and $\xi \cdot Tf(\mathfrak{w}) \geq k$. This proves $Tf \leq T_\xi f$. The second inequality follows immediately from (a).
(c) Consider first $r : X \nrightarrow Y$ and $g : Y \to Z$. Note that the diagrams

$$\begin{tikzcd} P_{\mathsf{V}}(X \times Y) \arrow[r, "\xi_{X\times Y}"] \arrow[d, "P_{\mathsf{V}}(1_X \times g)"'] & P_{\mathsf{V}}T(X \times Y) \arrow[r, "P_{\mathsf{V}}(\mathrm{can})"] \arrow[d, "P_{\mathsf{V}}T(1_X \times g)"] & P_{\mathsf{V}}(TX \times TY) \arrow[d, "P_{\mathsf{V}}(T1_X \times Tg)"] \\ P_{\mathsf{V}}(X \times Z) \arrow[r, "\xi_{X\times Z}"'] & P_{\mathsf{V}}T(X \times Z) \arrow[r, "P_{\mathsf{V}}(\mathrm{can})"'] & P_{\mathsf{V}}(TX \times TY) \end{tikzcd}$$

commute. Applying the upper path to $r \in P_{\mathsf{V}}(X \times Y)$ gives $Tg \cdot T_\xi r$, whereby applying the lower path gives $T_\xi(g \cdot r)$. Consider now $r : X \nrightarrow Y$ and $f : A \to X$. We obtain $T_\xi(r \cdot f) \leq T_\xi r \cdot Tf$ from the commutativity of the diagrams

$$\begin{tikzcd} A \times X \arrow[r, "f\times 1_Y"] \arrow[d, "r\cdot f"'] & X \times Y \arrow[dl, "r"] \\ \mathsf{V} & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T(A \times X) \arrow[r, "T(f\times 1_Y)"] \arrow[d, "\mathrm{can}"'] & T(X \times Y) \arrow[d, "\mathrm{can}"] \\ TA \times TX \arrow[r, "Tf\times T1_Y"'] & TX \times TY. \end{tikzcd}$$

We have actually equality provided that the right hand side diagram satisfies (BC), this proves the second part of the statement (see Corollary 2.7).
(d) We consider

$$\begin{tikzcd} P_{\mathsf{V}}(X\times Y)\times P_{\mathsf{V}}(Y\times Z) \arrow[r, "\xi_{X\times Y}\times\xi_{Y\times Z}"] \arrow[d, "\hat{\otimes}"'] \arrow[dr, phantom, "[1]"] & P_{\mathsf{V}}T(X\times Y)\times P_{\mathsf{V}}T(Y\times Z) \arrow[r, "P_{\mathsf{V}}\mathrm{can}\times P_{\mathsf{V}}\mathrm{can}"] \arrow[d, "\hat{\otimes}"] \arrow[dr, phantom, "[2]"] & P_{\mathsf{V}}(TX\times TY)\times P_{\mathsf{V}}(TY\times TZ) \arrow[d, "\hat{\otimes}"] \\ P_{\mathsf{V}}(X \times Y \times Z) \arrow[r, "\xi_{X\times Y\times Z}"'] \arrow[d, "P_{\mathsf{V}}(\pi)"'] \arrow[dr, phantom, "[3]"] & P_{\mathsf{V}}T(X \times Y \times Z) \arrow[r, "P_{\mathsf{V}}\,\mathrm{can}"'] \arrow[d, "P_{\mathsf{V}}T(\pi)"] \arrow[dr, phantom, "[4]"] & P_{\mathsf{V}}(TX \times TY \times TZ) \arrow[d, "P_{\mathsf{V}}(\pi)"] \\ P_{\mathsf{V}}(X \times Z) \arrow[r, "\xi_{X\times Z}"'] & P_{\mathsf{V}}T(X \times Z) \arrow[r, "P_{\mathsf{V}}\,\mathrm{can}"'] & P_{\mathsf{V}}(TX \times TZ), \end{tikzcd}$$

where $\hat{\otimes}$ sends $(a, b) \in P_{\mathsf{V}}T(X\times Y)\times P_{\mathsf{V}}T(Y\times Z)$ to

$$T(X \times Y \times Z) \to T(X \times Y) \times T(Y \times Z) \xrightarrow{a\times b} \mathsf{V} \times \mathsf{V} \xrightarrow{\otimes} \mathsf{V}. \tag{6}$$

Note that the lower path represents $T_\xi(\_ \cdot \_)$, whereby the upper path represents $T_\xi(\_) \cdot T_\xi(\_)$. The diagram [3] commutes since $(\xi_\_)$ is a natural transformation, and [4] commutes since the underlying diagram (without P_{V}) does so. Moreover, we remark that all maps in [3] and [4]

are order-preserving. We consider now diagram [2]. Let $a \in P_{\mathsf{V}}T(X \times Y)$ and $b \in P_{\mathsf{V}}T(Y \times Z)$. We put $\tilde{a} = P_{\mathsf{V}}\mathrm{can}(a)$ and $\tilde{b} = P_{\mathsf{V}}\mathrm{can}(b)$. Applying the upper path of [2] to (a, b) gives

$$TX \times TY \times TZ \xrightarrow{1_{TX} \times \Delta_{TY} \times 1_{TZ}} TX \times TY \times TY \times TZ \xrightarrow{\tilde{a} \times \tilde{b}} \mathsf{V} \times \mathsf{V} \xrightarrow{\otimes} \mathsf{V},$$

and the lower path gives $P_{\mathsf{V}}\mathrm{can}$ applied to (6). Since the diagram

$$\begin{tikzcd} T(X \times Y \times Z) \arrow[r] \arrow[d, "\mathrm{can}"'] & T(X \times Y) \times T(Y \times Z) \arrow[d] \\ TX \times TY \times TZ \arrow[r, "1_{TX} \times \Delta_{TY} \times 1_{TZ}"'] & TX \times TY \times TY \times TZ \end{tikzcd} \tag{7}$$

commutes we have "lower path $\leq$ upper path" in [2]. If in addition T satisfies (BC), then by applying T to the pullback diagram

$$\begin{tikzcd} X \times Y \times Z \arrow[r] \arrow[d] & Y \times Z \arrow[d] \\ X \times Y \arrow[r] & Y. \end{tikzcd}$$

we see that (7) satisfies (BC) as well and therefore [2] commutes. Finally, consider $r \in P_{\mathsf{V}}(X \times Y)$ and $s \in P_{\mathsf{V}}(Y \times Z)$. The upper respectively lower path in the following diagram corresponds to applying the lower respectively upper path of [1] to (r, s):

$$\begin{tikzcd} T(X \times Y \times Z) \arrow[r, "T(1_X \times \Delta_Y \times 1_Z)"] \arrow[dr] & T(X \times Y \times Y \times Z) \arrow[r, "T(r \times s)"] \arrow[d] & T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T\otimes"] \arrow[d] & T\mathsf{V} \arrow[dd, "\xi"] \\ & T(X \times Y) \times T(Y \times Z) \arrow[r, "Tr \times Ts"'] & T\mathsf{V} \times T\mathsf{V} \arrow[d, "\xi \times \xi"'] & \\ & & \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V}. \end{tikzcd}$$

The left hand side and the middle diagram commute, and ($\mathrm{Q}_{\otimes}$) postulates that we have "lower path $\leq$ upper path" on the right hand side, and therefore "upper path $\leq$ lower path" in [1]. If $\mathscr{T}$ satisfies even ($\mathrm{Q}_{\otimes}^{=}$), then [1] commutes.

(e) Let $r : X \times Y \to \mathsf{V}$ be a V-matrix. From

$$r(x, y) \leq \xi \cdot e_{\mathsf{V}} \cdot r(x, y) = \xi \cdot Tr \cdot e_{X \times Y}(x, y) \leq T_{\xi} r(e_X(x), e_Y(y))$$

follows the first inequality. Assume now that e satisfies (BC) and $1_{\mathsf{V}} = \xi \cdot e_{\mathsf{V}}$. Let $x \in X$, $\mathfrak{y} \in TY$ and $\mathfrak{w} \in T(X \times Y)$ with $T\pi_X(\mathfrak{w}) = e_X(x)$ and $T\pi_Y(\mathfrak{w}) = \mathfrak{y}$. Applying (BC) to the naturality square

$$\begin{tikzcd} X \times Y \arrow[r, "e_{X \times Y}"] \arrow[d, "\pi_X"'] & T(X \times Y) \arrow[d, "T\pi_X"] \\ X \arrow[r, "e_X"'] & TX \end{tikzcd}$$

yields $\mathfrak{w} = e_{X \times Y}(x, y)$ for some $y \in Y$. Hence $\mathfrak{y} = e_Y(y)$ and

$$\xi \cdot Tr(\mathfrak{w}) = \xi \cdot Tr \cdot e_{X \times Y}(x, y) = \xi \cdot e_{\mathsf{V}} \cdot r(x, y) = r(x, y).$$

To conclude the second inequality, we prove that

$$T_\xi T_\xi r(\mathfrak{X},\mathfrak{Y}) \leq T_\xi r(m_X(\mathfrak{X}), m_Y(\mathfrak{Y}))$$

for all $\mathfrak{X} \in T_\xi T_\xi X$ and $\mathfrak{Y} \in T_\xi T_\xi Y$. Recall that

$$T_\xi T_\xi r(\mathfrak{X},\mathfrak{Y}) = \bigvee\{\xi \cdot T(T_\xi r)(\mathfrak{P}) \mid \mathfrak{P} \in T(TX \times TY), T\pi_{TX}(\mathfrak{P}) = \mathfrak{X}, T\pi_{TY}(\mathfrak{P}) = \mathfrak{Y}\}.$$

Let $\mathfrak{P} \in T(TX \times TY)$ which projects to $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. Since the top square of

$$\begin{tikzcd}
TT(X \times Y) \arrow[r, "m_{X\times Y}"] \arrow[d, "TTr"'] & T(X \times Y) \arrow[d, "Tr"] \\
TT(\mathsf{V}) \arrow[r, "m_{\mathsf{V}}"] \arrow[d, "T\xi"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\
T\mathsf{V} \arrow[r, "\xi"'] & \mathsf{V}
\end{tikzcd}$$

commutes, it is enough to show

$$\bigvee_{\substack{\mathfrak{W} \in TT(X\times Y) \\ T\mathrm{can}(\mathfrak{W}) = \mathfrak{P}}} \xi \cdot T\xi \cdot TTr(\mathfrak{W}) \geq \xi \cdot T(T_\xi r)(\mathfrak{P})$$

which follows directly from ($\mathrm{Q}_\bigvee$). Assume now that m satisfies (BC) and $\xi \cdot T\xi = \xi \cdot m_{\mathsf{V}}$. Let $\mathfrak{X} \in TTX$, $\mathfrak{y} \in TY$ and $\mathfrak{p} \in T(X \times Y)$ with $T\pi_X(\mathfrak{p}) = m_X(\mathfrak{X})$ and $T\pi_X(\mathfrak{p}) = \mathfrak{y}$. We wish to find some $\mathfrak{P} \in T(TX \times TY)$ such that $T\pi_{TX}(\mathfrak{P}) = \mathfrak{X}$, $m_Y \cdot T\pi_{TY}(\mathfrak{P}) = \mathfrak{y}$ and

$$\xi \cdot Tr(\mathfrak{p}) \leq \xi \cdot T(T_\xi r)(\mathfrak{P}).$$

Since

$$\begin{tikzcd}
TT(X \times Y) \arrow[r, "m_{X\times Y}"] \arrow[d, "TT\pi_X"'] & T(X \times Y) \arrow[d, "T\pi_X"] \\
TTX \arrow[r, "m_X"'] & TX
\end{tikzcd}$$

satisfies (BC), there exists $\mathfrak{W} \in TT(X \times Y)$ such that $TT\pi_X(\mathfrak{W}) = \mathfrak{X}$ and $m_{X\times Y}(\mathfrak{W}) = \mathfrak{p}$ (and therefore $m_Y \cdot TT\pi_Y(\mathfrak{W}) = \mathfrak{y}$). With $\mathfrak{P} = T\mathrm{can}(\mathfrak{W})$ we have

$$T\pi_{TX}(\mathfrak{P}) = T\pi_{TX} \cdot T\mathrm{can}(\mathfrak{W}) = TT\pi_X(\mathfrak{W}) = \mathfrak{X}$$

and

$$m_Y \cdot T\pi_{TY}(\mathfrak{P}) = m_Y \cdot TT\pi_Y(\mathfrak{W}) = \mathfrak{y}$$

and, since $\xi \cdot Tr \leq T_\xi r \cdot \mathrm{can}$,

$$\begin{aligned}
\xi \cdot Tr(\mathfrak{p}) &= \xi \cdot Tr \cdot m_{X\times Y}(\mathfrak{W}) = \xi \cdot m_{\mathsf{V}} \cdot TTr(\mathfrak{W}) \\
&= \xi \cdot T\xi \cdot TTr(\mathfrak{W}) \leq \xi \cdot T(T_\xi r) \cdot T\mathrm{can}(\mathfrak{W}) = \xi \cdot T(T_\xi r)(\mathfrak{P}). \quad \square
\end{aligned}$$

Remarks 3.6. The map $\xi : T\mathsf{V} \to \mathsf{V}$ can be recovered from the extension T_ξ as follows. We regard the projection map $\pi_{\mathsf{V}} : 1 \times \mathsf{V} \to \mathsf{V}$ as the V-matrix $\pi_{\mathsf{V}} : 1 \mathrel{\mapstochar\!\!\rightarrow} \mathsf{V}$. Then $T_\xi \pi_{\mathsf{V}} \cdot !^\circ : 1 \mathrel{\mapstochar\!\!\rightarrow} T1 \mathrel{\mapstochar\!\!\rightarrow} T\mathsf{V}$ is essentially ξ since $T_\xi \pi_{\mathsf{V}} \cdot !^\circ(\star, \mathfrak{x}) = \xi(\mathfrak{x})$.

Of course, the extension $\mathrm{Id}_{1_{\mathsf{V}}}$ of the identity functor $\mathrm{Id} : \mathsf{Set} \to \mathsf{Set}$ is the identity on $\mathsf{V}\text{-}\mathsf{Mat}$. Choosing $1_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$ is not the only way to turn the identity monad $\mathbb{1}$ into a topological theory. For instance, for $\mathsf{V} = \{\bot, a, \top\}$ with $\bot < a < \top$ and $\otimes = \min$ we define

$$\xi(x) = \begin{cases} \bot & \text{if } x = \bot, \\ \top & \text{else.} \end{cases}$$

Then $(\mathbb{1}, \mathsf{V}, \xi)$ is a topological theory. The extension Id_ξ is the one considered in [CHT04, Remark 3.2] and, as observed there, Id_ξ is not the identity on $\mathsf{V}\text{-}\mathsf{Mat}$ thought it is the identity on functions. In general, the extension $T_{\mathsf{V}} = T_{\xi_{\mathsf{V}}}$ for $(\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ of Theorem 3.3 (b) is exactly the one obtained in [CH04].

We say that T_ξ *strictly extends* T if $T_\xi f = Tf$ (and hence $T_\xi(f^\circ) = Tf^\circ$) for each Set-map $f : X \to Y$. Note that $T_\xi f = T_\xi(f \cdot 1_X) = Tf \cdot T_\xi 1_X$, hence the extension T_ξ is strict if and only if $T_\xi 1_X = 1_{TX}$ for all X. Let $\Delta_X : X \to X \times X$ be the diagonal map and $k_X = k\cdot ! : X \to \mathsf{V}, x \mapsto k$. Then

$$1_X(x, y) = \bigvee_{\substack{z \in X, \\ \Delta_X(z) = (x,y)}} k$$

for all $(x, y) \in X \times Y$, hence $(\mathrm{Q}_{\bigvee})$ implies that

$$\xi \cdot T(1_X)(\mathfrak{w}) = \bigvee_{\substack{\mathfrak{x} \in TX, \\ T\Delta_X(\mathfrak{x}) = \mathfrak{w}}} \xi \cdot Tk_X(\mathfrak{x})$$

for all $\mathfrak{w} \in T(X \times Y)$. We conclude that $T_\xi 1_X(\mathfrak{x}, \mathfrak{y}) = \bot$ for $\mathfrak{x}, \mathfrak{y} \in TX$ with $\mathfrak{x} \neq \mathfrak{y}$, and that $T_\xi 1_X(\mathfrak{x}, \mathfrak{x}) = \xi \cdot Tk_X(\mathfrak{x}) \geq k$ with equality if we have equality in (Q_k). On the other hand, suppose that T_ξ is a strict extension of T and consider the identity function $1 \to 1$. Its corresponding V-matrix is $k : 1 \cong 1 \times 1 \to \mathsf{V}$. Since T_ξ is a strict extension of T, for each $\mathfrak{x} \in T1$ we have $k = T_\xi k(\mathfrak{x}, \mathfrak{x}) = \xi \cdot Tk(\mathfrak{x})$. We have seen that

Proposition 3.7. *Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory. Then T_ξ is a strict extension of T if and only if $\mathscr{T}$ satisfies $(\mathrm{Q}_k^{=})$.*

We are now ready to define models of topological theories.

Definition 3.8. Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory. A V-matrix $a : TX \mathrel{\mapstochar\!\!\rightarrow} X$ is called *reflexive* if $1_X \leq a \cdot e_X$, it is called *transitive* if $a \cdot T_\xi a \leq a \cdot m_X$. A *$\mathscr{T}$-algebra* is a pair (X, a) consisting of a set X and a reflexive and transitive V-matrix $a : TX \mathrel{\mapstochar\!\!\rightarrow} X$. We say that (X, a) is a *$\mathscr{T}$-graph* if a is only required to be reflexive.

A map $f : X \to Y$ between $\mathscr{T}$-algebras (resp. $\mathscr{T}$-graphs) (X, a) and (Y, b) is a *lax homomorphism* if $f \cdot a \leq b \cdot Tf$. The resulting category of $\mathscr{T}$-algebras ($\mathscr{T}$-graphs) and lax homomorphism we denote by $\mathscr{T}\text{-}\mathsf{Alg}$ ($\mathscr{T}\text{-}\mathsf{Graph}$).

Examples 3.9. (a) For each quantale V, $\mathscr{I}_{\mathsf{V}}$-algebras are precisely V-categories and lax homomorphism V-functors. In particular, $\mathscr{I}_2\text{-}\mathsf{Alg}$ is the category Ord of ordered sets and $\mathscr{I}_{\mathsf{P}_+}\text{-}\mathsf{Alg}$ is the category Met of (generalised) metric spaces [Law73].

(b) As already pointed out in [CT03], $\mathscr{L}_{\mathsf{V}}^{\otimes}$-algebras can be described as V-multicategories (see also [Bur71, Her00]).

(c) The main result of [Bar70] states that $\mathscr{U}_2$-Alg is isomorphic to the category Top of topological spaces. In [CH03] it is shown that $\mathscr{U}_{\mathsf{P}_+}$-$\mathsf{Alg}$ is isomorphic to the category App of approach spaces [Low97].

Remark 3.10. It is easy to see that with $f : (X, a) \to (Y, b)$ also $Tf : (TX, T_\xi a) \to (TY, T_\xi b)$ is a lax homomorphism. But note that $(TX, T_\xi a)$ need not be a $\mathscr{T}$-graph even when (X, a) is a $\mathscr{T}$-algebra. On the other hand, with (X, a) also $(TX, T_\xi a)$ is a V-category provided that T satisfies (BC).

The category $\mathscr{T}$-Alg (resp. $\mathscr{T}$-Graph) comes with a canonical forgetful functor to Set which sends (X, a) to X. In both cases this functor can be easily seen to be topological (see [CH03], and [AHS90] for the concept topological functor). In particular we obtain that both $\mathscr{T}$-Alg and $\mathscr{T}$-Graph are complete and cocomplete categories.

4. KLEISLI COMPOSITION AND THEORY MORPHISMS

A $\mathscr{T}$-algebra (X, a) can also be seen as a monoid in the Kleisli "category" $\mathscr{T}$-$\mathsf{Kleisli}$ of $\mathscr{T}$ which has sets as objects and a morphism $X \mathrel{\circ\!\!\!\longrightarrow} Y$ in $\mathscr{T}$-$\mathsf{Kleisli}$ is a V-matrix $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. Composition is given by *Kleisli composition*:

$$b \circ a := b \cdot T_\xi a \cdot m_X^\circ,$$

$$\begin{tikzcd}
TX \arrow[dd, "a", "|" marking] & TY \arrow[dd, "b", "|" marking] & & TX \arrow[r, "m_X^\circ", "|" marking] \arrow[ddr, "b\circ a"', "|" marking] & TTX \arrow[d, "T_\xi a", "|" marking] \\
& & \longmapsto & & TY \arrow[d, "b", "|" marking] \\
Y & Z & & & Z
\end{tikzcd}$$

for all $a : X \mathrel{\circ\!\!\!\longrightarrow} Y$ and $b : Y \mathrel{\circ\!\!\!\longrightarrow} Z$ in $\mathscr{T}$-$\mathsf{Kleisli}$. We have

$$a \circ e_X^\circ = a \cdot T_\xi e_X^\circ \cdot m_X^\circ = a \cdot T_\xi 1_X \geq a$$

and

$$e_X^\circ \circ a = e_X^\circ \cdot T_\xi a \cdot m_X^\circ \geq a \cdot e_X^\circ \cdot m_X^\circ = a,$$

that is, e_X° is a lax identity for "$\circ$". Moreover,

$$c \circ (b \circ a) \leq (c \circ b) \circ a$$

if T_ξ preserves composition, and

$$c \circ (b \circ a) \geq (c \circ b) \circ a$$

if $m : T_\xi T_\xi \to T_\xi$ is a (strict) natural transformation. Clearly, $a : X \mathrel{\circ\!\!\!\longrightarrow} X$ is reflexive if and only if $e_X^\circ \leq a$, and a is transitive if and only if $a \circ a \leq a$.

Examples 4.1. (a) For each quantale V, $\mathscr{I}_{\mathsf{V}}$-$\mathsf{Kleisli} = \mathsf{V}$-$\mathsf{Mat}$.

(b) Kleisli composition for $\mathscr{U}_2$ is associative and e_X° is a right unit for this composition. In [Hof05] it is shown that e_X° is also a left unit (precisely) if we restrict ourself to those $a : UX \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ where, for each $y \in Y$, $\{\mathfrak{x} \in UX \mid a(\mathfrak{x}, y) = \text{true}\}$ is closed in UX with respect to the Zariski closure. In fact, as also shown in [Hof05], this restriction of $\mathscr{U}_2$-$\mathsf{Kleisli}$ is 2-equivalent to the category CSet which has sets as objects, and a morphism from X to Y is a finitely additive map $c : PX \to PY$, that is, $c(\varnothing) = \varnothing$ and $c(A \cup B) = c(A) \cup c(B)$. As consequence we obtain at once that a monoid $a : X \relbar\joinrel\circ\joinrel\rightarrow X$ in $\mathscr{U}_2$-$\mathsf{Kleisli}$ "is the same thing" as a Kuratowski closure operator on X.

Definition 4.2. Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ and $\mathscr{T}' = (\mathbb{T}', \mathsf{V}', \xi')$ be topological theories. *A morphism* $(j, \varphi) : \mathscr{T}' \to \mathscr{T}$ *of topological theories* is a pair (j, φ) consisting of a monad morphism $j : \mathbb{T}' \to \mathbb{T}$ and a lax morphism of quantales $\varphi : \mathsf{V} \to \mathsf{V}'$ such that $\xi' \cdot T'\varphi \le \varphi \cdot \xi \cdot j_{\mathsf{V}}$.

(Mor)
$$\begin{tikzcd}
T'\mathsf{V} \arrow[r, "j_{\mathsf{V}}"] \arrow[d, "T'\varphi"'] & T\mathsf{V} \arrow[dd, "\xi"] \\
T'\mathsf{V}' \arrow[d, "\xi'"'] \arrow[r, phantom, "\le"] & \\
\mathsf{V}' & \mathsf{V} \arrow[l, "\varphi"']
\end{tikzcd}$$

If $(j', \varphi') : \mathscr{T}'' \to \mathscr{T}'$ and $(j, \varphi) : \mathscr{T}' \to \mathscr{T}$ are morphisms of topological theories, then so is the composite

$$(j, \varphi) \cdot (j', \varphi') = (j \cdot j', \varphi' \cdot \varphi) : \mathscr{T}'' \to \mathscr{T}.$$

The identity morphism on $\mathscr{T}$ is given by $(1_{\mathbb{T}}, 1_{\mathsf{V}})$. If $\mathsf{V} = \mathsf{V}'$ and $\varphi = 1_{\mathsf{V}}$, then condition (Mor) reduces to

$$\begin{tikzcd}
T'\mathsf{V} \arrow[r, "j_{\mathsf{V}}"] \arrow[dr, "\xi'"'] \arrow[dr, phantom, "\le", shift left=3] & T\mathsf{V} \arrow[d, "\xi"] \\
& \mathsf{V}.
\end{tikzcd}$$

On the other hand, if $\mathbb{T} = \mathbb{T}'$ and $j = 1_{\mathbb{T}}$, then (Mor) reduces to

$$\begin{tikzcd}
T\mathsf{V} \arrow[r, "T\varphi"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\ge"] & T\mathsf{V}' \arrow[d, "\xi'"] \\
\mathsf{V} \arrow[r, "\varphi"'] & \mathsf{V}'.
\end{tikzcd}$$

Examples 4.3. (a) Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory. Then $(e, 1_{\mathsf{V}}) : \mathscr{I}_{\mathsf{V}} \to \mathscr{T}$ is a theory morphism.

(b) $(1_{\mathbb{T}}, \varrho) : \mathscr{T}_2 \to \mathscr{T}_{\mathsf{P}_+}$ (see Example 1.2) is a theory morphism for each monad $\mathbb{T} = (T, e, m)$ where T is taut.

(c) $(1_{\mathbb{1}}, \lambda) : \mathscr{I}_2 \to \mathscr{I}_{\mathsf{P}_+}$ (see Example 1.2) is a theory morphism, but $(1_{\mathbb{U}}, \lambda) : \mathscr{U}_2 \to \mathscr{U}_{\mathsf{P}_+}$ is not.

Theorem 4.4. *Let $(j, \varphi) : \mathscr{T}' \to \mathscr{T}$ be a morphism of topological theories. Then the following statements hold.*

(a) $j : T'_{\xi'}\Phi \to \Phi T_\xi$ *is an op-lax natural transformation, where* $\Phi : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}'\text{-}\mathsf{Mat}$ *is the lax functor induced by* $\varphi : \mathsf{V} \to \mathsf{V}'$ *(see Section 1).*

(b) (j, φ) *induces a lax functor*

$$\Phi^j : \mathscr{T}\text{-}\mathsf{Kleisli} \to \mathscr{T}'\text{-}\mathsf{Kleisli}.$$
$$a : TX \longrightarrow\!\!\!\!\!\!+\ Y \mapsto \Phi(a) \cdot j_X : T'X \longrightarrow\!\!\!\!\!\!+\ Y$$

(c) $(j, \varphi)\text{-}\mathsf{Alg} : \mathscr{T}\text{-}\mathsf{Alg} \to \mathscr{T}'\text{-}\mathsf{Alg},\ (X, a) \mapsto (X, \Phi^j(a))$ *is a concrete functor.*

Proof. (a) Let $a : X \longrightarrow\!\!\!\!\!\!+\ Y$, $\mathfrak{x}' \in T'X$ and $\mathfrak{y} \in TY$. We have

$$\begin{aligned}
&j_Y \cdot T'_{\xi'}\Phi(a)(\mathfrak{x}', \mathfrak{y}) \\
&\quad = \bigvee \left\{ T'_{\xi'}\Phi(a)(\mathfrak{x}', \mathfrak{y}') \;\middle|\; \mathfrak{y}' \in T'Y, j_Y(\mathfrak{y}') = \mathfrak{y} \right\} \\
&\quad = \bigvee \left\{ \xi' \cdot T\varphi' \cdot T'a(\mathfrak{w}') \;\middle|\; \mathfrak{y}' \in T'Y, j_Y(\mathfrak{y}') = \mathfrak{y}; \mathfrak{w}' \in T'(X \times Y), \mathfrak{w}' \mapsto \mathfrak{x}', \mathfrak{w}' \mapsto \mathfrak{y}' \right\} \\
&\quad \le \bigvee \left\{ \varphi \cdot \xi \cdot j_{\mathsf{V}} \cdot T'a(\mathfrak{w}') \;\middle|\; \mathfrak{y}' \in T'Y, j_Y(\mathfrak{y}') = \mathfrak{y}; \mathfrak{w}' \in T'(X \times Y), \mathfrak{w}' \mapsto \mathfrak{x}', \mathfrak{w}' \mapsto \mathfrak{y}' \right\} \\
&\quad = \bigvee \left\{ \varphi \cdot \xi \cdot Ta(j_{X \times Y}(\mathfrak{w}')) \;\middle|\; \mathfrak{y}' \in T'Y, j_Y(\mathfrak{y}') = \mathfrak{y}; \mathfrak{w}' \in T'(X \times Y), \mathfrak{w}' \mapsto \mathfrak{x}', \mathfrak{w}' \mapsto \mathfrak{y}' \right\} \\
&\quad \le \varphi \left(\bigvee \left\{ \xi \cdot Ta(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = j_X(\mathfrak{x}'), T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\} \right) \\
&\quad = \Phi T_\xi a \cdot j_X(\mathfrak{x}', \mathfrak{y}).
\end{aligned}$$

(b) First note that $a \le a'$ implies $\Phi^j(a') \le \Phi^j(a)$ since Φ as well as $\_ \cdot j_X$ preserve the order. Since j is a monad morphism we have ${e_{X'}}^\circ \le e_X^\circ \cdot j_X$ from which follows $\Phi^j(e_X^\circ) \ge {e_{X'}}^\circ$. Finally, for $a : X \longrightarrow\!\!\!\!\!\!\circ\ Y$ and $b : Y \longrightarrow\!\!\!\!\!\!\circ\ Z$ we have

$$\begin{aligned}
\Phi^j(b \circ a) &= \Phi(b \cdot T_\xi a \cdot m_X^\circ) \cdot j_X \\
&\ge \Phi b \cdot \Phi T_\xi a \cdot m_X^\circ \cdot j_X \\
&\ge \Phi b \cdot \Phi T_\xi a \cdot j_{TX} \cdot T'j_X \cdot m_{X'}^\circ && (j \text{ is a monad morphism}) \\
&\ge \Phi b \cdot j_Y \cdot T'_{\xi'}\Phi a \cdot T'j_X \cdot m_{X'}^\circ && \text{(a)} \\
&\ge \Phi b \cdot j_Y \cdot T'_{\xi'}(\Phi a \cdot j_X) \cdot m_{X'}^\circ && \text{(Theorem 3.5)} \\
&= \Phi^j b \circ \Phi^j a.
\end{aligned}$$

(c) Follows immediately from (b). □

5. The $\mathscr{T}$-algebra V

It is well-known that each quantale V can be considered as a V-category with structure map $\hom : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$. Our next result implies that this structure on V can be lifted to a $\mathscr{T}$-algebra structure.

Lemma 5.1. *Let* $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ *be a topological theory,* (X, a) *and* (Y, b) *be* V*-categories and* (X, α), (Y, β) *be* $\mathbb{T}$*-algebras. Then the following assertions hold.*

(a) $(X, a \cdot \alpha)$ *is a* $\mathscr{T}$*-algebra provided that* $\alpha : (TX, T_\xi a) \to (X, a)$ *is a* V*-functor. If* $T : \mathsf{Set} \to \mathsf{Set}$ *satisfies (BC), then this condition is also necessary.*

(b) $f : (X, a \cdot \alpha) \to (Y, b \cdot \beta)$ *is a lax homomorphism if and only if* $f : (X, a) \to (Y, b)$ *is a* V*-functor and* $k \le b(\beta \cdot Tf(\mathfrak{x}), f \cdot \alpha(\mathfrak{x}))$ *for each* $\mathfrak{x} \in TX$.

Note: In case $(Y, b) = (\mathsf{V}, \hom)$ the second condition in (b) is equivalent to $\beta \cdot Tf \leq f \cdot \alpha$.

Proof. The proof of the first assertion can be found in [CH07]. To see (b), assume first that $f : (X, a \cdot \alpha) \to (Y, b \cdot \beta)$ is a lax homomorphism. Let $x, x' \in X$ and $\mathfrak{x} \in TX$. Then

$$a(x, x') = a(\alpha \cdot e_X(x), x') \leq b(\beta \cdot Tf(e_X(x)), f(x')) = b(\beta \cdot e_X(f(x)), f(x')) = b(f(x), f(x'))$$

and

$$k \leq a(\alpha(\mathfrak{x}), \alpha(\mathfrak{x})) \leq b(\beta \cdot Tf(\mathfrak{x}), f \cdot \alpha(\mathfrak{x})).$$

Assume now that f satisfies the two conditions in (b). Let $\mathfrak{x} \in TX$ and $x \in X$. Then

$$\begin{aligned} a(\alpha(\mathfrak{x}), x) &\leq b(\beta \cdot Tf(\mathfrak{x}), f \cdot \alpha(\mathfrak{x})) \otimes a(\alpha(\mathfrak{x}), x) \\ &\leq b(\beta \cdot Tf(\mathfrak{x}), f \cdot \alpha(\mathfrak{x})) \otimes b(f \cdot \alpha(\mathfrak{x}), f(x)) \\ &\leq b(\beta \cdot Tf(\mathfrak{x}), f(x)). \end{aligned}$$

□

As an immediate consequence we have

Corollary 5.2. *Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory.*

(a) *Each Eilenberg-Moore algebra (X, α) is a $\mathscr{T}$-algebra provided that T_ξ is a strict extension of T, i.e. if $(\mathrm{Q}_k^=)$.*
(b) *Consider $\hom_\xi = \hom \cdot \xi : T\mathsf{V} \nrightarrow \mathsf{V}$. Then $(\mathsf{V}, \hom_\xi)$ is a $\mathscr{T}$-algebra if and only if $(\mathrm{M}_\mathrm{e}^=)$ and $(\mathrm{M}_m^=)$.*

Proof. Surely $\alpha : (TX, T_\xi 1_X) \to (X, 1_X)$ is a V-functor if $T_\xi 1_X = 1_{TX}$, which proves (a). Lemma 3.2 implies that $\xi : (TV, T_\xi \hom) \to (\mathsf{V}, \hom)$ is a V-functor, this proves the first part of (b). It is left to show that $(\mathrm{M}_\mathrm{e}^=)$ and $(\mathrm{M}_m^=)$ if $(\mathsf{V}, \hom_\xi)$ is a $\mathscr{T}$-algebra. For each $x \in \mathsf{V}$ we have $k \leq \hom_\xi(e_\mathsf{V}(x), x)$ and hence $\xi \cdot e_\mathsf{V}(x) \leq x$. Let now $\mathfrak{X} \in T^2\mathsf{V}$. For each $\mathfrak{x} \in T\mathsf{V}$ and $x \in \mathsf{V}$ we have

$$T_\xi \hom_\xi(\mathfrak{X}, \mathfrak{x}) \otimes \hom_\xi(\mathfrak{x}, x) \leq \hom_\xi(m_\mathsf{V}(\mathfrak{X}), x).$$

Put $x = \xi \cdot T\xi(\mathfrak{X})$ and $\mathfrak{x} = T\xi(\mathfrak{X})$. Obviously we have $k \leq \hom_\xi(\mathfrak{x}, x)$, we show now that also $k \leq T_\xi \hom_\xi(\mathfrak{X}, \mathfrak{x})$. In order to see that, consider $\langle 1_\mathsf{V}, \xi \rangle : T\mathsf{V} \to T\mathsf{V} \times \mathsf{V}$ and put $\mathfrak{W} = T\langle 1_\mathsf{V}, \xi \rangle(\mathfrak{X})$. Clearly, $\mathfrak{W}$ projects to $\mathfrak{X}$ and $\mathfrak{x}$ respectively. Moreover, since $k \leq \hom_\xi \cdot \langle 1_\mathsf{V}, \xi \rangle$ we have $k \leq T_\xi \hom_\xi(\mathfrak{X}, \mathfrak{x})$. □

Corollary 5.3. *Let $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ be a topological theory where $\xi : T\mathsf{V} \to \mathsf{V}$ is an Eilenberg-Moore structure on V. We consider the $\mathscr{T}$-algebra $\mathsf{V} = (\mathsf{V}, \hom_\xi)$.*

(a) *For each set I, a map $f : \mathsf{V}^I \to \mathsf{V}$ is a lax homomorphism if and only if f is a V-functor and*

$$\begin{tikzcd} T(\mathsf{V}^I) \arrow[r, "Tf"] \arrow[d, "\langle \xi \cdot T\pi_i \rangle"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V}^I \arrow[r, "f"'] & \mathsf{V}. \end{tikzcd}$$

(b) *$\bigwedge : \mathsf{V}^I \to \mathsf{V}$ is a lax homomorphism.*
(c) *$\hom(v, \_) : \mathsf{V} \to \mathsf{V}$ is a lax homomorphism for each $v \in \mathsf{V}$.*

(d) *Assuming* $(\mathrm{Q}^{=}_{\otimes})$, $v \otimes \_ : \mathsf{V} \to \mathsf{V}$ *is a lax homomorphism for each* $v \in \mathsf{V}$ *which satisfies*

$$\begin{tikzcd} T1 \arrow[r, "Tv"] \arrow[d, "!"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "v"'] & \mathsf{V}. \end{tikzcd}$$

We do not state yet anything about $\bigvee : \mathsf{V}^I \to \mathsf{V}$. Indeed, as known (for instance) for topological spaces, it is in general not a lax homomorphism. However, in the next section we will show that it is so assuming that I is compact and V^I is equipped with the function spaces structure.

6. Closed objects

Throughout this section we consider *a topological theory* $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ *where* $\mathbb{T}$ *satisfies (BC).* Furthermore, we assume $(\mathrm{Q}^{=}_{k})$ to guarantee that T_ξ is a strict extension of T (see Proposition 3.7), and $(\mathrm{M}^{=}_{\mathrm{e}})$ and $(\mathrm{M}^{=}_{m})$ in order to have the $\mathscr{T}$-algebra $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ available (see Corollary 5.2). We wish to transport the tensor product $\otimes$ on V to $\mathscr{T}$-Alg (or $\mathscr{T}$-Graph) by putting $(X, a) \otimes (Y, b) = (X \times Y, c)$ and

$$c(\mathfrak{w}, (x, y)) = a(\mathfrak{x}, x) \otimes b(\mathfrak{y}, y), \tag{8}$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$, $\mathfrak{x} = T\pi_X(\mathfrak{w})$ and $\mathfrak{y} = T\pi_Y(\mathfrak{w})$. For lax homomorphisms $f : X \to X'$ and $g : Y \to Y'$, the map $f \otimes g : X \otimes Y \to X' \otimes Y'$, $(x, y) \mapsto (f(x), g(y))$ is a lax homorphism as well. The structure c on $X \times Y$ is easily seen to be reflexive, it is transitive provided that $(\mathrm{Q}^{=}_{\otimes})$. In fact, we have the following

Lemma 6.1. *Assume that the structure* c *on* $(\mathsf{V}, \hom_\xi) \otimes (\mathsf{V}, \hom_\xi)$ *is transitive. Then we have* $(\mathrm{Q}^{=}_{\otimes})$.

Proof. Let $\mathfrak{w} \in T(\mathsf{V} \times \mathsf{V})$ and put $x = \xi \cdot T\pi_1(\mathfrak{w})$ and $y = \xi \cdot T\pi_2(\mathfrak{w})$. By hypothesis we have, for each $\mathfrak{W} \in T^2(\mathsf{V} \times \mathsf{V})$,

$$T_\xi c(\mathfrak{W}, \mathfrak{w}) \otimes c(\mathfrak{w}, (x, y) \leq c(m_{\mathsf{V}\times\mathsf{V}}(\mathfrak{W}), (x, y)).$$

First observe that $c(\mathfrak{w}, (x, y) = k$. Let $h : 1 \to T(\mathsf{V} \times \mathsf{V})$ be the composite

$$1 \xrightarrow{k} \mathsf{V} \xrightarrow{\Delta_{\mathsf{V}}} \mathsf{V} \times \mathsf{V} \xrightarrow{e_{\mathsf{V}\times\mathsf{V}}} T(\mathsf{V} \times \mathsf{V})$$

and put $\mathfrak{W} = Th \cdot T!(\mathfrak{w})$, where $! : \mathsf{V} \times \mathsf{V} \to 1$. An easy calculation shows that

$$T_\xi c(\mathfrak{W}, \mathfrak{w}) \geq \xi \cdot T\otimes(\mathfrak{w}) \qquad \text{and} \qquad c(m_{\mathsf{V}\times\mathsf{V}}(\mathfrak{W}), (x, y)) = \xi \cdot T\pi_1(\mathfrak{w}) \otimes \xi \cdot T\pi_2(\mathfrak{w}),$$

hence $\otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle(\mathfrak{w}) \geq \xi \cdot T\otimes(\mathfrak{w})$. □

Therefore we assume from now on *that our given topological theory* $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ *is a strict theory.*

Examples 6.2. The following theories satisfy the conditions mentioned above.

(a) The identity theory $\mathscr{I}_{\mathsf{V}}$, for each quantale V (see Theorem 3.3 (a)).
(b) For each quantale V, the theory $\mathscr{L}^{\otimes}_{\mathsf{V}} = (\mathbb{L}, \mathsf{V}, \xi_\otimes)$ (see Theorem 3.3 (c)).
(c) Any topological theory $\mathscr{T} = (\mathbb{T}, \mathsf{V}, \xi)$ with a (BC)-monad $\mathbb{T}$, $\otimes = \wedge$ and ξ a (strict) Eilenberg-Moore algebra. In this case we obtain the categorical product on $\mathscr{T}$-Alg.

(d) The theory $\mathscr{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ (see Theorem 3.3 (b)). To see $(\mathrm{Q}^{=}_{\otimes})$, recall first that $\xi(\mathfrak{x}) = \sup\{u \in \mathsf{P}_+ \mid [u,\infty] \in \mathfrak{x}\}$ for each $\mathfrak{x} \in U\mathsf{P}_+$. Let $\mathfrak{w} \in U(\mathsf{P}_+ \times \mathsf{P}_+)$ and let $\mathfrak{x} = U\pi_1(\mathfrak{w})$ and $\mathfrak{y} = U\pi_2(\mathfrak{w})$. We have

$$\xi(\mathfrak{x}) + \xi(\mathfrak{y}) = \sup_{u:[u,\infty]\in\mathfrak{x}} u + \sup_{v:[v,\infty]\in\mathfrak{y}} v = \sup_{u:[u,\infty]\in\mathfrak{x}} \sup_{v:[v,\infty]\in\mathfrak{y}} u + v$$

since $u + \_ : \mathsf{P}_+ \to \mathsf{P}_+$ preserves non-empty suprema. Let $u, v \in \mathsf{P}_+$ such that $[u,\infty] \in \mathfrak{x}$ and $[v,\infty] \in \mathfrak{y}$. Then $[u,\infty] \times [v,\infty] \in \mathfrak{w}$ and therefore $[u+v,\infty] \in U{+}(\mathfrak{w})$, which implies $\xi \cdot U{+}(\mathfrak{w}) \geq u + v$.

We are interested in describing those objects X in $\mathscr{T}$-Alg where the functor $X \otimes \_ : \mathscr{T}\text{-}\mathsf{Alg} \to \mathscr{T}\text{-}\mathsf{Alg}$ has a (canonical) right adjoint. Such an object is called *closed* (for the tensor product $\otimes$). It is well-known that any V-category is closed in V-Cat (see [Law73]). This result can not be extended to $\mathscr{T}$-algebras in general as, for instance, the category of topological spaces is not Cartesian closed. Its exponentiable objects are described in many different ways, the most convenient for our purpose is the following characterisation.

Theorem 6.3 ([Möb81, Pis99])**.** *A topological space $(X, a : UX \rightarrowtail X)$ is exponentiable if and only of $a \cdot \hat{U}a = a \cdot m_X$.*

We are now going to show that this condition also characterises (canonically) closed objects for $\mathscr{T}$. In order to do so, fix a $\mathscr{T}$-algebra $X = (X, a)$. Recall first that a right adjoint to $X \otimes \_$ associates to each $\mathscr{T}$-algebra Y a $\mathscr{T}$-algebra Y^X such that we have, for each $\mathscr{T}$-algebra Z, a natural bijection

$$\frac{X \otimes Z \to Y}{Z \to Y^X.}$$

Choosing as test spaces $Z = 1$ with structure map e_1°, we see that

$$\frac{\dfrac{1 \to Y^X}{X \otimes 1 \to Y}}{\hat{X} \to Y.}$$

Here the structure $\hat{a}$ on $\hat{X} = X \otimes 1 = (X, \hat{a})$ is given by

$$\hat{a}(\mathfrak{x}, x) = \begin{cases} a(\mathfrak{x}, x) & \text{if } T!(\mathfrak{x}) = e_1(\star), \\ \bot & \text{else,} \end{cases}$$

for each $\mathfrak{x} \in TX$ and $x \in X$. Therefore the underlying set of Y^X is given by

$$\{f : \hat{X} \to Y \mid f \text{ is a lax homomorphism}\}.$$

The structure d on Y^X must be chosen so that the evaluation map

$$\mathrm{ev} : Y^X \times X \to Y, (h, x) \mapsto h(x)$$

is a lax homomorphism. In fact, we define d to be the largest such structure:

$$(9) \quad d(\mathfrak{p}, h) \quad = \quad \bigvee \left\{ v \in \mathsf{V} \;\middle|\; \forall \mathfrak{q} \in T\pi_{Y^X}^{-1}(\mathfrak{p}), x \in X \;\; a(T\pi_X(\mathfrak{q}), x) \otimes v \leq b(T\mathrm{ev}(\mathfrak{q}), h(x)) \right\},$$

for each $\mathfrak{p} \in T(Y^X)$ and $h \in Y^X$. Note that the supremum above is even a maximum since $\otimes$ distributes over suprema. Using the adjointness relation between $\otimes$ and hom we obtain

$$d(\mathfrak{p}, h) = \bigwedge_{\substack{\mathfrak{q}\in T(Y^X\times X), x\in X\\ \mathfrak{q}\mapsto\mathfrak{p}}} \hom(a(T\pi_X(\mathfrak{q}), x), b(T\mathrm{ev}(\mathfrak{q}), h(x))).$$

It is easy to see that d is reflexive and defines indeed the right adjoint $\_^X$ to $X \otimes \_$ in the category $\mathscr{T}$-Graph, it also does so in $\mathscr{T}$-Alg provided that d is transitive for each $\mathscr{T}$-algebra Y. Our first goal is to show that this is surely the case provided that X satisfies

$$a \cdot T_\xi a = a \cdot m_X. \tag{10}$$

The proof is very similar to the one of [Hof06, Theorem 3.5] which in turn is strongly motivated by [CHT03]. The main difficulty here is to obtain Lemma 3.4 of [Hof06] in our setting.

Lemma 6.4. *Let $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ and $b : TY \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ be V-matrices and define $c : T(X \times Y) \relbar\joinrel\mapstochar\joinrel\rightarrow X \times Y$ as in (8). Then, for all $\mathfrak{P} \in T^2(X \times Y)$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,*

$$\bigvee_{\mathfrak{p}\in T(X\times Y):\ \substack{T\pi_X(\mathfrak{p})=\mathfrak{x}\\ T\pi_Y(\mathfrak{p})=\mathfrak{y}}} T_\xi d(\mathfrak{P}, \mathfrak{p}) \geq T_\xi a(T^2\pi_X(\mathfrak{P}), \mathfrak{x}) \otimes T_\xi b(T^2\pi_Y(\mathfrak{P}), \mathfrak{y}).$$

Proof. Let $\mathfrak{P} \in T^2(X \times Y)$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$ and put $\mathfrak{X} = T^2\pi_X(\mathfrak{P})$ and $\mathfrak{Y} = T^2\pi_Y(\mathfrak{P})$. Consider (with $P = X \times Y$)

$$\begin{tikzcd}
T^2(P) \arrow[d, "T\mathrm{can}"'] & T(TP\times P) \arrow[l] \arrow[d] \arrow[rrd, bend left, "Tc"] & & \\
T(TX\times TY) \arrow[d] & T(TX\times X\times TY\times Y) \arrow[l] \arrow[r, "T(a\times b)"] \arrow[d] & T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T\otimes"] \arrow[d] & T\mathsf{V} \arrow[dd, "\xi"] \\
T^2X\times T^2Y & T(TX\times X)\times T(TY\times Y) \arrow[l] \arrow[r, "Ta\times Tb"] \arrow[d] & T\mathsf{V}\times T\mathsf{V} \arrow[d, "\xi\times\xi"'] & \\
 & TX\times TY & \mathsf{V}\times\mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V},
\end{tikzcd}$$

where the unlabelled arrows represent canonical maps. Both diagrams on the left hand side satisfy (BC), the upper right as well as the middle diagram commute and the diagram on the right hand side commutes by Axiom ($\mathrm{Q}_{\otimes}^{=}$). Hence

$$\begin{aligned}
&T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes T_\xi b(\mathfrak{Y}, \mathfrak{y})\\
&\quad= \bigvee \left\{ \xi \cdot Ta(\mathfrak{W}_1) \otimes \xi \cdot Tb(\mathfrak{W}_2) \;\middle|\; \begin{array}{l} \mathfrak{W}_1 \in T(TX \times X) : \mathfrak{W}_1 \mapsto \mathfrak{X}, \mathfrak{W}_1 \mapsto \mathfrak{x};\\ \mathfrak{W}_2 \in T(TY \times Y) : \mathfrak{W}_2 \mapsto \mathfrak{Y}, \mathfrak{W}_2 \mapsto \mathfrak{y} \end{array} \right\}\\
&\quad\leq \bigvee \left\{ \xi \cdot T\otimes \cdot T(a \times b)(\mathfrak{W}) \;\middle|\; \begin{array}{l} \mathfrak{W} \in T(TX \times X \times TY \times Y) :\\ \mathfrak{W} \mapsto (\mathfrak{x}, \mathfrak{y}), \mathfrak{W} \mapsto T\mathrm{can}(\mathfrak{P}) \end{array} \right\}\\
&\quad= \bigvee \left\{ \xi \cdot Td(\mathfrak{W}) \;\middle|\; \mathfrak{W} \in T(TP \times P) : \mathfrak{W} \mapsto (\mathfrak{x}, \mathfrak{y}), \mathfrak{W} \mapsto \mathfrak{P} \right\}\\
&\quad= \bigvee \left\{ T_\xi d(\mathfrak{P}, \mathfrak{p}) \;\middle|\; \mathfrak{p} \in T(X \times Y) : T\pi_X(\mathfrak{p}) = \mathfrak{x}, T\pi_Y(\mathfrak{p}) = \mathfrak{y} \right\}. \qquad \square
\end{aligned}$$

The following theorem can be proven as in [Hof06, Theorem 3.5]. Nevertheless, for the sake of readability, we include its proof here.

Theorem 6.5. *Let X be a $\mathscr{T}$-algebra satisfying* (10)*. Then, for each $\mathscr{T}$-algebra Y, the structure d on Y^X as defined in* (9) *is transitive.*

Proof. Assume that (X,a) satisfies (10) and let (Y,b) be a $\mathscr{T}$-algebra. Let $\mathfrak{P} \in T^2(Y^X)$, $\mathfrak{p} \in T(Y^X)$ and $h \in Y^X$. We have to show that, for each $x \in X$ and $\mathfrak{q} \in T(Y^X \times X)$ with $T\pi_1(\mathfrak{q}) = m_{Y^X}(\mathfrak{P})$,

$$T_\xi d(\mathfrak{P},\mathfrak{p}) \otimes d(\mathfrak{p},h) \otimes a(T\pi_2(\mathfrak{q}),x) \leq b(T\mathrm{ev}(\mathfrak{q}),h(x)).$$

Since m satisfies (BC), there exists $\mathfrak{Q} \in T(Y^X \times X)$ such that $m_{Y^X \times X}(\mathfrak{Q}) = \mathfrak{q}$ and $T^2\pi_1(\mathfrak{Q}) = \mathfrak{P}$. Then

$$\begin{aligned}
b(T\mathrm{ev}(\mathfrak{q}),h(x)) &= b(m_Y \cdot T^2\mathrm{ev}(\mathfrak{Q}),h(x)) \\
&\geq \bigvee_{\mathfrak{x}\in TX} \bigvee_{\substack{\mathfrak{q}'\in T(Y^X\times Y):\\ T\pi_1(\mathfrak{q}')=\mathfrak{p},\\ T\pi_2(\mathfrak{q}')=\mathfrak{x}}} T_\xi b(T^2\,\mathrm{ev}(\mathfrak{Q}),T\mathrm{ev}(\mathfrak{q}')) \otimes b(T\mathrm{ev}(\mathfrak{q}'),h(x)) \\
&\geq \bigvee_{\mathfrak{x}\in TX} \left(\bigvee_{\mathfrak{q}'\in T(Y^X\times Y):\substack{T\pi_1(\mathfrak{q}')=\mathfrak{p},\\ T\pi_2(\mathfrak{q}')=\mathfrak{x}}} T_\xi (d\otimes a)(\mathfrak{Q},\mathfrak{q}') \right) \otimes d(\mathfrak{p},h) \otimes a(\mathfrak{x},x) \\
&\geq \bigvee_{\mathfrak{x}\in TX} T_\xi d(\mathfrak{P},\mathfrak{p}) \otimes d(\mathfrak{p},h) \otimes T_\xi a(T^2\pi_2(\mathfrak{Q}),\mathfrak{x}) \otimes a(\mathfrak{x},x) \\
&= T_\xi d(\mathfrak{P},\mathfrak{p}) \otimes d(\mathfrak{p},h) \otimes a(T\pi_2(\mathfrak{q}),x).
\end{aligned}$$

□

Corollary 6.6. *Each $\mathbb{T}$-algebra (X,α) is closed in $\mathscr{T}$-*Alg.

Our next goal is to show that Condition (10) is also necessary for d being transitive. In order to do so we will make use of the $\mathscr{T}$-algebra V (see Corollary 5.2).

Lemma 6.7. *Consider the $\mathbb{T}$-algebra $TX = (TX, m_X)$. Then $a : TX \otimes X \to \mathsf{V}$ is a lax homomorphism.*

Proof. First note that the structure c on $TX \otimes X$ is given by

$$c(\mathfrak{W},(\mathfrak{x},x)) = \begin{cases} \bot & \text{if } \mathfrak{x} \neq m_X(T\pi_{TX}(\mathfrak{W})), \\ a(T\pi_X(\mathfrak{W}),x) & \text{if } \mathfrak{x} = m_X(T\pi_{TX}(\mathfrak{W})) \end{cases}$$

for $\mathfrak{W} \in T(TX \otimes X)$, $\mathfrak{x} \in TX$ and $x \in X$. Assume that $\mathfrak{x} = m_X(T\pi_{TX}(\mathfrak{W}))$. We have to show that

$$a(T\pi_X(\mathfrak{W}),x) \leq \hom(\xi \cdot Ta(\mathfrak{W}), a(\mathfrak{x},x))$$

which is equivalent to

$$\xi \cdot Ta(\mathfrak{W}) \otimes a(T\pi_X(\mathfrak{W}),x) \leq a(\mathfrak{x},x).$$

Now the assertion follows from

$$\xi \cdot Ta(\mathfrak{W}) \leq T_\xi a(T\pi_{TX}(\mathfrak{W}), T\pi_X(\mathfrak{W})).$$

□

Let $\mathfrak{X} \in TX$. We define a map

$$\varphi : X \to \mathsf{V}, \quad x \mapsto \bigvee_{\mathfrak{x}\in TX} T_\xi a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x).$$

Alternatively, with $i_{\mathfrak{X}} : 1 \to T^2X,\ \star \mapsto \mathfrak{X}$ we define a V-matrix $\psi : 1 \,{-\!\!\!\!\!\!\mapsto}\, X$ as the composite

$$\begin{tikzcd} 1 \arrow[r, "i_{\mathfrak{X}}"] \arrow[rrr, bend left, "\psi", "\shortmid"{marking}] & T^2X \arrow[r, "T_\xi a", "\shortmid"{marking}] & TX \arrow[r, "a", "\shortmid"{marking}] & X. \end{tikzcd}$$

We have $\varphi(x) = \psi(\star, x)$.

Lemma 6.8. *$\varphi : \hat{X} \to \mathsf{V}$ is a lax homomorphism.*

Proof. Let $x \in X$ and $\mathfrak{x} \in TX$ with $T!(\mathfrak{x}) = e_1(\star)$. We have to show that

$$a(\mathfrak{x}, x) \le \hom(\xi \cdot T\varphi(\mathfrak{x}), \varphi(x))$$

which is equivalent to

$$\xi \cdot T\varphi(\mathfrak{x}) \otimes a(\mathfrak{x}, x) \le \varphi(x).$$

Note that

$$\xi \cdot T\varphi(\mathfrak{x}) \otimes a(\mathfrak{x}, x) = T_\xi\psi(e_1(\star), \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a \cdot T_\xi\psi \cdot e_1(\star, x),$$

hence the assertion follows from

$$\begin{aligned} a \cdot T_\xi\psi \cdot e_1 &= a \cdot T_\xi a \cdot T_\xi^2 a \cdot Ti_{\mathfrak{X}} \cdot e_1 \\ &= a \cdot m_X \cdot T_\xi^2 a \cdot e_{T^2X} \cdot i_{\mathfrak{X}} \\ &= a \cdot T_\xi a \cdot m_{TX} \cdot e_{T^2X} \cdot i_{\mathfrak{X}} \\ &= a \cdot T_\xi a \cdot i_{\mathfrak{X}} = \psi. \end{aligned}$$

□

Theorem 6.9. *Let $X = (X, a)$ be a $\mathscr{T}$-algebra such that the structure d on V^X as defined in* (9) *is transitive. Then X satisfies* (10).

Proof. Fix some $\mathfrak{X} \in T^2X$. We have to show that, for each $x \in X$,

$$a(m_X(\mathfrak{X}), x) \le \bigvee_{\mathfrak{x}\in TX} T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x).$$

We consider

$$\mathrm{y} = \ulcorner a \urcorner : TX \to \mathsf{V}^X \qquad \text{and} \qquad \mathrm{y}_0 = \mathrm{y} \cdot e_X : X \to \mathsf{V}^X,$$

and put $\mathfrak{v} = T\mathrm{y}(\mathfrak{X})$ and $\mathfrak{V} = T^2\mathrm{y}_0(\mathfrak{X})$. First note that $Tm_X \cdot T^2e_X(\mathfrak{X}) = \mathfrak{X}$, hence

$$k = Tm_X(T^2e_X(\mathfrak{X}), \mathfrak{X}) \le T_\xi m_X(T^2e_X(\mathfrak{X}), \mathfrak{X})$$

and therefore $T_\xi d(\mathfrak{V}, \mathfrak{v}) \ge k$. Next we show that $d(\mathfrak{v}, \varphi) \ge k$. The evaluation map $\mathrm{ev} : \mathsf{V}^X \times X \to \mathsf{V}$ can be viewed as V-matrix $\mathrm{ev}' : \mathsf{V}^X \,{-\!\!\!\!\!\!\mapsto}\, X$, we have $\mathrm{ev}' \cdot \mathrm{y} = a$. Hence $T_\xi\mathrm{ev}' \cdot T\mathrm{y} = T_\xi a$, which implies

$$\xi \cdot T\mathrm{ev}(\mathfrak{w}) \le T_\xi a(\mathfrak{Y}, \mathfrak{y}) \tag{11}$$

for all $\mathfrak{y} \in TX$, $\mathfrak{Y} \in T^2Y$ and $\mathfrak{w} \in T(\mathsf{V}^X \times X)$ with $\mathfrak{w} \mapsto \mathfrak{y}$ and $\mathfrak{w} \mapsto T\mathrm{y}(\mathfrak{Y})$. Let $\mathfrak{w} \in T(\mathsf{V}^X \times X)$ and $x \in X$ with $\mathfrak{w} \mapsto \mathfrak{v}$. We have to show that

$$k \otimes a(T\pi_X(\mathfrak{w}), x) \leq \hom(\xi \cdot T\mathrm{ev}(\mathfrak{w}), \varphi(x))$$

which by adjointness is equivalent to

$$\xi \cdot T\mathrm{ev}(\mathfrak{w}) \otimes a(T\pi_X(\mathfrak{w}), x) \leq \varphi(x).$$

But this follows immediately from the inequality $\xi \cdot T\mathrm{ev}(\mathfrak{w}) \leq T_\xi(\mathfrak{X}, T\pi_X(\mathfrak{w}))$ which we deduce from (11).

Transitivity of d implies now $d(m_{\mathsf{V}^X}(\mathfrak{V}), \varphi) \geq k$. We put $\mathfrak{W} = T^2\langle \mathrm{y}_0, 1_X\rangle(\mathfrak{X})$, $\mathfrak{W}$ obviously projects to $\mathfrak{X}$ and $\mathfrak{V}$ respectively. From this follows

$$c(m_{\mathsf{V}^X \times X}(\mathfrak{W}), (\varphi, x)) \geq a(m_X(\mathfrak{X}), x)$$

for each $x \in X$, where c denotes the structure on $\mathsf{V}^X \otimes X$. Note that in the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle e_X, 1_X\rangle"] \arrow[d, "!"'] & TX \times X \arrow[r, "\mathrm{y} \times 1_X"] \arrow[d, "a"'] & \mathsf{V}^X \times X \arrow[dl, "\mathrm{ev}"] \\
1 \arrow[r, "k"'] & \mathsf{V} &
\end{tikzcd}$$

the right hand triangle commutes, the top line is equal to $\langle \mathrm{y}_0, 1_X\rangle$ and on the left hand side we have $a \cdot \langle e_X, 1_X\rangle \geq k \cdot !$. Hence

$$k \leq \xi \cdot T\xi \cdot T^2\,\mathrm{ev}(\mathfrak{W}) = \xi \cdot m_{\mathsf{V}} \cdot T^2\,\mathrm{ev}(\mathfrak{W}) = \xi \cdot T\mathrm{ev} \cdot m_{\mathsf{V}^X \times X}(\mathfrak{W})$$

and we conclude

$$\begin{aligned}
a(m_X(\mathfrak{X}), x) &\leq c(m_{\mathsf{V}^X \times X}(\mathfrak{W}), (\varphi, x)) \\
&\leq \hom(\xi \cdot T\mathrm{ev} \cdot m_{\mathsf{V}^X \times X}(\mathfrak{W}), \varphi(x)) \\
&\leq \hom(k, \varphi(x)) \\
&= \varphi(x) = \bigvee_{\mathfrak{x} \in TX} T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x)
\end{aligned}$$

for each $x \in X$. □

Finally, we have to study "how necessary" is the structure d on Y^X (see (9)), i.e. if $X \otimes \_$ might have a right adjoint without d being transitive. Unfortunately, we do not have a satisfying result available. However, to most of our examples one may apply the following theorem which is essentially taken from [Sch84, Theorem 3.3].

Proposition 6.10. *Assume that the inclusion functor $\mathscr{T}$-$\mathsf{Alg} \hookrightarrow \mathscr{T}$-$\mathsf{Graph}$ is finally dense. Then $X \otimes \_$ has a right adjoint if and only if the structure d of* (9) *is transitive.*

In many cases (ultrafilter monad, word monad), final density of $\mathscr{T}$-$\mathsf{Alg} \hookrightarrow \mathscr{T}$-$\mathsf{Graph}$ can be shown using "elementary structures" as in [Hof06, 1.9].

We finish this paper by showing an interesting application of exponentials to the study of compactness. Given a $\mathscr{T}$-algebra $X = (X, a)$, we define its *degree of compactness*

$$\mathrm{comp}(X) = \bigwedge_{\mathfrak{x} \in TX} \bigvee_{x \in X} a(\mathfrak{x}, x).$$

For short we call X *compact* if $\mathrm{comp}(X) \geq k$. We remark that compactness properties of lax algebras are also studied in [Sch05].

Proposition 6.11. *Let $X = (X, a)$ be a $\mathscr{T}$-algebra. Then the following assertions are equivalent.*

(a) *X is compact.*
(b) *$\bigvee : \mathsf{V}^X \to \mathsf{V}$ is a lax homomorphism.*
(c) *$\delta : TX \to \mathsf{V}$, $\mathfrak{x} \mapsto \bigvee_{x \in X} a(\mathfrak{x}, x)$ is a lax homomorphism.*

Here the exponential V^X is taken in $\mathscr{T}$-Graph.

Proof. Assume first that X is compact, i.e. $k \leq \bigvee_{x \in X} a(\mathfrak{x}, x)$, for each $\mathfrak{x} \in TX$. Let $\mathfrak{p} \in T(\mathsf{V}^X)$ and $\varphi \in \mathsf{V}^X$. Note that, since $\bigvee \varphi = \bigvee_{x \in X} \mathrm{ev}(\varphi, x)$, Axiom $(\mathrm{Q}_{\bigvee})$ implies

$$\xi \cdot T(\bigvee)(\mathfrak{p}) = \bigvee_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \xi \cdot T\mathrm{ev}(\mathfrak{q}).$$

Hence

$$\begin{aligned}
d(\mathfrak{p}, \varphi) &= \bigwedge_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \bigwedge_{x \in X} \hom(a(T\pi_X(\mathfrak{q}), x), \hom(\xi \cdot T\mathrm{ev}(\mathfrak{q}), \varphi(x))) \\
&\leq \bigwedge_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \bigwedge_{x \in X} \hom(a(T\pi_X(\mathfrak{q}), x), \hom(\xi \cdot T\mathrm{ev}(\mathfrak{q}), \bigvee \varphi)) \\
&= \bigwedge_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \hom(\bigvee_{x \in X} a(T\pi_X(\mathfrak{q}), x), \hom(\xi \cdot T\mathrm{ev}(\mathfrak{q}), \bigvee \varphi)) \\
&\leq \bigwedge_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \hom(\xi \cdot T\mathrm{ev}(\mathfrak{q}), \bigvee \varphi) \\
&= \hom(\bigvee_{\substack{\mathfrak{q} \in T(\mathsf{V}^I \times I) \\ \mathfrak{q} \mapsto \mathfrak{p}}} \xi \cdot T\mathrm{ev}(\mathfrak{q}), \bigvee \varphi) \\
&= \hom(\xi \cdot T(\bigvee)(\mathfrak{p}), \bigvee \varphi).
\end{aligned}$$

Assume now that $\bigvee : \mathsf{V}^X \to \mathsf{V}$ is a lax homomorphism. Note that δ can be written as the composite

$$TX \xrightarrow{\ulcorner a \urcorner} \mathsf{V}^X \xrightarrow{\bigvee} \mathsf{V},$$

which is a lax homomorphism by hypothesis and Lemma 6.7.
Supose now that δ is a lax homomorphism. Then we have

$$k \leq \hom(\xi \cdot T(\delta \cdot e_X)(\mathfrak{x}), \delta(\mathfrak{x}))$$

respectively

$$\xi \cdot T(\delta \cdot e_X)(\mathfrak{x}) \leq \bigvee_{x \in X} a(\mathfrak{x}, x),$$

for each $\mathfrak{x} \in TX$. Finally, since $k \leq \delta \cdot e_X$, from Axiom $(\mathrm{Q}_{\bigvee})$ and Axiom $(\mathrm{Q}_{\mathrm{k}})$ we conclude that $k \leq \bigvee_{x \in X} a(\mathfrak{x}, x)$. □

Corollary 6.12. *For each $\mathbb{T}$-algebra X, $\bigvee : \mathsf{V}^X \to \mathsf{V}$ is a lax homomorphism.*

In [Esc04] M.H. Escardó develops a "synthetic view" on topology with the aid of the Sierpinski space. Our description of compactness above permits us to import ideas from [Esc04] into our setting. As an example we present the following argumentation which should be compared with [Esc04, Theorem 9.15]. We call a map $f : X \to Y$ between $\mathscr{T}$-algebras $X = (X, a)$ and $Y = (Y, b)$ *closed* if $P_{\mathsf{V}}(f) : P_{\mathsf{V}}(X) \to P_{\mathsf{V}}(Y)$ sends lax homomorphisms to lax homomorphisms (see Example 2.3). We emphasize that we do not require f to be a lax homomorphism.

Proposition 6.13. *Let $X = (X, a)$ be a $\mathscr{T}$-algebra. Then X is compact if and only if the projection map*[3] *$\pi_Y : Y \otimes X \to Y$ is closed, for each $\mathscr{T}$-algebra $Y = (Y, b)$.*

Proof. Assume first that X is compact, i.e. $\bigvee : \mathsf{V}^X \to \mathsf{V}$ is a lax homomorphism. Let $\varphi : X \otimes Y \to \mathsf{V}$ be a lax homomorphism. Then the composite

$$Y \xrightarrow{\ulcorner \varphi \urcorner} \mathsf{V}^X \xrightarrow{\bigvee} \mathsf{V}$$

is a lax homomorphism as well, and an easy calculation shows that it is equal to $P_{\mathsf{V}}\pi_Y(\varphi)$. Assume now that $\pi_Y : Y \otimes X \to Y$ is closed, for each $\mathscr{T}$-algebra $Y = (Y, b)$. We choose $Y = TX$ and consider the lax homomorphism $a : TX \otimes X \to \mathsf{V}$. The assertion follows now from the fact that $\delta = P_{\mathsf{V}}\pi_Y(a)$. Alternatively, in case X is closed, we can choose $Y = \mathsf{V}^X$. Then the assertion follows from $\bigvee = P_{\mathsf{V}}\pi_Y(\mathrm{ev})$. □

Acknowledgements. I am indebted to Maria Manuel Clementino and Walter Tholen for many fruitful and enlightening discussions on the subject of this paper.

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