EXPONENATION FOR UNITARY STRUCTURES

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ABSTRACT. Motivated by the observation that both pretopologies and preapproach limits can be characterized as those convergence relations which have a unit for a suitable composition, we introduce the category Alg_u(T; V) of reflexive and unitary lax algebras, for a symmetric monoidal closed lattice V and a Set-monad T = (T, e, m). For T = U the ultrafilter monad, we characterize exponentiable morphisms in Alg_u(U; V). Further, we give a sufficient condition for an object to be exponentiable in the category Alg(U; V) of reflexive and transitive lax algebras. This specializes to known and new results for pretopological, preapproach and approach spaces.

INTRODUCTION

The category Top of topological spaces and continuous maps is undoubtedly the most important category in topology. However, many categorically defined constructions either cannot be carried out in Top or destroy properties of spaces or maps. In order to perform these constructions, topologists move (temporarily) outside Top into larger better behaved environments such as the category PrTop of pretopological spaces or the category PsTop of pseudotopological spaces. A pseudotopology on a set X is most easily described by a convergence relation \( \mathfrak{r} \rightarrow x \) between ultrafilters \( \mathfrak{r} \) on X and points \( x \in X \), where only the principal ultrafilter \( \mathfrak{x} \) is required to converge to \( x \). In [2] Barr showed that a pseudotopology is a topology exactly if it satisfies in addition the transitivity axiom

\[
\mathfrak{x} \rightarrow \mathfrak{r} \rightarrow x \Rightarrow m_X(\mathfrak{x}) \rightarrow x,
\]

for all \( x \in X \), \( \mathfrak{r} \in UX \) and \( \mathfrak{x} \in U^2X \). Here \( m_X \) is the X-component of the multiplication \( m \) of the ultrafilter monad \( U = (U, e, m) \), and the relation \( \rightarrow \) is naturally extended to a relation between \( U^2X \) and \( UX \). In fact, the latter defines an extension of the Set-monad U to a lax monad on the 2-category Rel of relations; and a topology is exactly a lax Eilenberg-Moore algebra for this extension. In [14] we observed that pretopologies can be described in a very

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similar way, namely as those convergence relations which satisfy, for all \( x \in X \) and \( \mathcal{X} \in U^2X \),

\[
\mathcal{X} \rightarrow x \quad \text{and} \quad \mathcal{X} \rightarrow \mathcal{X} \Rightarrow m_X(\mathcal{X}) \rightarrow x.
\]

After introducing the Kleisli composition \( a \ast b = a \cdot Ub \cdot m_X^\mathcal{X} \), for a relation \( a \) from \( UY \to Z \) and a relation \( b \) from \( UX \to Y \), we see that the latter axiom is equivalent to \( e_X^\mathcal{X} \ast a = a \). Here \( f^\circ \) denotes the inverse image relation of a function \( f : X \to Y \), that is, \( yf^\circ x \) if \( y = f(x) \). Since one always has \( a \ast e_X^\mathcal{X} = a \), we obtain that pretopologies are exactly those pseudotopologies for which \( e_X^\mathcal{X} \) is a unit for the Kleisli composition.

Combining the approach of [2] with [16], in several papers (see [6, 7, 9, 11], for instance) we develop the theory of lax Eilenberg-Moore algebras for extensions of \( \text{Set} \)-monads \( T \) to lax monads on bicategories of the form \( \text{Mat}(V) \), for a monoidal closed lattice (category) \( V \). This theory encompasses many important categories of topology: besides \( \text{Top} \) also the category \( \text{Met} \) of (generalized) metric spaces and the category \( \text{Ap} \) of approach spaces [17]. The description of pretopologies above opens the door to incorporate also this concept into our theory. In Section 2 we define the category \( \text{Alg}_u(U; V) \) of reflexive and \emph{unitary} lax algebras. We will show that, under mild conditions on \( V \), \( \text{Alg}_u(U; V) \) is the extensional topological hull of the category \( \text{Alg}(U; V) \) of reflexive and transitive lax algebras, generalizing known properties of the embeddings \( \text{Top} \hookrightarrow \text{PrTop} \) (see [13]) and \( \text{Ap} \hookrightarrow \text{PrAp} \) (see [19]).

The main purpose of this work is the study of exponentiability. Pisani [22] characterized exponentiable objects in \( \text{Top} \) by an interpolation property: a topological space \( X \) is exponentiable if and only if, for all \( \mathcal{X} \in U^2X \) and all \( x \in X \), \( m_X(\mathcal{X}) \rightarrow x \) implies the existence of \( \mathfrak{r} \in UX \) such that \( \mathcal{X} \rightarrow \mathfrak{r} \rightarrow x \). Hence for pretopological spaces we would expect that a space \( X \) is exponentiable if and only if, for all \( \mathcal{X} \in U^2X \) and all \( x \in X \), \( m_X(\mathcal{X}) \rightarrow x \) implies \( \mathcal{X} \rightarrow \mathfrak{r} \). It is not hard to see that this condition is equivalent to each point having a smallest neighborhood, which in turn is equivalent to the convergence structure being induced by a reflexive (binary) relation on \( X \). In [21] exponentiable pretopological spaces are characterized as those which satisfy these (equivalent) conditions. Motivated by this example, in Section 3 we formulate a condition and indeed prove that it characterizes exponentiable morphisms in \( \text{Alg}_u(U; V) \) (Theorem 3.5), under mild conditions on \( V \). Our result covers the characterization of exponentiable morphisms in \( \text{PrTop} \) obtained in [24], of exponentiable objects in \( \text{PrAp} \) obtained in [20], and it gives a new characterization of exponentiable morphisms in \( \text{PrAp} \).

The techniques used so far can be partially applied to transitive structures. Section 4 is devoted to this subject. Our main result here (Theorem 4.3) gives
a sufficient condition for an object in Alg(U; V) to be exponentiable. This theorem applies in particular to approach spaces.

1. Categories of lax algebras

1.1. Hypothesis. Throughout we assume that a complete lattice V equipped with a symmetric tensor product ⊗ is given. Moreover, this tensor has a unit element k and a right adjoint hom; that is, for each \( \alpha, \beta, \gamma \in V \),

\[ \alpha \otimes \beta \leq \gamma \iff \beta \leq \text{hom}(\alpha, \gamma). \]

Our two principal examples are the Boolean algebra \( 2 = \{ \text{false} \vdash \text{true} \} \) with tensor provided by “and” & and neutral element true; and the extended real half-line \( \mathbb{R}_+ = [0, \infty] \) ordered by the “greater or equal”-relation \( \geq \), with tensor defined by addition (where \( \infty + x = x + \infty = \infty \)) and 0 the neutral element. Its right adjoint is given by truncated minus: \( \text{hom}(x, y) = \max\{y - x, 0\} \).

1.2. V-matrices. The category Mat(V) of V-matrices (see [4, 11]) has sets as objects, and a morphism \( r : X \rightarrow Y \) in Mat(V) is a V-matrix \( r : X \times Y \rightarrow V \). Composition of V-matrices \( r : X \rightarrow Y \) and \( s : Y \rightarrow Z \) is defined as matrix multiplication

\[ s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z), \]

and the identity arrow \( 1_X : X \rightarrow X \) is the V-matrix which sends all diagonal elements \( (x, x) \) to \( k \) and all other elements to the bottom element \( \bot \) of V. The complete order of V induces a complete order on Mat(V)(X, Y): for V-matrices \( r, r' : X \rightarrow Y \) we define

\[ r \leq r' : \iff \forall x \in X \forall y \in Y \ r(x, y) \leq r'(x, y). \]

Hence Mat(V) is actually a 2-category. Moreover, it has an order-preserving involution \( \circ \) sending each \( r : X \rightarrow Y \) to its transpose \( r^\circ : Y \rightarrow X \), defined by \( r^\circ(y, x) = r(x, y) \). Returning to our main examples, we have \( \text{Mat}(2) \cong \text{Rel} \) while \( \text{Mat}(\mathbb{R}_+) \) is the 2-category whose morphisms \( a : X \rightarrow Y \) are generalized distances \( a : X \times Y \rightarrow \mathbb{R}_+ \) with composition given by

\[ b \cdot a(x, z) = \inf\{a(x, y) + b(y, z) \mid y \in Y\}; \]

\( 1_X : X \rightarrow X \) is the discrete distance sending the diagonal to 0 and all other pairs \( (x, x') \) to \( \infty \).

The category \textbf{Set} can be naturally embedded into Mat(V) by leaving objects unchanged and sending each map \( f : X \rightarrow Y \) to the V-matrix

\[ f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases} \]
In the sequel we will write \( f : X \rightarrow Y \) rather then \( f : X \twoheadrightarrow Y \) for a \( \mathbf{V} \)-matrix induced by a \( \mathbf{Set} \)-map in the sense above. We remark that each \( f : X \rightarrow Y \) satisfies the inequalities \( \text{id}_X \leq f^\circ \cdot f \) and \( f \cdot f^\circ \leq \text{id}_Y \), i.e. \( f \) is left adjoint to \( f^\circ \).

1.3. The ultrafilter monad. Lax algebras are defined relative to a lax extension of a \( \mathbf{Set} \)-monad \( T = (T, e, m) \) to \( \mathbf{Mat(V)} \). In this article we concentrate on the identity monad \( 1 = (\text{id}, \text{id}, \text{id}) \) and the ultrafilter monad \( U = (U, e, m) \).

Recall that the latter monad is induced by the dual adjunction

\[
\text{Bool} \xleftarrow{\eta} \mathbf{hom(2)} \xrightarrow{\varepsilon} \mathbf{Set}.
\]

Explicitly, the ultrafilter functor \( U : \mathbf{Set} \rightarrow \mathbf{Set} \) assigns to each set \( X \) the set \( UX \) of ultrafilters on \( X \) and to each function \( f : X \rightarrow Y \) the function \( Uf : UX \rightarrow UY \) which takes an ultrafilter \( x \in UX \) to the (ultra)filter generated by its \( f \)-image \( \{f[A] \mid A \in x\} \). The natural transformations \( e : \text{id} \rightarrow U \) and \( m : U^2 \rightarrow U \) are given by

\[
e_X(x) = \hat{x} = \{A \subset X \mid x \in A\} \quad \text{and} \quad m_X(x) = \{A \subset X \mid A^\# \in x\},
\]

for all sets \( X, x \in U^2X \) and \( x \in X \). Here \( A^\# \) denotes the set \( \{a \in UX \mid A \in a\} \).

In the sequel we will extend this notation to filters and write \( f^\# \) for \( \{A^\# \mid A \in f\} \). The set of all proper filters on a set \( X \) we will denote by \( FX \).

1.4. Extending to \( \mathbf{Mat(V)} \). By a lax extension of a monad \( T = (T, e, m) \) on \( \mathbf{Set} \) to \( \mathbf{Mat(V)} \) we mean an extension of the endofunctor \( T : \mathbf{Set} \rightarrow \mathbf{Set} \) to \( \mathbf{Mat(V)} \) which satisfies

\[
(1) \quad Tb \cdot Ta \leq T(b \cdot a),
(2) \quad a \leq a' \Rightarrow Ta \leq Ta',
(3) \quad e_Y \cdot a \leq Ta \cdot e_X,
(4) \quad m_Y \cdot T^2a \leq Ta \cdot m_X,
(5) \quad (Ta)^\circ = T(a^\circ) \quad \text{(and we write } Ta^\circ)\),
\]

for all \( a, a' : X \twoheadrightarrow Y \) and \( b : Y \twoheadrightarrow Z \). Note that we have automatically equality in (1) if \( a = f \) is a \( \mathbf{Set} \)-map. The identity monad \( 1 = (\text{id}, \text{id}, \text{id}) \) on \( \mathbf{Set} \) can be obviously “extended” to the identity monad on \( \mathbf{Mat(V)} \). In [7, 11] are shown equivalent ways how to extend \( U \) to \( \mathbf{Mat(V)} \) provided that \( V \) is \( \sqsubseteq \)-atomic. By that we mean that there is a binary relation \( \sqsubseteq \) on \( V \) which satisfies

\[
\bullet \quad \alpha \sqsubseteq \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma \quad \text{and}
\bullet \quad \alpha = \bigvee\{\alpha_0 \in V \mid \alpha_0 \sqsubseteq \alpha, \alpha_0 \text{ is a } \sqsubseteq \text{-atom}\},
\]

for all \( \alpha, \beta, \gamma \in V \). Here an element \( \alpha_0 \in V \) is called an \( \sqsubseteq \)-atom if, for each \( S \subset V \), \( \alpha_0 \sqsubseteq \bigvee S \) implies \( \alpha_0 \leq s \) for some \( s \in S \). We will write \( \text{At}(\alpha) \) for the
set of all atoms $\alpha_0 \supseteq \alpha$. It is easy to see that $2$ is $\leq$-atomic and $\mathbb{R}_+$ is $\triangleright$-atomic.

For $a : X \rightarrow Y$, $r \in U X$ and $\eta \in U Y$ one defines

$$U a(r, \eta) := \bigwedge_{A \in \mathcal{F} \atop B \in \mathcal{G} \atop y \in B} a(x, y) = \bigvee \{ \gamma \in V \mid r(U a_\gamma) \eta \}.$$ 

In the latter formula $a_\gamma$ denotes the relation defined by

$$x a_\gamma y \iff a(x, y) \geq \gamma;$$

and for a relation $r : X \rightarrow Y$ and ultrafilters $r \in U X$ and $\eta \in U Y$ we define

$$r(U r) \eta : \iff r[x] \subset \eta \iff r^a[\eta] \subset r.$$ 

In [7] it is shown that it holds indeed $\bigwedge_{A \in \mathcal{F}} \bigwedge_{x \in A} a(x, y) = \bigvee \{ \gamma \in V \mid r(U a_\gamma) \eta \}$, for all ultrafilters $r \in U X$ and $\eta \in U Y$. The argument used there still works when we substitute $\eta$ by a filter base $f$, hence we also have $\bigwedge_{A \in \mathcal{F} \atop B \in \mathcal{G} \atop y \in B} a(x, y) = \bigvee \{ \gamma \in V \mid a_\gamma^2[f] \subset r \}$. Besides the properties listed above, this extension satisfies

(6) $Ub \cdot Ua = U(b \cdot a)$ provided that $\otimes = \wedge$,
(7) $Ug \cdot Ua = U(g \cdot a)$ and $Ub \cdot Uf = U(b \cdot f)$,
(8) $m_Y \cdot U^2a = Ua \cdot m_X$,

for all $a : X \rightarrow Y$, $b : Y \rightarrow Z$, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. We remark that the last property is implied by the following useful fact: for each map $f : X \rightarrow Y$, $r \in U X$ and $\mathcal{G} \in U^2 Y$ with $U f(r) = m_Y(\mathcal{G})$, there exists $x \in U^2 X$ such that $m_X(x) = r$ and $U^2 f(x) = \mathcal{G}$.

Thought a monad might have more than one extension to Mat($V$) (see [9]), in this article we only consider the extension of $U$ (resp. 1) mentioned in this subsection.

1.5. Extension Lemma. This Lemma plays an essential role in many proofs using ultrafilters (see [22], for instance). It basically states that if a filter is in relation with an ultrafilter, then this filter can be extended to an ultrafilter with the same property. We shall now extend this lemma to $V$-matrices, provided that $V$ fulfils an additional condition. Concretely, we require that the atoms $\alpha \sqsubseteq \gamma$ below an element $\gamma$ behave well, either collectively – $At(\gamma)$ is up-directed – or individually – each $\alpha \in At(\gamma)$ is connected. Here a subset $A \subset V$ is called up-directed if any two elements of $A$ have an upper bound. An element $\alpha \in V$ is connected if $\alpha \leq \beta_1 \lor \beta_2$ implies $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.

Lemma. Let $a : X \times Y \rightarrow V$, $r \in U X$ and $f \in F Y$. Moreover, assume that $V$ is $\sqsubseteq$-atomic and that $\alpha \in V$ is such that

$$\alpha \leq \bigvee \{ \gamma \in V \mid a_\gamma^2[f] \subset r \} = \bigwedge_{A \in \mathcal{F} \atop B \in \mathcal{G} \atop y \in B} a(x, y),$$

...
with $\alpha$ connected or $\operatorname{At}(\alpha)$ up-directed. Then there exists an ultrafilter $\eta \in UY$ such that $\emptyset \subset \eta$ and $Ua(\emptyset, \eta) \geq \alpha$.

Proof. If $\alpha = \bot$, we choose any ultrafilter $\eta$ containing $\emptyset$. Assume $\alpha \neq \bot$. We define an ideal $j$ on $Y$ by

$$j := \{B \subset Y \mid \exists \alpha_0 \in \operatorname{At}(\alpha) \ a^{\circ}_{\alpha_0}[B] \notin \emptyset \} = \{B \subset Y \mid \exists A \in \emptyset \bigvee_{x \in A} \ a(x, y) \notin \alpha \}.$$  

Obviously, $j \cap \emptyset = \emptyset$, $\emptyset \in j$ and $j$ is down-directed. Let $B_1, B_2 \in j$. We consider first $\operatorname{At}(\alpha)$ up-directed. There exist $\alpha_1, \alpha_2 \in \operatorname{At}(\alpha)$ such that

$$a^{\circ}_{\alpha_1}[B_1] \notin \emptyset \quad \text{and} \quad a^{\circ}_{\alpha_2}[B_2] \notin \emptyset.$$  

Let $\alpha_0$ be any upper bound of $\alpha_1$ and $\alpha_2$. Then

$$a^{\circ}_{\alpha_0}[B_1 \cup B_2] = a^{\circ}_{\alpha_0}[B_1] \cup a^{\circ}_{\alpha_0}[B_2] \subset a^{\circ}_{\alpha_1}[B_1] \cup a^{\circ}_{\alpha_2}[B_2] \notin \emptyset$$

and therefore $B_1 \cup B_2 \in j$.

Assume now that $\alpha$ is connected. We choose $A_1$ and $A_2$ with $\bigvee_{x \in A_i} a(x, y) \notin \alpha$, for $i \in \{1, 2\}$. Connectedness of $\alpha$ implies

$$\bigvee_{x \in A_1} a(x, y) \lor \bigvee_{y \in B_1} a(x, y) \notin \alpha.$$  

From

$$\bigvee_{y \in B_1} a(x, y) \leq \bigvee_{x \in A_1} a(x, y) \lor \bigvee_{x \in A_2} a(x, y)$$

it follows $\bigvee_{y \in B_1 \cup B_2} a(x, y) \notin \alpha$, hence $B_1 \cup B_2 \in j$.

The Prime Ideal Theorem (see Section 2 of [15]) guarantees the existence of an ultrafilter $\eta \in UY$ such that $\emptyset \subset \eta$ and $\eta \cap j = \emptyset$, that is, $Ua(\emptyset, \eta) \geq \alpha$. □

Corollary. Assume that $V$ is $\sqsubset$-atomic and that, for each $\gamma \in V$, $\operatorname{At}(\gamma)$ is up-directed or each $\gamma_0 \in \operatorname{At}(\gamma)$ is connected. Let $a : X \times Y \to V$, $\emptyset \in UX$ and $\emptyset \in FY$. Then, for each $\alpha \in \operatorname{At}(\bigvee \{\gamma \in V \mid a^{\circ}_{\gamma}[\emptyset] \subset \emptyset\})$, there exists an ultrafilter $\eta_\alpha$ such that $\emptyset \subset \eta_\alpha$ and $Ua(\emptyset, \eta_\alpha) \geq \alpha$.

For $V = 2$ or $V = 2$, both possible conditions above are fulfilled. Note that $V = 2$ is a special case of $V = 2^X$, the power set of a set $X$, ordered by inclusion $\subset$ with $\emptyset = \emptyset$. Then $2^X$ is $\subset$-atomic, each atom is connected but in general $\operatorname{At}(A)$ is not up-directed.
1.6. **Lax algebras.** Let $T$ be a Set-monad laxly extended to Mat($V$). Our main interest lies in the study of categories with objects lax algebras $(X,a)$ – that is: sets $X$ equipped with a structure $a : TX \to X$ in Mat($V$) – and lax homomorphisms $f : (X,a) \to (Y,b)$ between them, i.e. maps $f : X \to Y$ satisfying $f \cdot a \leq b \cdot Tf$. A $V$-matrix $a : TX \to X$ is called

1. reflexive if $1_X \leq a \cdot e_X$,
2. transitive if $a \cdot Ta \leq a \cdot m_X$.

Note that these axioms are just lax versions of the Eilenberg-Moore axioms of a $T$-algebra. Expressed componentwise, they read as

1'. $k \leq a(e_X(x),x)$,
2'. $Ta(x,x) \otimes a(x,x) \leq a(m_X(x),x)$,

for all $X \in U^2$, $x \in UX$ and $x \in X$. In the sequel we shall say that a lax algebra $(X,a)$ is reflexive, transitive, ... , if $a$ is so. We denote by Alg($T; V$) the category of lax algebras and lax homomorphisms, by Alg($T,e; V$) its full subcategory of reflexive lax algebras and by Alg($T; V$) its full subcategory of reflexive and transitive lax algebras. If $T = I$, a reflexive lax algebra is a reflexive $V$-graph and a reflexive and transitive lax algebra is a $V$-category; moreover, a lax homomorphism is a $V$-functor (see [16]). We write $V$-RGph instead of Alg(Id, id; $V$) and $V$-Cat instead of Alg(1; $V$). Note that $V$ equipped with the $V$-matrix hom : $V \times V \to V$ is a $V$-category.

We obtain full embeddings Alg($T; V$) $\hookrightarrow$ Alg($T,e; V$) $\hookrightarrow$ Alg($T; V$). All three categories are topological (in the sense of [1]) over Set with respect to their canonical forgetful functor, and in each case the initial structure $a$ on a set $X$ with respect to a source $(f_i : X \to (X_i, a_i))_{i \in I}$ is given by

$$a = \bigwedge_{i \in I} f_i^* \cdot a_i \cdot Tf_i.$$ 

Therefore we see that all inclusion functors above have a left adjoint. We also remark that $T$ extends to a functor $T : \text{Alg}(T; V) \to \text{Alg}(T; V)$ which preserves initial morphisms $f : (X,a) \to (Y,b)$ since $T(f^* \cdot b \cdot Tf) = Tf^* \cdot Tb \cdot T^2 f$.

1.7. **Examples.** A 2-category is just a preordered set and a 2-functor is an order-preserving map, whereby an $\mathbb{R}_+$-category is a (generalized) metric space and an $\mathbb{R}_+$-functor is a non-expansive map.

In [2] Barr describes the topological convergence relations $x \to x$ between ultrafilters $x$ on a set $X$ and points $x \in X$ as exactly those relations which satisfy the lax Eilenberg-Moore axioms for the extension of $U$ to $\text{Rel} \cong \text{Mat}(2)$. In other words: Alg($U; 2$) $\cong$ Top.

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1that is: $a$ is initial for $f : X \to (Y,b)$
Approach spaces were introduced by Lowen (see [17]) as a common framework for both topological and metric structures. Being precise, an approach space is a pair \((X, \delta)\) consisting of a set \(X\) and a function \(\delta : 2^X \times X \to \mathbb{R}_+\) subject to

1. \(\delta(\{x\}, x) = 0,\)
2. \(\delta(\emptyset, x) = \infty,\)
3. \(\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\},\)
4. \(\delta(A, x) \leq \delta(A^{(\varepsilon)}, x) + \varepsilon;\)

for each \(A, B \subset X, x \in X\) and \(\varepsilon \in \mathbb{R}_+.\) Here \(A^{(\varepsilon)} = \{x \in X \mid \delta(A, x) \leq \varepsilon\}.\) A function \(\delta : 2^X \times X \to \mathbb{R}_+\) satisfying the axioms above is called approach distance on \(X.\) If \(\delta\) takes only the values 0 and \(\infty,\) these are the axioms of a topological closure operator considering \(A^{(0)} = A.\) For \(\delta : 2^X \times X \to \mathbb{R}_+\) and \(\delta' : 2^Y \times Y \to \mathbb{R}_+,\) a map \(f : X \to Y\) is called non-expansive if \(\delta(A, x) \geq \delta'(f[A], f(x))\) for each \(A \subset X\) and \(x \in X.\) Approach spaces and non-expansive maps are the objects and morphisms of the category \(\text{Ap}.\) Each \(\delta : 2^X \times X \to \mathbb{R}_+\) defines a map \(a : U X \times X \to \mathbb{R}_+\) by

\[a(x, x) = \sup_{A \in 2^x} \delta(A, x),\]

and vice versa, each \(a : U X \times X \to \mathbb{R}_+\) defines a function \(\delta : 2^X \times X \to \mathbb{R}_+\) by

\[\delta(A, x) = \inf_{A \in U} a(x, x).\]

In [6] it is shown that the functions \(a : U X \times X \to \mathbb{R}_+\) coming from a approach distance are precisely those satisfying the lax Eilenberg-Moore axioms for the lax extension of \(U\) to \(\text{Mat}(\mathbb{R}_+).\) Hence we have \(\text{Alg}(U; \mathbb{R}_+ \cong \text{Ap}.\)

1.8. Kleisli composition. Given a lax extension of a \(\text{Set}\)-monad \(T\) to \(\text{Mat}(V),\) the Kleisli “category” has as objects sets and a morphism from \(X\) to \(Y\) is a \(V\)-matrix \(a : TX \twoheadrightarrow Y.\) Composition is given by Kleisli composition

\[b \ast a := b \cdot Ta \cdot m_X^a,\]

for all \(b : TY \to Z\) and \(a : TX \to Y.\) For each \(a : TX \to Y\) it holds \(a \ast e_X^a = a\) and \(e_Y^a \ast a \geq a,\) that is, the \(V\)-matrix \(e_X^a\) is a lax identity for this composition. Moreover, we have

\[c \ast (b \ast a) \leq (c \ast b) \ast a\]

or

\[c \ast (b \ast a) \geq (c \ast b) \ast a\]
provided that $T : \text{Mat}(V) \rightarrow \text{Mat}(V)$ is a functor or $m$ extends to a (strict) natural transformation respectively. Of course, for $T = 1$ the Kleisli category coincides with $\text{Mat}(V)$. Reflexivity and transitivity of $a : TX \rightarrow X$ can be now equivalently expressed by the inequalities

$$e_X^a \leq a, \quad a \ast a \leq a.$$ 

1.9. Elementary structures. To derive properties of a lax algebra $(X, a)$, it will be often useful to “isolate” values $\gamma = a(\mathfrak{r}, x)$. For the ultrafilter monad $U$ and $\sqsubseteq$-atomic $V$, this can be done using the following structure. Let $X$ be a set and $\mathfrak{r} \in X$, $x \in X$ and $\gamma \in V$. We put $Y = X_1 + X_2$, where $X_1 = X_2 = X$, and define a structure $a_{\mathfrak{r}, x}^\gamma : UY \times Y \rightarrow V$ by putting

$$a_{\mathfrak{r}, x}^\gamma(y, y) = \begin{cases} k & \text{if } \eta = \hat{y}, \\ \gamma & \text{if } \eta = \mathfrak{r} \in UX_1 \text{ and } y = x \in X_2, \\ \bot & \text{else}, \end{cases}$$

for each $\eta \in UY$ and $y \in Y$. In this subsection we will always consider $\mathfrak{r}$ as an ultrafilter on the component $X_1$ of $Y$ and $x \in X_2$. It is clear from the definition that $a_{\mathfrak{r}, x}^\gamma$ is reflexive. We are now going to show that it is also transitive. In order to see that, let $Y \in U2Y$, $\eta \in UY$ and $y \in Y$. We consider first the case $\eta = \hat{y}$. If

$$Ub(\mathcal{Y}, \hat{y}) = \bigwedge_{B \in \mathcal{Y}} \bigvee_{b \in B} b(b, y) > \bot,$$

it follows $\{\hat{y}, \mathfrak{r}\} \in \mathcal{Y}$ which implies $\mathcal{Y} = \hat{y} \text{ or } \mathcal{Y} = \mathfrak{r}$. Note that the latter case is only possible if $y = x$. Hence in both cases we have

$$Ub(\mathcal{Y}, \hat{y}) = b(m_Y(\mathcal{Y}), y).$$

We now consider $\eta = \mathfrak{r}$ and $y = x$. Here

$$Ub(\mathcal{Y}, \mathfrak{r}) = \bigwedge_{B \in \mathcal{Y}} \bigvee_{b \in B} b(b, z) > \bot$$

implies $\mathcal{Y} = Ue_Y(x)$ and therefore

$$Ub(\mathcal{Y}, \mathfrak{r}) \otimes b(\mathfrak{r}, x) = k \otimes \gamma = \gamma = b(m_Y(\mathcal{Y}), y).$$

2. Unitary structures

2.1. Pretopological spaces. Pretopological spaces were essentially introduced by Choquet in [5]; for more informations see also [3, 13]. A pretopology $a$ on a set $X$ is a relation $\mathfrak{r} \rightarrow x$ between ultrafilters and points which fulfils, for each $x \in X$ and $\mathfrak{r} \in UX$,

$$\mathfrak{r} \rightarrow x \quad \text{and} \quad \mathfrak{r} \rightarrow x \quad \text{whenever} \quad \bigcap_{y \rightarrow x} \eta \subseteq \mathfrak{r}.$$
This can be equivalently expressed by saying that to each point \( x \in X \) is associated a neighborhood filter \( \mathcal{V}(x) \subset \mathcal{B} \) and defining \( x \to x \) precisely if \( \mathcal{V}(x) \subset \mathcal{B} \). A pretopological space is a set \( X \) equipped with a pretopology; together with continuous maps they form the category \( \text{PrTop} \). Recall that the Zariski closure on \( UX \) is defined by \( x \in A \iff \bigcap A \subset x \), for all \( x \in UX \) and \( A \subset UX \). Hence the second condition above just states that the set \( \{ y \in UX \mid y \to x \} \) is closed in \( UX \) with respect to the Zariski closure. In [14] we observed that this is equivalent to \( e_X^\circ a = a \).

2.2. Preapproach spaces \([18, 19]\). A preapproach limit \( a \) on a set \( X \) is a function \( a : UX \times X \to \mathbb{R}_+ \) which fulfills, for each \( x \in X \) and \( \xi \in UX \),

\[
a(\xi, x) = 0 \quad \text{and} \quad a(\xi, x) \leq \sup_{i \in I} a(\eta_i, x) \quad \text{whenever} \quad \bigcap_{i \in I} \eta_i \subset \xi.
\]

A preapproach space is a pair \( (X, a) \) consisting of a set \( X \) and a preapproach limit \( a \) on \( X \). \( \text{PrAp} \) denotes the category of preapproach spaces and non-expansive maps. As in the case of pretopological spaces, the second axiom is equivalent to \( e_X^\circ a = a \).

2.3. Definition and basic properties. Guided by the two previous examples, we introduce the following

**Definition.** A \( \mathbb{V} \)-matrix \( a : TX \twoheadrightarrow X \) is called unitary provided that \( e_X^\circ a = a \).

Therefore a unitary \( \mathbb{V} \)-matrix \( a : TX \twoheadrightarrow X \) satisfies \( e_X^\circ a = a \). We let \( \text{Alg}_u(T; \mathbb{V}) \) denote the category of reflexive and unitary lax algebras and lax homomorphisms. The two examples above tell us that \( \text{Alg}_u(U, 2) \cong \text{PrTop} \) and \( \text{Alg}_u(U, \mathbb{R}_+) \cong \text{PrAp} \). The canonical forgetful functor \( | \cdot | : \text{Alg}_u(T; \mathbb{V}) \to \text{Set} \) is topological: the initial structure \( a \) on \( X \) with respect to a source \( (f_i : X \to (X_i, a_i))_{i \in I} \) is also given by

\[
a = \bigwedge_{i \in I} f_i^\circ a_i \cdot T f_i.
\]

We infer that the full embeddings \( \text{Alg}(T; \mathbb{V}) \hookrightarrow \text{Alg}_u(T; \mathbb{V}) \) and \( \text{Alg}_u(T; \mathbb{V}) \hookrightarrow \text{Alg}(T, e; \mathbb{V}) \) are right adjoints. In the latter case the reflector \( R \) has a simple description provided that \( T \) preserves composition of \( \mathbb{V} \)-matrices with \( \text{Set} \)-maps: \( R(X, a) = (X, e_X^\circ a) \).

2.4. Embedding \( \mathbb{V}-\text{Cat} \). Assume that \( \mathbb{V} \) is \( \square \)-atomic and let \( X \) be a set. We define order-preserving maps

\[
\begin{array}{ccc}
\{ r : X \twoheadrightarrow X \} & \xleftarrow{\phi} & \{ a : UX \twoheadrightarrow X \} \\
\end{array}
\]
by putting
\[ \phi(r) = e^\phi_X \cdot Ur \quad \text{and} \quad \psi(a) = a \cdot e_X. \]

Componentwise, this translates to
\[ \phi(r)(x, x) = \bigwedge_{A \in \Gamma} \bigvee_{y \in A} r(y, x) \quad \text{and} \quad \psi(a)(y, x) = a(y, x). \]

It follows that \( r = \psi \phi(r) \) for each \( r : X \to X \). For each unitary \( a \) we have \( a \geq \phi \psi(a) \): \( \phi \psi(a) = e^\phi_X \cdot Ua \cdot Ue_X \leq e^\phi_X \cdot Ua \cdot m^\phi_X = e^\phi_X \cdot a = a \). Further, for all \( r : X \to X \), \( r \in UX \), \( x \in X \) and \( \gamma \in V \),
\[ \phi(r)(x, x) \geq \gamma \iff \forall \gamma_0 \in At(\gamma) \ r^\gamma_{\gamma_0}(x) \in \mathfrak{x} \iff \forall \gamma_0 \in At(\gamma) \ x \phi(r_{\gamma_0})(x). \]

Hence we have \( \phi(r) = \bigcap_{\gamma_0 \in At(\gamma)} \phi(r_{\gamma_0}) \), and therefore \( \phi(r_\gamma) \leq \phi(r) \). We also obtain \( \phi(s \wedge r) = \phi(s) \wedge \phi(r) \) for all \( s, r : X \to X \): Since \( \phi \) preserves the order, we have \( \phi(s \wedge r) \leq \phi(s) \wedge \phi(r) \). Let \( r \in UX \), \( x \in X \) and \( \gamma \in At(\phi(s) \wedge \phi(r))(x, x) \). We have \( s^\gamma_\gamma(x) \in \mathfrak{x} \) and \( r^\gamma_\gamma(x) \in \mathfrak{x} \) and then
\[ (s \wedge r)^\gamma_\gamma(x) = s^\gamma_\gamma(x) \cap r^\gamma_\gamma(x) \in \mathfrak{x}. \]

**Lemma.** For each \( r, s : X \to X \) and \( a : UX \to X \),
1. \( a \cdot m_X \leq e^\phi_X \cdot Ua \) implies \( a \leq \phi \psi(a) \).
2. \( \phi(s) \cdot U \phi(r) \leq \phi(s \cdot r) \cdot m_X \).
3. \( e^\phi_X \cdot U \phi(r) = \phi(r) \cdot m_X \).
4. \( \phi(s) \cdot U \phi(r) = \phi(s \cdot r) \cdot m_X \) provided that \( \otimes = \wedge \) and, for each \( \gamma \in V \),
\( At(\gamma) \) is up-directed or each \( \gamma_0 \in At(\gamma) \) is connected.

**Proof.** (1) From \( a \cdot m_X \leq e^\phi_X \cdot Ua \) we obtain
\[ a = a \cdot m_X \cdot Ue_X \leq e^\phi_X \cdot Ua \cdot Ue_X = \phi \psi(a). \]

(2) \( \phi(s) \cdot U \phi(r) = e^\phi_X \cdot Us \cdot Ue^\phi_X \cdot U^2 r \)
\[ \leq e^\phi_X \cdot Us \cdot m_X \cdot U^2 r \]
\[ = e^\phi_X \cdot Us \cdot Ur \cdot m_X \]
\[ \leq e^\phi_X \cdot U(s \cdot r) \cdot m_X = \phi(s \cdot r) \cdot m_X. \]

(3) Let \( X \in U^2 X \), \( x \in X \) and \( \gamma_0 \in At(\phi(r)(m_X(X), x)) \). It holds \( r^\gamma_{\gamma_0}(x) \in m_X(X) \), and therefore \( r^\gamma_{\gamma_0}(x) \# \in X \). Consequently, each \( A \in X \) contains some \( a \) with \( \phi(r)(a, x) \geq \gamma_0 \). We obtain \( Ua(X, x) \geq \gamma_0 \).

(4) We assume now that \( \otimes = \wedge \) and that \( At(\gamma) \) is up-directed or each \( \gamma_0 \in At(\gamma) \) is connected, for each \( \gamma \in V \). Let \( X \in U^2 X \), \( x \in X \) and \( \gamma_0 \in At(\phi(s \cdot r)(m_X(X), x)) \). As above we have \( (s \cdot r)^\gamma_{\gamma_0}(x) \# \in X \). Hence for each \( \gamma_1 \in At(\gamma_0) \) it holds \( (r^\gamma_{\gamma_1} \cdot s^\gamma_{\gamma_1})(x) \# \in X \). Consequently, for each \( A \in X \), there exist some \( a \in A \) and \( y \in s^\gamma_{\gamma_1}(x) \) such that \( \phi(r)(a, y) \geq \gamma_1 \). Corollary 1.5 implies that, for each
\[ \alpha \in \text{At}(\gamma_1), \] there exists \( r_\alpha \in UX \) with \( U\phi(r)(X, r_\alpha) \geq \alpha \) and \( \phi(s)(r_\alpha, x) \geq \gamma_1 \). Therefore \( \phi(s) \cdot U\phi(r)(X, x) \geq \gamma_1 \).

The lemma above implies that \( \phi \) is a lax homomorphism with respect to the Kleisli composition:

\[ \phi(\text{id}_X) = e_X^\circ \quad \text{and} \quad \phi(s) \cdot \phi(r) \leq \phi(s \cdot r). \]

It is not hard to see that the same is true for \( \psi \):

\[ \psi(e_X^\circ) = \text{id}_X \quad \text{and} \quad \psi(b) \cdot \psi(a) \leq \psi(b \ast a). \]

We conclude that \( \phi(r) \) is unitary and \( \phi(r) \) resp. \( \psi(a) \) is reflexive or transitive if \( r \) resp. \( a \) is so. Hence \( \phi \) and \( \psi \) induce pairs of adjoint functors

\[ \mathbf{V-RGph} \rightleftarrows \mathbf{Alg}_d(U; V) \quad \text{and} \quad \mathbf{V-Cat} \rightleftarrows \mathbf{Alg}(U; V), \]

where the left adjoint inclusion functor preserves finite limits. In particular, \( \mathbf{V} \) becomes an \( \mathbf{Alg}(U; V) \)-object \( (V, h) \) with \( h := \phi(hom) \). Finally, we can characterize those structures \( a : UX \to X \) coming from a \( \mathbf{V} \)-matrix \( r : X \to X \):

\[ a = \phi\psi(a) \iff a \cdot m_X = e_X^\circ \cdot Ua. \]

2.5. **Extensional topological hulls.** Among other results it is shown in [13] that \( \text{PrTop} \) is the extensional topological hull of \( \text{Top} \), that is, the smallest extensional topological category containing \( \text{Top} \) nicely. We present here the results and definitions necessary for this paper; for more information we refer to [13].

Let \( A \) be a category and \( S \) be a pullback-stable class of morphisms of \( A \). A \( S \)-partial map from \( X \) to \( Y \) is a pair \( (X \leftarrow Z \rightarrow Y) \) where \( s \in S \). We say that \( A \) has \( S \)-partial map classifiers if, for every \( Y \in A \), there is an \( S \)-morphism \( \text{true}_Y : Y \to Y^* \) such that every \( S \)-partial map \( (X \leftarrow Z \rightarrow Y) \) from \( X \) to \( Y \) can be uniquely completed so that the diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow s & & \downarrow \text{true}_Y \\
X & \longrightarrow & Y^*
\end{array}
\]

is a pullback. If \( A \) is a construct (i.e., \( A \) is equipped with a forgetful functor \( |\cdot| : A \to \text{Set} \)) and \( S = \{\text{embeddings}\} \), we simply say partial map instead of \( S \)-partial map. A topological construct which has partial map classifiers is called extensional. By an extensional topological hull of a construct \( A \) we understand a full concrete embedding \( E : A \hookrightarrow B \) such that \( B \) is an extensional topological construct and \( E \) is finally dense, and for each full, concrete, finally dense embedding \( F : A \to B' \) with \( B' \) extensional there exists a unique full concrete embedding \( G : B \to B' \) with \( G \cdot E = F \). There is, up to isomorphism,
at most one extensional topological hull of a category \( A \). It is characterized by the following theorem.

**Theorem.** The extensional topological hull \( B \) of a construct \( A \) is characterized by the following properties:

1. \( B \) is an extensional topological construct.
2. \( A \) is a finally dense full concrete subcategory of \( B \).
3. \( \{ X^* \mid X \in A \} \) is initially dense in \( B \).

We will now use the theorem above to show that \( \text{Alg}_u(U; V) \) is the extensional topological hull of \( \text{Alg}(U; V) \), provided that \( V \) satisfies further properties.

### 2.6. One-point extensions

Assume that \( V \) is \( \sqsubseteq \)-atomic and let \((X, a) \in \text{Alg}_u(U; V)\). We put \( X^* = X + \{ * \} \) and extend \( a \) to \( X^* \) by defining

\[
a(x, * ) = \top, \quad \text{and} \quad a(\ast, x) = \top,
\]

for all \( x \in UX \) and \( x \in X^* \). It is easy to see that \((X^*, a)\) is again unitary and that \((X, a) \leftrightarrow (X^*, a)\) classifies partial morphisms to \((X, a)\). We conclude that \( \text{Alg}_u(U; V) \) is extensional.

### 2.7. Initially dense object

Assume that \( V \) is \( \sqsubseteq \)-atomic and \( k = \top \). Let \((X, a) \in \text{Alg}_u(U; V)\). For each \( A \subseteq X \) and \( x \in X \) we define a map \( g_{A,x} : X \rightarrow V^* \) by putting

\[
g_{A,x}(y) = \begin{cases} \bigvee_{A \in a} a(a, y) & \text{if } y \in A \cup \{x\}, \\ * & \text{else;} \end{cases}
\]

for each \( y \in X \). We are going to show that \( g_{A,x} \) is a lax homomorphism. If \( \bigvee_{A \in a} a(a, x) = k \), then \( g_{A,x}[X] \subset \{k, *\} \) and hence \( g_{A,x} \) is a lax homomorphism. Otherwise, put \( \gamma := \bigvee_{A \in a} a(a, x) < k \) and let \( \eta \in UX \) and \( y \in X \). We only have to consider the case \( U g_{A,x}(\eta) = k \) and \( g_{A,x}(y) = \gamma \), that is: \( A \in \eta \) and \( y = x \). Hence we have

\[
k \otimes a(\eta, x) = a(\eta, x) \leq g_{A,x}(x)
\]

from which follows

\[
a(\eta, x) \leq \text{hom}(k, g_{A,x}(x)) = h(U g_{A,x}(\eta), g_{A,x}(x)).
\]

**Lemma.** Assume that \( k = \top \) and, for each \( \gamma \in V \), \( \text{At}(\gamma) \) is up-directed or each \( \gamma_0 \in \text{At}(\gamma) \) is connected. For each \((X, a) \in \text{Alg}_u(U; V), x \in UX\) and \( x \in X \),

\[
a(x, x) = \bigwedge_{A \in x} h(U g_{A,x}(x), g_{A,x}(x)).
\]
Proof. Note that $Ug_{A,x}(\overline{x}) = \overline{k}$ for each $\overline{x} \in A^\#$, hence we have to show

$$a(\overline{x}, x) \geq \bigwedge_{A \in \mathcal{F}} g_{A,x}(x) = \bigwedge_{A \in \mathcal{F}} \bigvee_{\alpha \in A^\#} a(\alpha, x) =: \gamma.$$ 

Applying Corollary 1.5 to $\overline{x}^#$ and $\overline{\cdot}^x$, for each $\alpha \in \text{At}(\gamma)$ there exists an ultrafilter $\mathcal{X}_\alpha \in U^2X$ such that $\overline{x}^# \subset \mathcal{X}_\alpha$ and $\alpha \leq Ua(\mathcal{X}_\alpha, \overline{\cdot})$. Since $a$ is unitary, we have $\gamma \leq a(\overline{x}, x)$. □

Corollary. If $k = \top$ and, for each $\gamma \in V$, $\text{At}(\gamma)$ is up-directed or each $\gamma_0 \in \text{At}(\gamma)$ is connected, then $(V^*, h)$ (see end of 2.4 and 2.6) is initially dense in $\text{Alg}_u(U; V)$.

2.8. Theorem. $\text{Alg}_u(U; V)$ is the extensional topological hull of $\text{Alg}(U; V)$ provided that $k = \top$, $V$ is $\sqsubseteq$-atomic and, for each $\gamma \in V$, $\text{At}(\gamma)$ is up-directed or each $\gamma_0 \in \text{At}(\gamma)$ is connected.

Proof. Combine 1.9, 2.6 and 2.7 and apply Theorem 2.5. □

3. Exponentiable morphisms in $\text{Alg}_u(U; V)$

3.1. Partial products. Recall that, by definition, a morphism $f : X \to Y$ in a finitely complete category $X$ is exponentiable if the pullback functor

$$X \times_Y - : X/Y \to X/Y$$

has a right adjoint. In [12] it is shown that this is equivalent to the existence of partial products over $f$, that is, for each $Z \in X$ there is a diagram

$$
\begin{array}{ccc}
Z & \overset{\text{ev}}{\leftarrow} & P \times_Y X \\
& \pi_1 \downarrow & \pi_2 \downarrow \\
& P & X \\
\end{array}
\xrightarrow{f} 
\begin{array}{ccc}
P & \overset{p}{\rightarrow} & Y
\end{array}
$$

such that for every diagram

$$
\begin{array}{ccc}
Z & \overset{\text{ev}'}{\leftarrow} & P' \times_Y X \\
& \pi_1 \downarrow & \pi_2 \downarrow \\
& P' & X \\
\end{array}
\xrightarrow{f} 
\begin{array}{ccc}
P' & \overset{p'}{\rightarrow} & Y
\end{array}
$$

there exists a unique $t : P' \to P$ with $p \cdot t = p'$ and $\text{ev} \cdot (\text{id}_X \times_Y t) = \text{ev}'$. Considering $P' = (1, e_0^i)$, we see that, in any of the categories $\text{Alg}(U, e; V)$, $\text{Alg}_u(U; V)$ or $\text{Alg}(U; V)$, $P$ should have as underlying set

$$P = \{(s, y) \mid y \in Y, s : (X_y, a_y) \to (Z, c)\}$$
where \((X_y, a_y)\) is the domain of the pullback of \(y : (1, c^2) \to (Y, b)\) along \(f\), \(p : P \to Y\) is the projection map and \(ev : P \times Y X \to Z\) is the evaluation map \((s, x) \mapsto s(x)\).

### 3.2. Exponentiation in \(\text{Alg}(U, e; V)\)

In [10] it is shown that the category \(\text{Alg}(U, e; V)\) is locally cartesian closed provided that \(V\) is so, that is, a Heyting algebra. For \(f : (X, a) \to (Y, b)\) and \((Z, c)\) in \(\text{Alg}(U, e; V)\), the structure \(d\) on the partial product \(P\) is defined as (with \(Q := P \times_Y X\))

\[
d(p, (s, y)) = \bigvee \left\{ \gamma \in V \mid \gamma \leq b(Up(p), y) \land \forall q \in U_{\pi_1}^{-1}(p), x \in f^{-1}(y) a(U_{\pi_2}(q), x) \land \gamma \leq c(Uev(q), s(x)) \right\},
\]

for every \(p \in UP\) and \((s, y) \in P\).

### 3.3. Partial products coincide

Assume that \(V\) is \(\sqsubset\)-atomic and a Heyting algebra. Let \(f : (X, a) \to (Y, b)\) and \((Z, c)\) in \(\text{Alg}_u(U; V)\) and assume that the partial product \((P, d)\) of \((Z, c)\) over \(f\) in \(\text{Alg}_u(U; V)\) exists. In 3.1 we observed that it has the same underlying set as the partial product \((P, d')\) of \((Z, c)\) over \(f\) in \(\text{Alg}(U, e; V)\). An argument similar to the one in [8], 3.3, shows that also the structures \(d\) and \(d'\) coincide: we just replace the two-element \(V\)-category \((\{0, 1\}, e)\) used in [8] by the elementary structure described in 1.9.

Note that the same argument shows that partial products taken in \(\text{Alg}(U; V)\) and \(\text{Alg}(U, e; V)\) coincide as well.

### 3.4. Lemma

Assume that \(V\) is \(\sqsubset\)-atomic. For each pullback

\[
\begin{array}{ccc}
(P, d) & \xrightarrow{\pi_1} & (X, a) \\
\pi_2 & \downarrow & \downarrow f \\
(Y, b) & \xrightarrow{g} & (Z, c)
\end{array}
\]

in \(\text{Alg}(U; V)\), \(\mathfrak{P} \in U^2P\), \(x \in X\) and \(y \in Y\) with \(f(x) = g(y)\),

\[
Ud(\mathfrak{P}, (\cdot, y)) = Ua(U^2_{\pi_1}(\mathfrak{P}), x) \land Ub(U^2_{\pi_2}(\mathfrak{P}), y).
\]

**Proof.** Since \(U_{\pi_1}\) and \(U_{\pi_2}\) are lax homomorphisms, we only have to show

\[
Ud(\mathfrak{P}, (\cdot, y)) \geq Ua(\mathfrak{X}, x) \land Ub(\mathfrak{Y}, y),
\]

where \(\mathfrak{X} = U^2_{\pi_1}(\mathfrak{P})\) and \(\mathfrak{Y} = U^2_{\pi_2}(\mathfrak{P})\). Let \(\alpha \in \text{At}(Ua(\mathfrak{X}, x))\) and \(\beta \in \text{At}(Ub(\mathfrak{Y}, y))\). We have \(\{a \in UX \mid a(\cdot, x) \geq \alpha\} \in \mathfrak{X}\) and \(\{b \in UX \mid b(\cdot, y) \geq \beta\} \in \mathfrak{Y}\) and therefore \(\{\varnothing \in UP \mid Ud(\varnothing, (\cdot, y)) \geq \alpha \land \beta\} \in \mathfrak{P}\). \(\square\)
3.5. **Theorem.** If \( V \) is \( \sqsubseteq \)-atomic and a Heyting algebra, then \( f : (X,a) \to (Y,b) \) in \( \text{Alg}_a(U;V) \) is exponentiable if and only if,

\[
(\diamond) \quad \forall \mathfrak{X} \in U^2 X \forall x \in X \quad U(a(\mathfrak{X},x)) \geq Ur(Uf(\mathfrak{X}),f(x)) \land \alpha(m_X(\mathfrak{X},x)).
\]

**Proof.** We are first going to show that the condition \((\diamond)\) is sufficient for exponentiability. Assume that \( f : (X,a) \to (Y,b) \) in \( \text{Alg}_a(U;V) \) satisfies \((\diamond)\) and let \((Z,c) \in \text{Alg}_a(U;V)\). We have to show that the partial product (see 3.2)

\[
(Z,c) \quad \xrightarrow{\text{ev}} \quad (Q,a \land d) \quad \xrightarrow{\pi_2} \quad (X,a)
\]

\[
\quad \xrightarrow{\pi_1} \quad f \quad \xrightarrow{p} \quad (Y,b)
\]

of \((Z,c)\) over \(f\) taken in \( \text{Alg}(U,e;V) \) actually lies in \( \text{Alg}_a(U;V) \). To see that, let \( P \in U^2 P \) and \((s,y) \in P\). According to the definition of the structure \( d \) on \( P \),

\[
Ud(\mathfrak{P},(s,y)) \leq d(m_P(\mathfrak{P}), (s,y))
\]

holds precisely if

\[
Ud(\mathfrak{P},(s,y)) \leq b(Up(m_P(\mathfrak{P})), y)
\]

and, for each \( q \in UQ \) with \( U\pi_1(q) = m_P(\mathfrak{P}) \) and each \( x \in f^{-1}(y)\),

\[
Ud(\mathfrak{P},(s,y)) \land a(U\pi_2(q), x) \leq c(U \text{ev}(q), s(x)).
\]

The first inequality follows from

\[
Ud(\mathfrak{P},(s,y)) \leq Ub(U^2p(\mathfrak{P}), y) \leq b(m_Y(U^2p(\mathfrak{P})), y) = b(Up(m_P(\mathfrak{P})), y).
\]

From \( U\pi_1(q) = m_P(\mathfrak{P}) \) follows that there exists a \( \mathcal{Q} \in U^2Q \) with (see remark at the end of 1.4)

\[
m_Q(\mathcal{Q}) = q \quad \text{and} \quad U^2\pi_1(\mathcal{Q}) = \mathfrak{P}.
\]

We obtain

\[
c(U \text{ev}(q), s(x)) = c(m_Z(U^2 \text{ev}(\mathcal{Q})), s(x))
\]

\[
\geq Uc(U^2 \text{ev}(\mathcal{Q}), s(x))
\]

\[
\geq Um(\alpha) \land (\Omega, (s,x))
\]

\[
= Ua(U^2\pi_2(\mathcal{Q}), x) \land Ud(\mathfrak{P}, (s,y))
\]

\[
\geqUb(U^2(p \cdot \pi_1)(\mathcal{Q}), y) \land a(m_X(U^2\pi_2(\mathcal{Q})), x) \land Ud(\mathfrak{P}, (s,y))
\]

\[
= a(U\pi_2(q), x) \land Ud(\mathfrak{P}, (s,y)).
\]
To prove the necessity of condition (a), let \( f : (X, a) \to (Y, b) \) be exponentiable in \( \text{Alg}_u(U; V) \) and consider the partial product of \((V^*, h)\) over \( f \) in \( \text{Alg}_u(U; V) \)

\[
(V^*, h) \xrightarrow{ev} (Q, a \land d) \xrightarrow{\pi_2} (X, a) \\
\xrightarrow{\pi_1} (P, d) \xrightarrow{p} (Y, b).
\]

For each \( x \in X \) we define a lax homomorphism (with \( y = f(x) \))

\[
\delta_x : X_y \to V^*
\]

\[
z \mapsto \begin{cases} 
  k & \text{if } z = x, \\
  * & \text{if } z \neq x.
\end{cases}
\]

It induces a map \( X \to P \), \( x \mapsto (\delta_x, f(x)) \); for every \( A \subseteq X \), \( x \in UX, \ldots \), we will write \( A^x, y^x, \ldots \), for its image under this map.

For \( X \in U^2X \) and \( x_0 \in X \) we put \( \mathcal{Y} = U^2f(X) \) and \( y_0 = f(x_0) \). We have to show that, for each \( A \in \mathcal{X} \),

\[
\bigvee_{a \in A} a(a, x_0) \geq Ub(\mathcal{Y}, y_0) \land a(m_X(X), x_0).
\]

Given \( A \in \mathcal{X} \), we define a lax homomorphism

\[
\lambda_A : X_{y_0} \to V^*
\]

\[
x \mapsto \begin{cases} 
  \bigvee_{a \in A} a(a, x_0) & \text{if } x = x_0, \\
  * & \text{else.}
\end{cases}
\]

**Claim 1.** For each \( B \in \mathcal{X} \) and each atom \( \beta \sqsubset Ub(\mathcal{Y}, y_0) \), there exists \( a \in B \) such that \( d(a^\mathcal{Y}, (\lambda_A, y_0)) \geq \beta \).

**Proof (Claim 1).** Since \( Uf(A \cap B) \in \mathcal{Y} \), there exists some \( a \in A \cap B \) with \( b(Uf(a), y_0) \geq \beta \). We will show that \( d(a^\mathcal{Y}, (\lambda_A, y_0)) \geq \beta \). To see that, let \( q \in UQ \) with \( U\pi_1(q) = a^\mathcal{X} \). We have to show that

\[
h(U ev(q), \lambda_A(x_0)) \geq \beta \land a(U\pi_2(q), x_0).
\]

From \( U\pi_1(q) = a^\mathcal{X} \) follows \( U ev(q) = \bullet \) or \( U ev(q) = * \). In the latter case we have \( h(U ev(q), \lambda_A(x_0)) = k \). If \( U ev(q) = \bullet \), then \( U\pi_2(q) = a \) and hence \( \lambda_A(x_0) \geq a(U\pi_2(q), x_0) \). This finishes the proof of Claim 1.

We obtain \( d(m_P(\mathcal{X}^\mathcal{Y}), (\lambda_A, y_0)) \geq Ud(\mathcal{X}^\mathcal{Y}, (\lambda_A, y_0)) \geq Ub(\mathcal{Y}, y_0) \).

**Claim 2.** There exists \( \Omega \in U^2Q \) such that

- \( U^2\pi_1(\Omega) = \mathcal{X}^\mathcal{Y} \),
- \( U^2\pi_2(\Omega) = \mathcal{X} \), and
- \( \Delta = \{ q \in UQ \mid ev(q) = \bullet \} \in \Omega \).
Proof (Claim 2). We have to show that, for all $\mathcal{B}, \mathcal{C} \in \mathfrak{X}$,

$$U\pi_1^{-1}(\mathcal{B}) \cap U\pi_1^{-1}(\mathcal{C}) \cap \Delta \neq \emptyset.$$ 

We take any $a \in \mathcal{B} \cap \mathcal{C}$. For all $A, B \in a$ and $z \in A \cap B$ we have

$$(\delta_{z}, z) \in \pi_1^{-1}(A^\Delta) \cap \pi_2^{-1}(B) \cap \{(\psi, x) \in Q \mid \psi(x) = k\},$$

hence $\pi_1^{-1}(A^\Delta) \cap \pi_2^{-1}(B) \cap \{(\psi, x) \in Q \mid \psi(x) = k\} \mid A, B \in a$ is a filter on $Q$ and any $q \in U\pi_0$ containing it belongs to $U\pi_1^{-1}(\mathcal{B}) \cap U\pi_1^{-1}(\mathcal{C}) \cap \Delta$. This finishes the proof of Claim 2.

We conclude that

$$(d \land a)(m_Q(\mathfrak{Q}), (\lambda_A, x_0)) \geq U_{b}(\mathfrak{Y}, y_0) \land a(m_X(\mathfrak{X}), x_0)$$

and therefore

$$\lambda_A(x_0) = \hom(k, \lambda_A(x_0)) = \varphi(m_{U}U^2\varphi(\mathfrak{Q})), \lambda_A(x_0)) \geq$$

$$\geq \varphi(m_{U}U^2\varphi(\mathfrak{Q})), \lambda_A(x_0) \geq \varphi(m_{U}U^2\varphi(\mathfrak{Q})), \lambda_A(x_0).$$

\[\square\]

3.6. Corollary. Assume that $V$ is $\sqcap$-atomic and a Heyting algebra.

1. An object $(X, a) \in \Alg_a(U; V)$ is exponentiable if and only if $a = \bar{\varphi}(a)$.

2. Each $f : (X, a) \to (Y, b)$ in $\Alg_a(U; V)$ with exponentiable domain is exponentiable.

3. Each initial $f : (X, a) \to (Y, b)$ is exponentiable in $\Alg_a(U; V)$.

Proof. Theorem 3.5 implies that $(X, a)$ is exponentiable if and only if $a \cdot m_X = e^X \cdot U a$ which, as shown in 2.4., is equivalent to $a = \bar{\varphi}(a)$. If $(X, a)$ is exponentiable we have

$$U_{a}(X, x) \geq a(m_X(\mathfrak{X}), x) \geq U_{b}(Uf(\mathfrak{X}), f(x)) \land a(m_X(\mathfrak{X}), x);$$

and if $i : (X, a) \hookrightarrow (Y, b)$ is initial

$$U_{a}(X, x) \geq U_{b}(Uf(\mathfrak{X}), f(x)) \geq U_{b}(Uf(\mathfrak{X}), f(x)) \land a(m_X(\mathfrak{X}), x).$$

\[\square\]

3.7. Examples. Exponentiable objects in $\PrTop$ and $\PrAp$ are characterized in [21] (see also [23]) and [20] respectively as exactly the finitely generated ones, i.e. the structure is induced by a reflexive function $r : X \times X \to 2$ resp. $d : X \times X \to \mathbb{R}_+$. Hence both results are special cases of Corollary 3.6. For a pretopological space $X$, this is equivalent to each point $x$ of $X$ having a smallest neighborhood $V$. Note that any (ultra)filter which contains $V$ necessarily contains all neighborhoods of $x$ and hence converges to $x$. The corresponding map version of this result is proved in [24]. It states that a map $f : X \to Y$ between pretopological spaces is exponentiable if and only if it is
fibrewise finitely generated, that is, each \( x \in X \) has a neighborhood \( V \) such that each tied filter \((g, f(x))\) which contains \( V \) converges to \( x \). Here a tied filter \((g, y)\) is a filter \( g \) together with a limit point \( y \) of \( f[g] \). Of course, it is enough to consider only ultrafilters. Our Theorem 3.5 specializes to

**Theorem 1.** A continuous map \( f : X \to Y \) between pretopological spaces is exponentiable if and only if, for each \( \mathcal{X} \in U^2X \) and \( x \in X \), \( \mathcal{X} \to x \) whenever \( m_X(\mathcal{X}) \to x \) and \( U^2f(\mathcal{X}) \to f(x) \).

The condition on \( f : X \to Y \) in the theorem above must be equivalent to \( f : X \to Y \) being fibrewise finitely generated. This can be also seen directly as follows. Assume first that \( f : X \to Y \) is fibrewise finitely generated and let \( \mathcal{X} \in U^2X \) and \( x \in X \) be such that, with \( \mathcal{Y} := U^2f(\mathcal{X}) \) and \( y := f(x) \), \( m_X(\mathcal{X}) \to x \) and \( \mathcal{Y} \to \mathcal{Y} \). By hypothesis, there is some neighborhood \( V \) of \( x \) such that every ultrafilter \( \mathfrak{a} \) on \( X \) with \( V \in \mathfrak{a} \) and \( Uf(\mathfrak{a}) \to y \) converges to \( x \). From \( m_X(\mathcal{X}) \to x \) follows \( V \in m_X(\mathcal{X}) \), hence \( V^\# \in \mathcal{X} \). Let \( \mathcal{A} \in \mathcal{X} \). We have \( V^\# \cap A \in \mathcal{X} \), hence for some \( \mathfrak{a} \in V^\# \cap \mathcal{A} \) it holds \( Uf(\mathfrak{a}) \to y \) and therefore \( \mathfrak{a} \to x \). We have shown that \( \mathcal{X} \to x \). Assume now that \( f : X \to Y \) is not fibrewise finitely generated. Hence there is some \( x \in X \) such that, for each neighborhood \( V \) of \( x \), there exists an ultrafilter \( \mathfrak{a} \in UX \) such that \( V \in \mathfrak{a} \), \( Uf(\mathfrak{a}) \to y := f(x) \) and \( \mathfrak{a} \to x \). We define

\[
\mathcal{A} := \{ \mathfrak{a} \in UX \mid V \in \mathfrak{a} \& Uf(\mathfrak{a}) \to y \& \mathfrak{a} \to x \}.
\]

Then \( \mathfrak{G} := \{ W^\# \mid W \text{ is a neighborhood of } x \} \cup \{ \mathcal{A} \} \) is a filter base on \( UX \); applying 1.5 twice we obtain first an ultrafilter \( \mathcal{Y} \in U^2Y \) such that \( Uf(\mathfrak{G}) \subset \mathcal{Y} \) and \( \mathcal{Y} \to \mathcal{Y} \), and then \( \mathcal{X} \in U^2X \) with \( \mathfrak{G} \subset \mathcal{X} \) and \( U^2f(\mathcal{X}) = \mathcal{Y} \). Hence \( m_X(\mathcal{X}) \to x \) but \( \mathcal{X} \not\to x \).

Theorem 3.5 gives also a characterization of exponentiable maps in \( \text{PrAp} \):

**Theorem 2.** A non-expansive map \( f : (X, a) \to (Y, b) \) between preapproach spaces is exponentiable if and only if, for each \( \mathcal{X} \in U^2X \) and \( x \in X \),

\[
Ua(\mathcal{X}, x) \leq \max(Ub(U^2f(\mathcal{X}), f(x)), a(m_X(\mathcal{X}), x)).
\]

4. Exponentiability in \( \text{Alg}(U; V) \)

4.1. **Preamble.** In this section we give a sufficient condition for an object to be exponentiable in \( \text{Alg}(U; V) \). The proof will be very similar to the one of Theorem 3.5, we “just” substitute principal by arbitrary ultrafilters. Unfortunately, due to this complication, we are not able to derive a result about maps because we are only able to prove Lemma 4.2 – the corresponding version of Lemma 3.4 – for products.

Throughout this section we assume that \( V \) is \( \sqsubset \)-atomic and a Heyting algebra and that, for each \( \gamma \in V \), \( \text{At}(\gamma) \) is up-directed or each \( \gamma_0 \in \text{At}(\gamma) \) is connected.
4.2. Lemma. Let

\[(X, a) \xrightarrow{\pi_1} (P, d) \xrightarrow{\pi_2} (Y, b)\]

be a product in \(\operatorname{Alg}(U; V)\). For every \(\mathfrak{P} \in U^2P, \mathfrak{x} \in UX\) and \(\eta \in UY\),

\[
\bigvee_{p \in UP: U\pi_1(p) = \mathfrak{x}, U\pi_2(p) = \eta} Ud(\mathfrak{P}, p) = Ua(U^2\pi_1(\mathfrak{P}), \mathfrak{x}) \land Ub(U^2\pi_2(\mathfrak{P}), \eta).
\]

Proof. Let \(\alpha \in \text{At}(Ua(U^2\pi_1(\mathfrak{P}), \mathfrak{x}))\) and \(\beta \in \text{At}(Ub(U^2\pi_2(\mathfrak{P}), \eta))\). For \(A \in \mathfrak{x}\) and \(B \in \eta\), we have \(a_\alpha^0[A] \in U^2\pi_1(\mathfrak{P})\) and \(b_\beta^0[B] \in U^2\pi_2(\mathfrak{P})\) and therefore

\[
d_{\alpha \land \beta}[A \land \beta] \supset U\pi_1^{-1}[a_\alpha^0[A]] \cap U\pi_2^{-1}[b_\beta^0[B]] \in \mathfrak{P}.
\]

The Extension Lemma guarantees now the existence of \(p \in UP\) such that \(U\pi_1(p) = \mathfrak{x}, U\pi_2(p) = \eta\) and \(Ud(\mathfrak{P}, p) \geq \alpha \land \beta\). \(\square\)

4.3. Theorem. Under the conditions 4.1, an object \((X, a)\) in \(\operatorname{Alg}(U; V)\) is exponentiable provided that

\[(\ast) \quad \forall \mathfrak{x} \in U^2X, x \in X \forall \gamma_1, \gamma_0 \in V
\]

\[
\bigvee_{\mathfrak{x} \in UX} (Ua(\mathfrak{x}, \mathfrak{x}) \land \gamma_1) \otimes (a(\mathfrak{x}, x) \land \gamma_0) \geq a(m_X(\mathfrak{x}, x, \mathfrak{x})) \land (\gamma_1 \otimes \gamma_0).
\]

Proof. Assume that \((X, a)\) fulfills \((\ast)\) and let \((Y, b) \in \operatorname{Alg}_u(U; V)\). We will show that the structure \(d\) on \(P\) defined in 3.2 is transitive. To see that, we put \(Q = P \times X\) and let \(\mathfrak{P} \in U^2P, \mathfrak{p} \in UP\) and \(s \in P\). According to the definition of \(d\), we have to show that, for each \(x \in X\) and \(q \in UQ\) with \(U\pi_1(q) = m_P(\mathfrak{P})\),

\[
(Ud(\mathfrak{P}, p) \otimes d(p, s)) \land a(U\pi_2(q), x) \leq b(\text{ev}(q), s(x)).
\]

Assume now that \(x \in X\) and \(q \in UQ\) are given such that \(U\pi_1(q) = m_P(\mathfrak{P})\). As in the proof of Theorem 3.5, for such \(q\) there exists \(\Omega \in UQ\) such that \(m_Q(\Omega) = q\) and \(U^2\pi_1(\Omega) = \mathfrak{P}\). The previous lemma implies that, for each \(\mathfrak{x} \in UX\),

\[
\bigvee_{q' \in UQ: U\pi_1(q') = \mathfrak{x}} Ud(\mathfrak{P}, p) = Ud(\mathfrak{P}, p) \land a(U^2\pi_2(\mathfrak{P}), \mathfrak{x}).
\]

Moreover, for each \(q' \in UQ\) with \(U\pi_2(q') = \mathfrak{x}\) and \(U\pi_1(q') = p\) it holds

\[
b(U\text{ev}(q'), s(x)) = b(m_Y(U^2\text{ev}(\Omega)), s(x))
\]

\[
\geq Ub(U^2\text{ev}(\mathfrak{Q}), b(U\text{ev}(q'), s(x))) \otimes b(U\text{ev}(q'), s(x))
\]

\[
\geq U(d \land a)(\mathfrak{Q}, q) \otimes (d(p, s) \land a(\mathfrak{x}, x)).
\]
Combining the inequalities above we obtain

\[
b(U \text{ev}(q), s(x)) \geq \bigvee_{r \in UX} \left( \bigvee_{q' \in U Q, U_{\pi_1}(q') = p} (U(d \wedge a)(\Xi, q')) \otimes (d(p, s) \wedge a(x, x)) \right) \geq \bigvee_{r \in UX} (U(d(\mathcal{P}, p) \wedge U(a(U^2\pi_2(\Omega), r))) \otimes (d(p, s) \wedge a(x, x)) \geq a(m_X(U^2\pi_2(\Omega), x) \wedge (Ud(\mathcal{P}, p) \otimes d(p, s))) = a(U\pi_2(q), x) \wedge (Ud(\mathcal{P}, p) \otimes d(p, s)).
\]

\[\square\]

4.4. **Corollary.** In addition to 4.1, assume that \(\otimes = \wedge\).

1. \((X, a)\) in \(\text{Alg}(U; V)\) is exponentiable provided that \(a \cdot Ua = a \cdot m_X\).
2. \((X, \phi(r))\) is exponentiable in \(\text{Alg}(U; V)\), for each \(V\)-category \((X, r)\).

**Proof.** (1) follows immediately from the theorem above. To see (2), let \((X, r)\) \(\in V\)-\textbf{Cat}. We apply Lemma 2.4 to \(r \cdot r = r\) and obtain \(\phi(r) \cdot U\phi(r) = \phi(r) \cdot m_X\). \(\square\)

4.5. **Application to approach spaces.** As for metric spaces (see [8]), the condition \((\otimes)\) can be simplified in the case of approach spaces.

**Theorem.** Let \((X, a)\) be an approach space. Then the following assertions are equivalent.

1. \((X, a)\) satisfies \((\otimes)\).
2. For \(x \in U^2X\), \(x \in X\) and \(\gamma_1, \gamma_0 \in \mathbb{R}_+\) with \(\gamma_1 + \gamma_0 = a(m_X(x), x)\),
   \[
   \inf_{r \in UX} (Ua(x, x) \wedge a(x) \wedge \gamma_0) \leq a(m_X(x), x).
   \]
3. For \(x \in U^2X\), \(x \in X\) with \(a(m_X(x), x) < \infty\) and \(\gamma_1, \gamma_0 \in [0, \infty)\) with \(\gamma_1 + \gamma_0 = a(m_X(x), x)\), for each \(\varepsilon > 0\) there exists an ultrafilter \(\mathfrak{r} \in UX\) such that
   \[
   Ua(x, x) \leq \gamma_1 + \varepsilon \quad \text{and} \quad a(x, x) \leq \gamma_0 + \varepsilon.
   \]
4. For \(\gamma_1, \gamma_0 \in [0, \infty)\) and \(\varepsilon > 0\),
   \[
a_{\gamma_0 + \varepsilon} \cdot (Ua)_{\gamma_1 + \varepsilon} \geq a_{\gamma_0 + \gamma_1} \cdot m_X.
   \]

**Corollary.** \((X, \phi(r))\) is exponentiable in \(\text{Ap}\) if and only if \((X, r)\) is exponentiable in \(\text{Met}\).

**Proof.** Let \((X, r)\) be exponentiable in \(\text{Met}\). According to [8], this is equivalent to

\[
\forall \gamma_0, \gamma_1 \in \mathbb{R}_+ \forall \varepsilon > 0 \ r_{\gamma_0 + \varepsilon} \cdot r_{\gamma_1 + \varepsilon} \geq r_{\gamma_0 + \gamma_1 + \varepsilon}.
\]
Let \( \gamma_1, \gamma_0 \in [0, \infty) \) and \( \varepsilon > 0 \). We apply 2.4 and obtain
\[
\phi(r_{\gamma_0+\varepsilon}) \cdot U \phi(r_{\gamma_1+\varepsilon}) \geq \phi(r_{\gamma_0+\gamma_1+\varepsilon}) \cdot m_X.
\]
For the right hand side we have
\[
\phi(r_{\gamma_0+\varepsilon}) \cdot U \phi(r_{\gamma_1+\varepsilon}) \leq \phi(r)_{\gamma_0+\varepsilon} \cdot U (\phi(r)_{\gamma_1+\varepsilon}) \leq \phi(r)_{\gamma_0+\varepsilon} \cdot (U \phi(r))_{\gamma_1+\varepsilon},
\]
and for the left hand side \( \phi(r_{\gamma_0+\gamma_1+\varepsilon}) \cdot m_X \geq \phi(r)_{\gamma_0+\gamma_1} \cdot m_X \). The reverse implication follows immediately from the fact that \( \text{Met} \) is a full and coreflective subcategory of \( \text{Ap} \) closed under finite limits (see 2.4).

We finish this paper by exhibiting an exponentiable approach space whose structure is not induced by a metric or a topology, namely the Sierpinski approach space \( P \). Recall from [17] that \( P \) has as underlying set \( \mathbb{R}_+ = [0, \infty] \) and structure \( p \) defined by \( p(\bar{x}, x) := \text{hom}(l(\bar{x}), x) = \max\{x - l(\bar{x}), 0\} \), for each ultrafilter \( \bar{x} \in U^P \) and each \( x \in P \) and \( l(\bar{x}) = \inf_{A \in \bar{x}} \sup_{y \in A} y \). Since \( \text{hom}(\cdot, x) \) is a contravariant adjoint functor, it holds
\[
\text{hom}(\inf_{y \in A} y, x) = \sup_{y \in A} \text{hom}(y, x)
\]
for all \( x \in P \) and \( A \subset P \). If \( x \in [0, \infty) \) and \( A \neq \emptyset \), then we also have
\[
\text{hom}(\sup_{y \in A} y, x) = \inf_{y \in A} \text{hom}(y, x).
\]
In particular we see that the subspace \([0, \infty)\) of \( P \) is finitely generated. Therefore, in order to show that \( P \) is exponentiable, we only need to consider \( \bar{x} \in U^2P \)
and \( x \in P \) with \( p(m_X(\bar{x}), x) < \infty \), where \( \bar{x} = \infty \) or \( x = \infty \). Since \( x = \infty \) implies \( l(m_X(\bar{x})) = \infty \), we cover both cases by assuming from now on that \( \bar{x} \in U^2P \) and \( x \in P \) are given with \( l(m_X(\bar{x})) = \infty \). Note that
\[
l(m_X(\bar{x})) = \infty \iff \forall y \in [0, \infty) \, [y, \infty] \in m_X(\bar{x})
\]
\[
\iff \forall y \in [0, \infty) \, [y, \infty]^\# \in \bar{x}.
\]
Let \( \mathcal{I} := \{[y, \infty] \mid y \in [0, \infty)\} \). Then \( \sup_{x \in \mathcal{I}, y \in P} \inf_{a \in A, z \in P} p(a, z) = 0 \). From Lemma 1.5 we infer that there exists some ultrafilter \( \bar{x} \in U^P \) with \( \mathcal{I} \subset \bar{x} \) and \( Up(\bar{x}, x) = 0 \). By construction we have \( p(\bar{x}, x) = 0 \), which completes the proof.

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