Exponentiability in categories of lax algebras

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Abstract

For a complete cartesian-closed category $V$ with coproducts, and for any pointed endofunctor $T$ of the category of sets satisfying a suitable Beck-Chevalley-type condition, it is shown that the category of lax reflexive $(T,V)$-algebras is a quasitopos. This result encompasses many known and new examples of quasitopoi.

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Key words: lax algebra, partial product, locally cartesian-closed category, quasitopos.

0 Introduction

Failure to be cartesian closed is one of the main defects of the category of topological spaces. But often this defect can be side-stepped by moving temporarily into the quasitopos hull of $\text{Top}$, the category of pseudotopological (or Choquet) spaces, see for example [11, 14, 7]. A pseudotopology on a set $X$ is most easily described by a relation $x \rightarrow x$ between ultrafilters $x$ on $X$ and points $x$ in $X$, the only requirement for which is the reflexivity condition $\hat{x} \rightarrow x$ for all $x \in X$, with $\hat{x}$ denoting the principal ultrafilter on $x$. In this setting, a topology on $X$ is a pseudotopology which satisfies the transitivity condition

$$\mathcal{X} \rightarrow \eta \& \eta \rightarrow z \Rightarrow m(\mathcal{X}) \rightarrow z$$

for all $z \in X$, $\eta \in UX$ (the set of ultrafilters on $X$) and $\mathcal{X} \in UUX$; here the relation $\rightarrow$ between $UX$ and $X$ has been naturally extended to a relation between $UUX$ and $UX$, and $m = m_X : UUX \rightarrow UX$ is the unique map that gives $U$ together with $e_X(x) = \hat{x}$ the structure of a monad $U = (U, e, m)$. Barr [2] observed that the two conditions, reflexivity and transitivity, are precisely the two basic laws of a lax Eilenberg-Moore algebra when one extends the Set-monad $U$ to a lax monad of Rel(Set), the category of sets with relations as morphisms. In [9] Barr’s presentation of topological spaces was extended to include Lawvere’s presentation of metric spaces as $V$-categories with $V = \mathbb{R}_+$, the extended real half-line. Thus, for any symmetric monoidal category $V$ with coproducts preserved by the tensor product, and for any Set-monad $T$ that suitably extends from Set-maps to all $V$-matrices (or “$V$-relations”, with

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ordinary relations appearing for $V = 2$, the two-element chain), the paper [9] develops the
notion of reflexive and transitive $(T, V)$-algebra, investigates the resulting category Alg$(T, V)$,
and presents many examples, in particular $\text{Top} = \text{Alg}(U, 2)$.

The purpose of this paper is to show that dropping the transitivity condition leads us to a
quasitopos not only in the case of $\text{Top}$, but rather generally. In order to define just reflexive
$(T, V)$-algebras, one indeed needs neither the tensor product of $V$ (just the “unit” object) nor
the “multiplication” of the monad $T$. Positively speaking then, we start off with a category $V$
with coproducts and a distinguished object $I$ in $V$ and any pointed endofunctor $T$ of $\text{Set}$
and define the category $\text{Alg}(T, V)$. Our main result says that when $V$ is complete and locally
cartesian closed and a certain Beck-Chevalley condition is satisfied, also $\text{Alg}(T, V)$ is locally
cartesian closed (Theorem 2.7).

Defining reflexive $(T, V)$-algebras for the “truncated” data $T, V$ entails a considerable depar-
ture from [9], as it is no longer possible to talk about the bicategory $\text{Mat}(V)$ of $V$-matrices. The
missing tensor product prevents us from being able to introduce the (horizontal) matrix com-
position; however, “whiskering” by $\text{Set}$-maps (considered as 1-cells in $\text{Mat}(V)$) is still well-defined
and well-behaved, and this is all that is needed in this paper.

We explain the relevant properties of $\text{Mat}(V)$ in Section 1 and define the needed Beck-
Chevalley condition. Briefly, this condition says that the comparison map that “measures” the
extent to which the $T$-image of a pullback diagram in $\text{Set}$ still is a pullback diagram must be a
lax epimorphism when considered a 1-cell in $\text{Mat}(V)$. Having presented our main result, at the
end of Section 2 we show that this condition is equivalent to asking $T$ to preserve pullbacks or,
if $V$ is thin (i.e., a preordered class), to transform pullbacks into weak pullback diagrams (barring
trivial choices for $I$ and $V$). In certain cases, (BC) turns out to be even a necessary condition
for local cartesian closedness of $\text{Alg}(T, V)$, see 2.10. In Section 3 we show how to construct
limits and colimits in $\text{Alg}(T, V)$ in general, and Section 4 presents the construction of partial
map classifiers, leading us to the theorem stated in the Abstract. A list of examples follows in
Section 5.

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their respective articles [15] and [3].

1 $V$-matrices

1.1 Let $V$ be a category with coproducts and a distinguished object $I$. A $V$-\textit{matrix} (or $V$-
relation) $r$ from a set $X$ to a set $Y$, denoted by $r : X \rightarrow Y$, is a functor $r : X \times Y \rightarrow V$,
i.e. an $X \times Y$-indexed family $(r(x, y))_{x,y}$ of objects in $V$. With $X, Y$ fixed, such $V$-matrices
form the objects of a category $\text{Mat}(V)(X, Y)$, the morphisms $\varphi : r \rightarrow s$ of which are natural
transformations, i.e. families $(\varphi_{x,y} : r(x, y) \rightarrow s(x, y))_{x,y}$ of morphisms in $V$; briefly,
\[ \text{Mat}(V)(X, Y) = V^{X \times Y}. \]
1.2 Every Set-map $f : X \to Y$ may be considered as a $V$-matrix $f : X \nrightarrow Y$ when one puts

$$f(x, y) = \begin{cases} I & \text{if } f(x) = y, \\ 0 & \text{else,} \end{cases}$$

with 0 denoting a fixed initial object in $V$. This defines a functor

$$\text{Set}(X, Y) \longrightarrow \text{Mat}(V)(X, Y),$$

of the discrete category Set$(X, Y)$, and the question is: when do we obtain a full embedding, for all $X$ and $Y$? Precisely when

(*) $V(I, 0) = \emptyset$ and $|V(I, I)| = 1$,

as one may easily check. In the context of a cartesian-closed category $V$, we usually pick for $I$ a terminal object $1$ in $V$, and then condition (*) is equivalently expressed as

(**) $0 \not\cong 1$,

preventing $V$ from being equivalent to the terminal category.

1.3 While in this paper we do not need the horizontal composition of $V$-matrices in general, we do need the composites $sf$ and $gr$ for maps $f : X \to Y$, $g : Y \to Z$ and $V$-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, defined by

$$(sf)(x, z) = s(f(x), z),$$

$$(gr)(x, z) = \sum_{y : g(y) = z} r(x, y),$$

for $x \in X$, $z \in Z$; likewise for morphisms $\varphi : r \to r'$ and $\psi : s \to s'$. Hence, we have the “whiskering” functors

$$-f : \text{Mat}(V)(Y, Z) \to \text{Mat}(V)(X, Z),$$

$$g- : \text{Mat}(V)(X, Y) \to \text{Mat}(V)(X, Z).$$

The horizontal composition with Set-maps from either side is associative up to coherent isomorphisms whenever defined; hence, if $h : U \to X$ and $k : Z \to V$, then

$$(sf)h = s(fh) \text{ and } k(gr) \cong (kg)r.$$  

Although Mat$(V)$ falls short of being a bicategory, even a sesquicategory [15], we refer to sets as 0-cells of Mat$(V)$, $V$-matrices as its 1-cells, and natural transformations between them as its 2-cells.

1.4 The transpose $r^\circ : Y \nrightarrow X$ of a $V$-matrix $r : X \nrightarrow Y$ is defined by $r^\circ(y, x) = r(x, y)$ for all $x \in X$, $y \in Y$. Obviously $r^{\circ\circ} = r$, and with

$$(sf)^\circ = f^\circ s^\circ, \ (gr)^\circ = r^\circ g^\circ$$
we can also introduce whiskering by transposes of \textbf{Set}-maps from either side, also for 2-cells.

A \textbf{Set}-map \( f : X \to Y \) gives rise to 2-cells

\[ \eta : 1_X \to f^\circ f, \quad \varepsilon : f f^\circ \to 1_Y \]

satisfying the triangular identities \((\varepsilon f)(f \eta) = 1_f, (f^\circ \varepsilon)(f \eta f^\circ) = 1_f\).

1.5 For a functor \( T : \text{Set} \to \text{Set} \), we denote by \( \kappa : TW \to U \) the comparison map from the \( T \)-image of the pullback \( W := Z \times_Y X \) of \((g,f)\) to the pullback \( U := T Z \times T X \) of \((Tg,Tf)\)

\begin{equation}
\begin{aligned}
TW & \xrightarrow{\kappa} U \\
\downarrow & \quad \downarrow \\
TZ & \xrightarrow{Tf} TY.
\end{aligned}
\end{equation}

We say that the \textbf{Set}-functor \( T \) satisfies the \textit{Beck-Chevalley Condition (BC)} if the 1-cell \( \kappa \) is a lax epimorphism; that is, if the “whiskering” functor \(-\kappa : \text{Mat}(V)(TW,S) \to \text{Mat}(V)(U,S)\) is full and faithful, for every set \( S \).

In the next section we will relate this condition with other known formulations of the Beck-Chevalley condition.

## 2 Local cartesian closedness of \( \text{Alg}(T,V) \)

2.1 Let \((T,e)\) be a pointed endofunctor of \textbf{Set} and \( V \) category with coproducts and a distinguished object \( I \). A \textit{lax (reflexive) \( (T,V) \)-algebra} \((X,a,\eta)\) is given by a set \( X \), a 1-cell \( a : TX \to X \) and a 2-cell \( \eta : 1_X \to ae_X \) in \( \text{Mat}(V) \). The 2-cell \( \eta \) is completely determined by the \( V \)-morphisms

\[ \eta_x := \eta_{e,x} : I \xrightarrow{} a(e_X(x),x), \]

\( x \in X \). As we shall not change the notation for this 2-cell, we write \((X,a)\) instead of \((X,a,\eta)\).

A \textit{(lax) homomorphism} \((f,\varphi) : (X,a) \to (Y,b)\) of \((T,V)\)-algebras is given by a map \( f : X \to Y \) in \textbf{Set} and a 2-cell \( \varphi : fa \to b(Tf) \) which must preserve the units: \( (\varphi e_X)(f \eta) = \eta f \). The 2-cell \( \varphi \) is completely determined by a family of \( V \)-morphisms

\[ f_{e,x} : a(x,x) \xrightarrow{} b(Tf(x),f(x)), \]

\( x \in X, r \in TX \), and preservation of units now reads as \( f_{e,x} x \eta_x = \eta_{f(x)} \) for all \( x \in X \). For simplicity, we write \( f \) instead of \((f,\varphi)\), and when we write

\[ f_{e,x} : a(x,x) \xrightarrow{} b(\eta,y) \]

\textit{this automatically entails} \( \eta = Tf(x) \) and \( y = f(x) \); these are the \( V \)-\textit{components} of the homomorphism \( f \). Composition of \((f,\varphi)\) with \((g,\psi) : (Y,b) \to (Z,c)\) is defined by

\[ (g,\psi)(f,\varphi) = (gf,(\psi(Tf))(g\varphi)) \]
which, in the notation used more frequently, means

\[(g f)_{x,x} = (a(x, x) \xrightarrow{f_{x,x}} b(\eta, y) \xrightarrow{g_{y,y}} c(3, z)).\]

We obtain the category \(\text{Alg}(T, V)\) (denoted by \(\text{Alg}(T, e; V)\) in [9]).

2.2 Let \(V\) be finitely complete. The pullback \((W, d)\) of \(f : (X, a) \to (Z, c)\) and \(g : (Y, b) \to (Z, c)\) in \(\text{Alg}(T, V)\) is constructed by the pullback \(W = X \times_Z Y\) in \(\text{Set}\) and a family of pullback diagrams in \(V\), as follows:

\[
d(w, w) \xrightarrow{f_{w,w}} b(\eta, y) \\
\xrightarrow{g_{w,w}} a(x, x) \xrightarrow{f_{x,x}} c(3, z)
\]

for all \(w \in W\); hence,

\[
d(w, w) = a(Tg'(w), g'(w)) \times_c b(Tf'(w), f'(w))
\]

in \(V\), where \(g' : W \to X\) and \(f' : W \to Y\) are the pullback projections in \(\text{Set}\). For each \(w = (x, y) \in W\), we define \(\eta_w := <\eta_x, \eta_y>\).

2.3 Every set \(X\) carries the discrete \((T, V)\)-structure \(e^X_X\). In fact, the 2-cell \(\eta : 1_X \to e^X_X e_X\) making \((X, e^X_X)\) a \((T, V)\)-algebra is just the unit of the adjunction \(e_X \vdash e^X_X\) in \(\text{Mat}(V)\). Now \(X \mapsto (X, e^X_X)\) defines the left adjoint of the forgetful functor \(\text{Alg}(T, V) \longrightarrow \text{Set}\) since every map \(f : X \to Y\) into a \((T, V)\)-algebra \((Y, b)\) becomes a homomorphism \(f : (X, e^X_X) \to (Y, b)\); indeed the needed 2-cell \(f e^X_X \to b(Tf)\) is obtained from the unit 2-cell \(\eta : 1 \to be_Y\) with the adjunction \(e_X \vdash e^X_X\); it is the mate of \(f \eta : f \to be_Y f = b(Tf)e_X\). In pointwise notation, for

\[
f_{x,x} : e^X_X(x, x) \longrightarrow b(\eta, y)
\]

one has \(f_{x,x} = 1_I\) if \(e_X(x) = x\); otherwise its domain is the initial object \(0\) of \(V\), i.e. it is trivial.

2.4 We consider the discrete structure in particular on a one-element set \(1\). Then, for every \((T, V)\)-algebra \((X, a)\), an element \(x \in X\) can be equivalently considered as a homomorphism \(x : (1, e^X_1) \to (X, a)\) whose only non-trivial component is the unit \(\eta_x : I \to a(e_X(x), x)\).

2.5 Assume \(V\) to be complete and locally cartesian closed. For a homomorphism \(f : (X, a) \to (Y, b)\) and an additional \((T, V)\)-algebra \((Z, c)\) we form a substructure of the partial product of the underlying \(\text{Set}\)-data (see [10]), namely

\[
Z \xleftarrow{e^X_X} Q \xrightarrow{q} X \\
\xrightarrow{f'} \downarrow \quad \downarrow f \\
\xrightarrow{p'} P \longrightarrow Y,
\]

5
Continuing in the notation of 2.5 and 2.6, we equip

\[ P = Z^f = \{(s, y) \mid y \in Y, s : (X_y, a_y) \to (Z, c)\}, \]

\[ Q = Z^f \times_Y X = \{(s, x) \mid x \in X, s : (X_{f(x)}, a_{f(x)}) \to (Z, c)\}, \]

where \((X_y = f^{-1}y, a_y)\) is the domain of the pullback

\[ i_y : (X_y, a_y) \longrightarrow (X, a) \]

of \(y : (1, e_0^y) \to (Y, b)\) along \(f\). Of course, \(p\) and \(q\) are projections, and \(\text{ev}\) is the evaluation map. We must find a structure \(d : TP \to P\) which, together with a 2-cell \(\eta\), will make these maps morphisms in \(\text{Alg}(T, V)\).

For \((s, y) \in P\) and \(p \in TP\), in order to define \(d(p, (s, y))\), consider each pair \(x \in X\) and \(q \in TQ\) with \(f(x) = y\) and \(Tf'(q) = p\) and form the partial product

\[ c(z, s(x)) \xrightarrow{\text{ev}_{q,a}} c(z, s(x))f_{t,x} \times_b a(x, x) \longrightarrow a(x, x) \]

in \(V\), where \(z = \text{ev}(q)\), and then the multiple pullback \(d(p, (s, y))\) of the morphisms \(\tilde{p}_{q,a}\) in \(V\), as in:

\[ d(p, (s, y)) \]

2.6 We define the 2-cell \(\eta : 1_P \to de_P\) componentwise. Let \((s, y) \in P\) and consider each \(x \in X\) and \(q \in TQ\) with \(f(x) = y\) and \(Tf'(q) = e_P(s, y) = T(s, y)e_1\) (where \((s, y) : 1 \to P\)). Consider the pullback \(j_y : X_y \to Q\) of \((s, y) : 1 \to P\) along \(f'\) in \(\text{Set}\); whence, \(j_y(x) = s(x)\). By (BC) there is \(z \in TX_y\) such that \(Tj_y(z) = q\) and \(Tf'(z) = e_1(\ast)\) (where \(! : X_y \to 1\) and \(\ast\) is the only point of \(1\)). Since \(\text{ev}j_y = s\), we may form the diagram

\[ c(z, s(x)) \xrightarrow{s_{z,x}} a_g(x, x) \xrightarrow{(Ig)_{t,x}} a(x, x) \]

in \(V\), where \(z = \text{ev}(q) = Ts(\ast)\), and the square is a pullback. The universal property of (3) guarantees the existence of \(\tilde{\eta}_{q,a} : I \to c(z, s(x))f_{t,x}\) such that \(\tilde{p}_{q,a}\tilde{\eta}_{q,a} = \eta_y\) and \(\text{ev}_{q,a}(\tilde{\eta}_{q,a} \times_b 1) = s_{z,x}\). Then, with the multiple pullback property, the morphisms \(\tilde{\eta}_{q,a}\) define jointly \(\eta_{(s, y)} : I \to d(e_P(s, y), (s, y))\).

2.7 Theorem. If the pointed \(\text{Set}\)-functor \(T\) satisfies (BC) and \(V\) is complete and locally cartesian closed, then also \(\text{Alg}(T, V)\) is locally cartesian closed.

Proof. Continuing in the notation of 2.5 and 2.6, we equip \(Q\) with the lax algebra structure \(r : TQ \to Q\) that makes the square of diagram (2) a pullback diagram in \(\text{Alg}(T, V)\). Then the
2-cell defined by
\[
r(q, (s, x)) \xrightarrow{\pi_q \times 1} c(\zeta, s(x)) f_{s,x} \times h_a(\xi, x) \xrightarrow{\bar{e}_{\pi_q,x}} c(\zeta, s(x))
\]

makes \( ev : (Q, r) \to (Z, c) \) a homomorphism.

In order to prove the universal property of the partial product, given any other pair \((h : (L, u) \to (Y, b), k : (M, v) \to (Z, c))\), where \( M := L \times_Y X \), we consider the map \( t : L \to P \), defined by \( t(l) := (s_l, h(l)) \), with
\[
((X_{h(l)}, a_{h(l)})) \xrightarrow{s_l} (Z, c) = ((X_{h(l)}, a_{h(l)}) \xrightarrow{j_l} (M, v) \xrightarrow{k} (Z, c)),
\]
where \( j_l \) is the pullback of \( l : (1, e \circ 1) \to (L, u) \) along \( f'' : (M, v) \to (L, u) \). We remark that in the commutative diagram
\[
\begin{array}{c}
Z & \xrightarrow{\text{ev}} & Q & \xrightarrow{q} & X \\
\downarrow{k} & & \downarrow{h_l} & & \downarrow{f} \\
M & \xrightarrow{j_l} & X_{h(l)} & \xrightarrow{h_l} & Y \\
\downarrow{f''} & & \downarrow{t} & & \downarrow{p} \\
L & \xleftarrow{l} & 1 & \xrightarrow{h(l)} & Y
\end{array}
\]
every vertical face of the cube is a pullback in \( \text{Set} \).

Now, for each \( l \in L \) and \( l \in L \) we define \( t_{l,l} : u(l, l) \to d(Tl(l), t(l)) \) componentwise. Since \( evt' = k \) we observe that \( Tk \) factors through the comparison map \( \kappa : TM \to TL \times_{TP} TQ \), defined by the diagram
\[
\begin{array}{c}
TM & \xrightarrow{\kappa} & Tt' \\
\downarrow{\kappa} & & \downarrow{Tf''} \\
TL \times_{TP} TQ & \xrightarrow{\pi_2} & TQ \\
\downarrow{Tf''} & & \downarrow{Tt} \\
TL & \xleftarrow{Tt} & TP \\
\end{array}
\]
that is \( Tk = (Tev)(Tt') = (Tev)\pi_2\kappa \). Since also \( kv \) factors through \( \kappa \), i.e., \( kv = k\bar{k}\kappa \), with (BC) we conclude that the 2-cell \( kv \to c(Tk) \) is of the form
\[
M \xrightarrow{\kappa} TL \times_{TP} TQ \xrightarrow{\varphi} Z.
\]
For each \( x \in X \) and \( q \in TQ \) such that \( f(x) = h(l) \) and \( Tf''(q) = Tt(l) \), let \( m \in TM \) be such that \( (Tf'')(m) = l \) and \( (Tt')(m) = q \). In the diagram
\[
\begin{array}{c}
c(\zeta, s_l(x)) \xrightarrow{k_{m, (l, x)}} v(m, (l, x)) \xrightarrow{a(\xi, x)} \\
u(l, l) \xrightarrow{h_{l,l}} b(\eta, y) \xrightarrow{f_{s,x}}
\end{array}
\]
in \( V \) one has \( \mathfrak{I} = (T \mathfrak{e} \mathfrak{v})(q) \) and the morphism \( k_{m,(l,x)} \) depends only on \( q \) and \( l \). Moreover, the square is a pullback, hence there is a \( V \)-morphism \( \tilde{t}_{\mathfrak{I}l} : u(l, l) \rightarrow c(q, s_l(x))f_{l,x} \) such that \( \tilde{p}_{q,x} \tilde{t}_{\mathfrak{I}l} = h_{l} \) and \( k_{m,(l,x)}(\tilde{t}_{\mathfrak{I}l} \times_h 1) = ev_{q,x} \). With the multiple pullback property, the morphisms \( \tilde{t}_{\mathfrak{I}l} \) define the unique \( 2 \)-cell that makes \( t : (L, u) \rightarrow (P, d) \) a homomorphism.

If in the proof we take for \((Y, b)\) the terminal object of \( \text{Alg}(T, V) \), that is, the pair \((1, T)\) where the lax structure \( \top \) is constantly equal to the terminal object of \( V \), we conclude:

**2.8 Corollary.** If the pointed \( \text{Set} \)-functor \( T \) satisfies \((\text{BC})\) and \( V \) is complete and cartesian closed, then also \( \text{Alg}(T, V) \) is cartesian closed.

We explain now the strength of our Beck-Chevalley condition.

**2.9 Proposition.** For \( T \) and \( V \) as in \ref{sec:2.5}, let \( V(I, 0) = \emptyset \). Then:

(a) If \( T \) satisfies \((\text{BC})\), then \( T \) transforms pullbacks into weak pullbacks. The two conditions are actually equivalent when \( V \) is thin \( \text{i.e.} \) a preordered class.

(b) If \( V \) is not thin, satisfaction of \((\text{BC})\) by \( T \) is equivalent to preservation of pullbacks by \( T \).

(c) If \( V \) is cartesian closed, with \( I = 1 \) the terminal object, then \( T \) satisfies \((\text{BC})\) if and only if \( (Tf)^{\circ}Tg = Tk(Th)^{\circ} \), for every pullback diagram

\[
\begin{array}{ccc}
W & \xrightarrow{k} & X \\
\downarrow{h} & & \downarrow{f} \\
Z & \xrightarrow{g} & Y
\end{array}
\]

in \( \text{Set} \).

**Proof.** (a) Let \( \kappa : TW \rightarrow U \) be the comparison map of diagram \((1)\). By \((\text{BC})\) the \( 2 \)-cell \( \kappa_{\mathfrak{I}} : \kappa \rightarrow \kappa \kappa^{\circ} \kappa \) is the image by \( -\kappa \) of a \( 2 \)-cell \( \sigma : 1_U \rightarrow \kappa \kappa^{\circ} \). Hence, for each \( u \in U \) there is a \( V \)-morphism \( I \rightarrow \kappa \kappa^{\circ} \{u\} = \sum_{w \in TW : \kappa(w) = u} \kappa(w, u) \). Therefore the set \( \{w \in TW | \kappa(w) = u\} \) cannot be empty, that is, \( \kappa \) is surjective.

If \( V \) is thin and \( \kappa \) is surjective, there is a (necessarily unique) \( 2 \)-cell \( 1_U \rightarrow \kappa \kappa^{\circ} \). Then each \( 2 \)-cell \( \psi : \kappa r \rightarrow \kappa s \) induces a \( 2 \)-cell \( \varphi : r \rightarrow s \) defined by

\[
\begin{array}{ccc}
r & \xrightarrow{r} & r \kappa^{\circ} \\
\downarrow{r \sigma} & \xrightarrow{r \kappa^{\circ}} & \psi \kappa^{\circ} \\
\downarrow{\kappa \kappa^{\circ} \psi} & \xrightarrow{s \kappa^{\circ} \psi} & s \kappa^{\circ} \\
\downarrow{s \sigma} & \xrightarrow{s \kappa^{\circ}} & s
\end{array}
\]

whose image under \( -\kappa \) is necessarily \( \psi \).

(b) If \( T \) preserves pullbacks, then \( \kappa \) is an isomorphism and \((\text{BC})\) holds.

Conversely, let \( T \) satisfy \((\text{BC})\) and let \( \kappa : TW \rightarrow U \) be a comparison map as in \((1)\). We consider \( w_0, w_1 \in TW \) with \( \kappa(w_0) = \kappa(w_1) \) and \( V \)-morphisms \( \alpha, \beta : v \rightarrow v' \) with \( \alpha \neq \beta \), and define \( r : U \times U \rightarrow V \) by \( r(u, u') = v \) and \( s : U \times U \rightarrow V \) by \( s(u, u') = v' \). The \( 2 \)-cell \( \psi : r \kappa \rightarrow s \kappa \), with \( \psi_{w,u} = \alpha \) if \( w = w_0 \) and \( \psi_{w,u} = \beta \) elsewhere, factors through \( \kappa \) only if \( w_0 = w_1 \).

(c) For any commutative diagram \((4)\) there is a \( 2 \)-cell \( kh^{\circ} \rightarrow f^{\circ} g \), defined by

\[
kh^{\circ} \xrightarrow{\eta kh^{\circ}} f^{\circ} fh^{\circ} = f^{\circ} g h^{\circ} \xrightarrow{f^{\circ} ge} f^{\circ} g,
\]
For \( v \alpha \) constantly equal to \( I \) respectively, constantly equal to \( \alpha \) Therefore we define \( \varepsilon \) by:

\[
\varepsilon_{w_0, w_1} = \begin{cases} 1 + \alpha & \text{if } w = w_0, \\ 1 + \beta & \text{elsewhere.} \end{cases}
\]

The square is a pullback. Hence the morphism \((\text{id}, \varepsilon)\) factors through the partial product via \( t \times_Y \text{id} \), with \( t : Z \to P \). Since the 2-cell of \( t \times_Y \text{id} \) is obtained by a pullback construction and \( \kappa(w_0) = \kappa(w_1) \), its 2-cell "identifies" \( w_0 \) and \( w_1 \), hence \( \varepsilon_{w_0, w} = \varepsilon_{w_1, w} \), that is, \( 1 + \alpha = 1 + \beta \). Therefore \( \alpha = \beta \), by extensivity of \( V \).
3 (Co)completeness of the category \(\text{Alg}(T, V)\)

3.1 We assume \(V\) to be complete and cocomplete. The construction of limits in \(\text{Alg}(T, V)\) reduces to a combined construction of limits in \(\text{Set}\) and \(V\), as we show next.

The limit of a functor

\[
\begin{align*}
F : D & \to \text{Alg}(T, V) \\
D & \mapsto (FD, a_D) \\
D \xrightarrow{f} E & \mapsto (FD, a_D) \xrightarrow{Ff} (FE, a_E)
\end{align*}
\]

is constructed in two steps.

First we consider the composition of \(F\) with the forgetful functor into \(\text{Set}\)

\[
\begin{array}{ccc}
D & \xrightarrow{F} & \text{Alg}(T, V) & \xrightarrow{p^D} & \text{Set} \\
\end{array}
\]

and construct its limit in \(\text{Set}\)

\[
(L \xrightarrow{p^D} FD)_{D \in D}.
\]

Then, we define the \((T, V)\)-algebra structure \(a : TL \to L\), that is the map \(a : TX \times X \to V\), pointwise. For every \(l \in TL\) and \(l \in L\), we consider now the functor

\[
\begin{align*}
F_l : D & \to V \\
D & \mapsto a_D(Tp^D(l), p^D(l)) \\
D \xrightarrow{f} E & \mapsto a_D(Tp^D(l), p^D(l)) \xrightarrow{Ff_{tp^D(l), l}} a_E(Tp^E(l), p^E(l))
\end{align*}
\]

and its limit in \(V\)

\[
(a(l, l) \xrightarrow{p^D_{l,l}} a_D(Tp^D(l), p^D(l)))_{D \in D}.
\]

This equips \(p^D : (L, a) \to (FD, a_D)\) with a 2-cell \(p^D a \to a_D Tp^D\).

By construction

\[
(L, a) \xrightarrow{p^D} (FD, a_D)
\]

is a cone for \(F\). To check that it is a limit, let

\[
(Y, b) \xrightarrow{g^D} (FD, a_D)
\]

be a cone for \(F\). By construction of \((L, p^D)\), there exists a map \(t : Y \to L\) such that \(p^D t = g^D\) for each \(D \in D\). For each \(\eta \in TY\) and \(y \in Y\),

\[
b(\eta, y) \xrightarrow{g^D_{\eta, y}} a_D(Tp^D(Tt(\eta)), p^D(t(y)))
\]

is a cone for the functor \(F_{Tt(\eta), t(y)}\). Hence, by construction of \(a(Tt(\eta), t(y))\), there exists a unique \(V\)-morphism \(t_{\eta, y}\) making the diagram

\[
\begin{array}{ccc}
b(\eta, y) & \xrightarrow{g^D_{\eta, y}} & a_D(Tp^D(Tt(\eta)), p^D(t(y))) \\
\end{array}
\]


commutative. These $\mathbf{V}$-morphisms define pointwise the unique 2-cell $gb \to p^D a$.

For each $l \in L$, $\eta_l : I \to a(e_L(l), l)$ is the morphism induced by the cone

$$(\eta^D_{p^D(l), p^D(l)} : I \to a_D(e_{p^D(l)}, p^D(l))_{D \in D}).$$

### 3.2 Cocompleteness.
To construct the colimit of a functor $F : D \to \text{Alg}(T, \mathbf{V})$ we first proceed analogously to the limit construction. That is, we form the colimit in $\text{Set}$

$$(FD \xrightarrow{i_D} Q)_{D \in D}$$

of the functor (5).

To construct the structure $c : TQ \to Q$, for each $q \in TQ$ and $q \in Q$, we consider the functor $F^q,q : D \to \mathbf{V}$, with

$$F^q,q(D) = \sum \{ a_D(x, x) | Ti^D(x) = q, i^D(x) = q \},$$

and, for $f : D \to E$, the morphism $F^q,q(f) : F^q,q(D) \to F^q,q(E)$ is induced by

$$a_D(x, x) \xrightarrow{Ff_x} a_E(Tf(x), f(x)) \xrightarrow{\sum} \{ a_E(y, y) | Ti^E(y) = q, i^E(y) = q \} = F^q,q(E).$$

and denote by $\tilde{c}(q, q)$ the colimit of $F^q,q$. If $q \neq e_Q(q)$ for $q \in Q$, then $\tilde{c}(q, q)$ is in fact the structure $c(q, q)$ on the colimit. For $q = e_Q(q)$, the multiple pushout

$$\xymatrix{ a_D(e_{p^D(x)}, x) \ar[rd]_{\eta_x} \ar[rr]^{i^D_{p^D(x), q}} & & \tilde{c}(e_Q(q), q) \ar[ld]_{\eta_x} \ar[dl]_{a_D(e_{p^D(x)}, x)} \ar[dd]_c \ar[ur]_{i^D_{e_{p^D(x)}, q}} \ar[ur]_c \\
I & & c(e_Q(q), q),}
$$

defines $c(e_Q(q), q)$, with $D \in D$ and $x \in FD$ such that $i^D(x) = q$.

### 4 Representability of partial morphisms

4.1 Let $S$ be a pullback-stable class of morphisms of a category $C$. An $S$-partial map from $X$ to $Y$ is a pair $(X \xleftarrow{s} U \rightarrow Y)$ where $s \in S$. We say that $S$ has a classifier if there is a morphism $\text{true} : 1 \to \tilde{1}$ in $S$ such that every morphism in $S$ is, in a unique way, a pullback of $\text{true}$; $C$ has $S$-partial map classifiers if, for every $Y \in C$, there is a morphism $\text{true}_Y : Y \to \tilde{Y}$ in $S$ such that every $S$-partial map $(X \xleftarrow{s} U \rightarrow Y)$ from $X$ to $Y$ can be uniquely completed so that the diagram

$$\begin{array}{ccc}
U & \rightarrow & Y \\
\downarrow & & \downarrow_{\text{true}_Y} \\
X & \rightarrow & \tilde{Y},
\end{array}$$

is a pullback.
From Corollary 4.6 of [10] it follows that:

4.2 Proposition. If $S$ is a pullback-stable class of morphisms in a finitely complete locally cartesian-closed category $C$, then the following assertions are equivalent:

(i) $S$ has a classifier;

(ii) $C$ has $S$-partial map classifiers.

4.3 Our goal is to investigate whether the category $\text{Alg}(T, V)$ has $S$-partial map classifiers, for the class $S$ of extremal monomorphisms. For that we first observe:

4.4 Lemma. An $\text{Alg}(T, V)$-morphism $s : (U, c) \to (X, a)$ is an extremal monomorphism if and only if the map $s : U \to X$ is injective and, for each $u \in TU$ and $u \in U$, $s_{u,u} : c(u, u) \to a(x, x)$ is an isomorphism in $V$.

4.5 Proposition. In $\text{Alg}(T, V)$ the class of extremal monomorphisms has a classifier.

Proof. For $\tilde{1} = (1 + 1, \tilde{\top})$, where $\tilde{\top}$ is pointwise terminal, we consider the inclusion $\text{true} : 1 \to \tilde{1}$ onto the first summand. For every extremal monomorphism $s : (U, c) \to (X, a)$, we define $\chi_U : (X, a) \to \tilde{1}$ with $\chi_U : X \to 1 + 1$ the characteristic map of $s(U)$, and the 2-cell constantly $! : a(x, x) \to 1$. Then the diagram below

$$
\begin{array}{ccc}
(U, s) & \xrightarrow{i} & 1 \\
\downarrow s & & \downarrow \text{true} \\
(X, a) & \xrightarrow{\chi_U} & \tilde{1}.
\end{array}
$$

is a pullback diagram; it is in fact the unique possible diagram that presents $s$ as a pullback of true. \qed

Using Theorem 2.7 and Proposition 4.5, we conclude that:

4.6 Theorem. If the pointed $\text{Set}$-functor $T$ satisfies (BC) and $V$ is a complete and cocomplete locally cartesian closed category, then $\text{Alg}(T, V)$ is a quasitopos.

4.7 Remark. Representability of (extremal mono)-partial maps can also be proved directly, and in this way one obtains a slight improvement of Theorem 4.6: $\text{Alg}(T, V)$ is a quasi-topos whenever $T$ satisfies (BC) and $V$ is a complete and cocomplete cartesian closed category, not necessarily locally so.

5 Examples.

5.1 We start off with the trivial functor $T$ which maps every set to a terminal object $1$ of $\text{Set}$. $T$ preserves pullbacks. Choosing for $I$ the top element of any (complete) lattice $V$ we obtain with $\text{Alg}(T, V)$ nothing but the topos $\text{Set}$. This shows that local cartesian closedness of $V$ is
not a necessary condition for local cartesian closedness of Alg(\(T, V\)). We also note that \(T\) does not carry the structure of a monad.

If, for the same \(T\), we choose \(V = \text{Set}\), then Alg(\(T, \text{Set}\)) is the formal coproduct completion of the category \(\text{Set}_*\) of pointed sets, i.e. Alg(\(T, \text{Set}\)) \(\cong\) Fam(\(\text{Set}_*\)).

5.2 Let \(T = \text{Id}, e = \text{id}\). Considering for \(V\) as in \([9]\) the two-element chain \(2\), the extended half-line \(\mathbb{R}_+ = [0, \infty)\) (with the natural order reversed), and the category \(\text{Set}\), one obtains with Alg(\(T, V\)) the category of

- sets with a reflexive relation
- sets with a fuzzy reflexive relation
- reflexive directed graphs,

respectively.

More generally, if we let \(TX = X^n\) for a non-negative integer \(n\), with the same choices for \(V\) one obtains

- sets with a reflexive \((n + 1)\)-ary relation
- sets with a fuzzy reflexive \((n + 1)\)-ary relation
- reflexive directed “multigraphs” given by sets of vertices and of edges, with an edge having an ordered \(n\)-tuple of vertices as its source and a single edge as its target; reflexivity means that there is a distinguished edge \((x, \cdots, x) \to x\) for each vertex \(x\).

Note that the case \(n = 0\) encompasses Example 5.1.

5.3 For a fixed monoid \(M\), let \(T\) belong to the monad \(T\) arising from the adjunction

\[
\text{Set}^M \xleftarrow{i_{\text{Set}}} \text{Set,}
\]

i.e. \(TX = M \times X\) with \(e_X(x) = (0, x)\), with 0 neutral in \(M\) (writing the composition in \(M\) additively). \(T\) preserves pullbacks. The quasitopos Alg(\(T, \text{Set}\)) may be described as follows. Its objects are “\(M\)-normed reflexive graphs”, given by a set \(X\) of vertices and sets \(a(x, y)\) of edges from \(x\) to \(y\) which come with a “norm” \(v_{x,y} : a(x, y) \to M\) for all \(x, y \in X\); there is a distinguished edge \(1_x : x \to x\) with \(v_{x,x}(1_x) = 0\). Morphisms must preserve the norm. Of course, for trivial \(M\) we are back to directed graphs as in 5.2.

It is interesting to note that if one forms Alg(\(T, \text{Set}\)) for the (untruncated) monad \(T\) (see \([9]\), then Alg(\(T, \text{Set}\)) is precisely the comma category \(\text{Cat}/M\), where \(M\) is considered a one-object category; its objects are categories which come with a norm function \(v\) for morphisms satisfying \(v(gf) = v(g) + v(f)\) for composable morphisms \(f, g\).

5.4 Let \(T = U\) be the ultrafilter functor, as mentioned in the Introduction. \(U\) transforms pullbacks into weak pullback diagrams. Hence, for \(V = 2\) we obtain with Alg(\(T, 2\)) the quasitopos of pseudotopological spaces, and for \(V = \mathbb{R}_+\) the quasitopos of (what should be called) quasi-approach spaces (see \([9, 8]\)). If we choose for \(V\) the extensive category \(\text{Set}\), then the resulting
category Alg(U, Set) is a rather naturally defined supercategory of the category of ultracategories (as defined in [9]) but fails to be locally cartesian closed, according to 2.9(b) and 2.10.

References


