TRIQUOTIENT MAPS VIA ULTRAFILTER CONVERGENCE

MARIA MANUEL CLEMENTINO AND DIRK HOFMANN

Abstract. In this paper we characterize triquotient maps as those that are surjective on chains of convergent ultrafilters, extending the result known for triquotient maps between finite topological spaces.

1. Introduction

Triquotient maps, introduced by Michael [11], fit very nicely among classes of special quotient maps:

– proper maps and open maps are triquotient maps;
– triquotient maps are effective descent maps, which in turn are biquotient maps.

A recent study of Janelidze and Sobral on the behaviour of the mentioned classes of morphisms, when defined between finite topological spaces, led to very interesting characterizations based on point convergence (see Theorem 2.2). Among these characterizations, it was established the following

**Theorem I.** If $X$ and $Y$ are finite topological spaces, a continuous map $f : X \to Y$ is a triquotient map if and only if it is surjective on chains of convergent points.

To pass from the finite to the infinite case one must replace points by (ultra)filters – or (ultra)nets – and, besides the case of effective descent and triquotient maps (and partially local homeomorphisms), the characterizations are straightforward. To establish a general characterization of triquotient maps that includes the former Theorem, some new notions and techniques are needed.

This is the central part of this paper: we introduce and study a new category, defined via ultrafilter convergence, endowed with a special endofunctor that is used to define chains of convergent ultrafilters, and that finally leads to the

**Theorem II.** A continuous map $f : X \to Y$ is a triquotient map if and only if it is surjective on chains of convergent ultrafilters.

Moreover, these characterizations turn out to be very effective on proving stability properties for special kinds of limits, since initial structures – in particular limit structures – are easily described by convergent ultrafilters. This gives rise to unified proofs of results obtained separately, and will be the subject of a forthcoming note.

1991 Mathematics Subject Classification. 54C10, 54A20, 54B30, 18A20, 18B30.

Key words and phrases. Biquotient map, effective descent map and triquotient map, convergent structure.

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra. The first author also thanks Project PRAXIS XXI 2/2.1/MAT/46/94.
Acknowledgment. We thank George Janelidze, Manuela Sobral and Walter Tholen for valuable discussions on the subject of this paper.

2. Basic definitions and results

For a topological space $X$ we denote its topology by $\mathcal{O}X$. For $x \in X$, $\mathcal{O}(x)$ denotes the set of open subsets of $X$ containing $x$. If $x$ and $y$ are points of $X$, by $y \rightarrow x$ we mean that $x \in \{y\}$.

Definitions 2.1. A topological continuous map $f : X \rightarrow Y$ is:

1. a biquotient map if, whenever $y \in Y$ and $A$ is an open covering of $f^{-1}(y)$, then finitely many $f(A)$, with $A \in A$, cover some neighbourhood of $y$ in $Y$;
2. effective descent (descent) if the pullback functor $f^* : \textbf{Top}/Y \rightarrow \textbf{Top}/X$, that assigns to each $g : Z \rightarrow Y$ its pullback along $f$, is (pre)monadic: see [8];
3. a triquotient map if there exists a map $(\_)^\sharp : \mathcal{O}X \rightarrow \mathcal{O}Y$ such that:
   - (T1) $(\forall U \in \mathcal{O}X) \ U^\sharp \subseteq f(U)$,
   - (T2) $X^\sharp = Y$,
   - (T3) $(\forall U, V \in \mathcal{O}X) \ U \subseteq V \Rightarrow U^\sharp \subseteq V^\sharp$,
   - (T4) $(\forall U \in \mathcal{O}X) \ (\forall y \in U^\sharp) \ (\forall S \subseteq \mathcal{O}X \text{ directed}) \ f^{-1}(y) \cap U \subseteq \bigcup S \Rightarrow \exists S \in \Sigma : y \in S^\sharp$;
4. proper (perfect) if it is closed and has compact fibres (and Hausdorff, i.e. if $f(x) = f(x')$ and $x \neq x'$ there exist $U \in \mathcal{O}(x)$ and $V \in \mathcal{O}(x')$ with $U \cap V = \emptyset$).

Concerning the notion of triquotient map, we note that (T3) is implied by (T4).

We also remark that every proper map $f : X \rightarrow Y$ is triquotient: take $U^\sharp := Y - f(X - U)$ for $U \in \mathcal{O}X$, as well as every open map: take $U^\sharp := f(U)$. Plewe showed that – both on topological spaces and locales – triquotient maps are effective descent (see [12]). These latter are descent maps, which are exactly the biquotient maps (see [12]) introduced independently by Michael [10], Hájek [5], by the name of limit lifting maps, and Day and Kelly [4], as universal quotient maps.

The following results may be found in [7] and [3]:

Theorem 2.2. If $X$ and $Y$ are finite topological spaces and $f : X \rightarrow Y$ is a continuous map, then:

1. $f$ is a biquotient map if and only if, for each $y_1 \rightarrow y_0$ in $Y$, there exists $x_1 \rightarrow x_0$ in $X$ with $f(x_i) = y_i$ for $i = 0, 1$;
2. $f$ is effective descent if and only if, for each $y_2 \rightarrow y_1 \rightarrow y_0$ in $Y$, there exists $x_2 \rightarrow x_1 \rightarrow x_0$ in $X$ with $f(x_i) = y_i$ for $i = 0, 1, 2$;
3. $f$ is a triquotient map if and only if, for each chain $y_n \rightarrow \ldots \rightarrow y_0$ ($n \in \mathbb{N}$) in $Y$, there exists a chain $x_n \rightarrow \ldots \rightarrow x_0$ in $X$ with $f(x_i) = y_i$ for each $i = 0, 1, \ldots, n$;
4. $f$ is proper (perfect) if and only if, for each $x_1 \in X$ and $f(x_1) \rightarrow y_0$ in $Y$, there exists (a unique) $x_0$ in $X$ with $x_1 \rightarrow x_0$ and $f(x_0) = y_0$;
5. $f$ is open (local homeomorphism) if and only if, for each $x_0 \in X$ and $y_1 \rightarrow f(x_0)$ in $Y$, there exists (a unique) $x_1$ in $X$ with $x_1 \rightarrow x_0$ and $f(x_1) = y_1$.

The statements 1, 4, and partially 5 can be easily generalized using ultrafilters, while a possible generalization of 2 is the well-known Reiterman-Tholen characterization of topological effective descent maps [13].
Theorem 2.3. If $f : X \to Y$ is a continuous map, then:

1. $f$ is a biquotient map if and only if, for each ultrafilter $b \to y$ in $Y$, there exists an ultrafilter $a \to x$ in $X$ with $f(a) = b$ and $f(x) = y$:

   \begin{align*}
   &X \xrightarrow{f} Y \\
   &a \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
We denote by $(X,r)$ ultrarelational spaces $(X,\tau)$ is a set ultrarelational spaces $An$ topological spaces – is the fact that principal ultrafilters do not need to converge. ultrarelational spaces, whose particularity – that distinguishes them from pseudo-

give rise to the combination of 2 sorts of convergent ultrafilters while n-chains, for maps, but in a higher order: 2-chains for effective descent maps between finite spaces combination of ultrafilters convergence, as it is already the case of effective descent

We remarked that the notation introduced for points, $y$ with $(X,r)$ the principal ultrafilter defined by $y$ we preferred them to nets, and investigated similar characterizations for effective descent and triquotient maps. This is the aim of the next sections.

First we recall some known facts on ultrafilters. If $X$ is a set, the set $\mathcal{U}(X)$ of ultrafilters on $X$ may be endowed with the Zariski closure, becoming a compact Hausdorff space (see [9]). For a map $f : X \to Y$ and $a \in \mathcal{U}(X)$, $f(a)$ denotes the filter generated by $\{ f(A) \mid A \in a \}$, which is automatically an ultrafilter since $a$ is. The map $\mathcal{U}f : \mathcal{U}(X) \to \mathcal{U}(Y)$, $a \mapsto f(a)$ is continuous.

3. The category URS

In order to characterize triquotient maps using convergence, we will need a combination of ultrafilters convergence, as it is already the case of effective descent maps, but in a higher order: 2-chains for effective descent maps between finite spaces give rise to the combination of 2 sorts of convergent ultrafilters while n-chains, for triquotient maps, will give rise to infinite chains of convergent ultrafilters.

To make the description as simple as possible, we introduce the category of ultrarelational spaces, whose particularity – that distinguishes them from pseudo-topological spaces – is the fact that principal ultrafilters do not need to converge. Definition 3.1. An ultrarelation on a set $X$ is a subset $r \subseteq \mathcal{U}(X) \times X$. An ultrarelational space is a set $X$ equipped with an ultrarelation $r$ on $X$. Given ultrarelational spaces $(X,r)$ and $(Y,s)$, a map $f : X \to Y$ is continuous if, for each $(a,x) \in r$, $(f(a), f(x)) \in s$.

We denote by $\text{URS}$ the category of ultrarelational spaces and continuous maps.

We will often use the more suggestive notation $a \to x$ instead of $(a,x) \in r$.

The category $\text{URS}$ is equipped with a canonical faithful functor $\mathcal{U} : \text{URS} \to \text{Set}$ sending $(X,r)$ to $X$. The construct $(\text{URS}, \mathcal{U})$ is topological (in the sense of [1]) and therefore concretely complete and cocomplete. It contains $\text{Top}$ as a full and concrete subcategory: each topology $\tau$ on $X$ defines an ultrarelation $r_{(X,\tau)}$ by $r_{(X,\tau)} = \{(a,x) \mid a \to x \text{ w.r.t. the topology } \tau \}$. We remark that the notation introduced for points, $y \to x$ if $x \in \overline{\{y\}}$, is consistent with the notation for ultrafilters: $y \to x$ if and only if $\eta(y) \to x$, where $\eta(y)$ denotes the principal ultrafilter defined by $y$.

For an ultrarelational space $(X,r)$, we consider the projection map $p_{(X,r)} : r \to X$, with $(a,x) \mapsto x$, and define the following ultrarelation $R_{(X,r)}$ on $r$:

$$R_{(X,r)} = \{ (\mathcal{A}, (a,x)) \in \mathcal{U}(r) \times r \mid p_{(X,r)}(\mathcal{A}) = a \}.$$  

We denote the ultrarelational space $(r, R_{(X,r)})$ by $\text{Ult}(X,r)$. Obviously, the map $p_{(X,r)} : \text{Ult}(X,r) \to (X,r)$ is continuous, by definition of the ultrarelational structure on $r$.

Some extra conditions on these spaces will give us back well-known structures:

Definition 3.2. An ultrarelational space $(X,r)$ is called

1. weak reflexive if, for each $x \in X$, there exists an $a \in \mathcal{U}(X)$ such that $(a,x) \in r$;
(2) reflexive if, for each \( x \in X \), \((\dot{x}, x) \in r\);
(3) fibre-closed if, for each \( x \in X \), \( \{a \in \mathcal{U}(X) \mid (a, x) \in r\} \) is closed in \( \mathcal{U}(X) \) with respect to the Zariski topology;
(4) transitive if the map
\[
\mu_{(X,r)} : R_{(X,r)} \rightarrow \mathcal{U}(X) \times X, \quad (\mathfrak{A}, (a, x)) \mapsto \left( \bigcup_{A \in \mathfrak{A}} (a', x) \right)
\]
factors via the inclusion \( r \hookrightarrow \mathcal{U}(X) \times X \).

An ultrarelational space \((X, r)\) is weak reflexive if and only if \( p_{(X,r)} : \text{Ult}(X, r) \rightarrow (X, r)\) is surjective. Hence \( \text{Ult}(X, r)\) is weak reflexive provided that \((X, r)\) is: for an \((a, x)\), \( p_{(X,r)}^{-1}(a) \) is a filter base and any ultrafilter \( \mathfrak{A} \) in \( \text{Ult}(X, r)\) containing it converges to \((a, x)\). Moreover, \( \text{Ult}(X, r)\) is always fibre-closed.

The choice of the name transitive needs some justification. Assume that a chain \( x_2 \rightarrow x_1 \rightarrow x_0 \) in \( X \) is given. Hence \((\mathfrak{A}, (a, x_0)) \in R_{(X,r)}\) with \( a = \eta(x_1) \) and \( \mathfrak{A} = \eta(\eta(x_2), x_1) \). Then
\[
\eta(x_2) \subseteq \bigcup_{A \in \mathfrak{A}} (a', x_1)
\]
and therefore \( \mu_{(X,r)}(\eta(x_2), x_1), (\eta(x_1), x_0)) = (\eta(x_2), x_0) \).

These properties define full subcategories of \( \mathcal{URS} \): the category \( \text{PsTop} \) (\( \text{PrTop} \); \( \text{Top} \)) of pseudotopological (pretopological; topological) spaces is concretely isomorphic to the full subcategory of \( \mathcal{URS} \) consisting of all reflexive (reflexive and fibre-closed; reflexive and transitive) ultrarelational spaces.

4. The functor \( \text{Ult} \)

Each ultrarelational continuous map \( f : (X, r) \rightarrow (Y, s) \) induces a map
\[
\text{Ult}(f) : r \rightarrow s, \quad (a, x) \mapsto (f(a), f(x)),
\]
and the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p_{(X,r)}} & & \downarrow{p_{(Y,s)}} \\
\text{Ult}(f) & \xrightarrow{\text{Ult}(f)} & \text{Ult}(f)
\end{array}
\]
commutes. It is then clear that \( \text{Ult}(f) : \text{Ult}(X, r) \rightarrow \text{Ult}(Y, s)\) is continuous. Moreover, the equalities \( \text{Ult}(1) = 1 \) and \( \text{Ult}(f \circ g) = \text{Ult}(f) \circ \text{Ult}(g)\) hold, hence \( \text{Ult} : \mathcal{URS} \rightarrow \mathcal{URS} \) is a functor and \( (p_{(X,r)})(X,r) \in \mathcal{Ob} \mathcal{URS} \) : \( \text{Ult} \rightarrow \text{Id}_{\mathcal{URS}} \) is a natural transformation.

Note that we have \( \text{Ult}(p_{(X,r)}) = p_{\text{Ult}(X,r)} \) for each ultrarelational space \((X, r)\), that is, \((\text{Ult}, p)\) is a well copointed endofunctor (see [6]).

Hence, we may define endofunctors \( \text{Ult}^\alpha \) and natural transformations \( p_\beta^\alpha \) for ordinal numbers \( \alpha, \beta \) with \( \beta \leq \alpha \), by:
- \( \text{Ult}^0 = \text{Id}_{\mathcal{URS}}, \text{Ult}^1 = 1_{\mathcal{URS}} \);
- \( \text{Ult}^\alpha + 1 = \text{Ult}(\text{Ult}^\alpha), \text{Ult}^\alpha \cdot \text{Ult}^\beta = p_\beta^\alpha \cdot \text{Ult}^\alpha \) and \( \text{Ult}^\alpha + 1 = 1_{\mathcal{URS}} \), for \( \beta \leq \alpha \);
- \( \text{Ult}^\lambda = \lim_{\beta \leq \alpha} p_{\lambda}^\beta, p_{\lambda}^\beta = \text{the limit projection and } p_{\lambda}^1 = 1_{\mathcal{URS}} \), for every limit ordinal \( \lambda \) and every \( \beta < \lambda \).

From now on, since we usually work with only one ultrarelation on a set \( X \), for an ultrarelational space we relax our notation and write \( X \) instead of \((X, r)\).
Also, we will denote \( \text{Ult}^\alpha(X) \) by \( X_\alpha \) and \( \text{Ult}^\alpha(f) \) by \( f_\alpha \), for every continuous map \( f : X \to Y \) between ultrarelational spaces.

This transfinite construction can be easily described: for each ultrarelational space \( X \) and each ordinal \( \alpha \),
\[
X_\alpha = \{(a_\beta)_{\beta \in \alpha}, x) \in \prod_{\beta \in \alpha} \mathcal{U}(X_\beta) \times X \mid a_0 \to x \text{ and } (\forall \gamma \leq \beta < \alpha) (p_\beta^\alpha)(x)(a_\beta) = a_\gamma, \}
\]
for each \( \beta \leq \alpha \), the projection \((p_\beta^\alpha)_X : X_\alpha \to X_\beta\) is defined by
\[
(p_\beta^\alpha)_X((a_\gamma)_{\gamma \in \beta}, x) = ((a_\gamma)_{\gamma \in \beta}, x),
\]
and the ultrarelational structure in \( X_\alpha \) is defined by
\[
a_\alpha \to ((a_\beta)_{\beta \in \alpha}, x) \iff (\forall \beta \in \alpha) (p_\beta^\alpha)(x)(a_\beta) = a_\beta.
\]

Finally, if \( f : X \to Y \) is a continuous map, then, for each ordinal \( \alpha \) and each
\[
((a_\beta)_{\beta \in \alpha}, x) \in X_\alpha, \ f_\alpha((a_\beta)_{\beta \in \alpha}, x) = (((f_\beta(a_\beta))_{\beta \in \alpha}, f(x)).
\]

We remark that, for each ordinal \( \alpha \) and each ultrarelational space \( X \), an element of \( X_{\alpha+1} \) is given by an ultrafilter \( a_\alpha \in \mathcal{U}(X_\alpha) \) and an element \( x \in X \) such that \((p_\alpha^\beta)(x)(a_\alpha) \to x\). The map \( f_{\alpha+1} : X_{\alpha+1} \to Y_{\alpha+1} \) is surjective if and only if, for each ultrafilter \( b_\alpha \) on \( Y_\alpha \) and each \( y \in Y \) such that \((p_\beta^\alpha)(y)(b_\alpha) \to y\), there exist an ultrafilter \( a_\alpha \) on \( X_\alpha \) and an \( x \in f^{-1}(y) \) such that \((p_\alpha^\beta)(x)(a_\alpha) \to x \) and \( f_\alpha(a_\alpha) = b_\alpha \).

Hence, by Theorem 2.3, if \( X \) and \( Y \) are topological spaces, a continuous map \( f : X \to Y \) is a biquotient map if and only if \( f_1 \), and then also \( f_0 \), is surjective.

Next we will show that this kind of surjectivity condition characterizes also effective descent and triquotient maps in \( \text{Top} \). Therefore, we introduce the following

**Definition 4.1.** If \( \alpha \) is an ordinal number, an ultrarelational continuous map \( f : X \to Y \) is said to be \( \alpha \)-surjective if \( f_\beta : X_\beta \to Y_\beta \) is surjective for every \( \beta \in \alpha \). The map \( f \) is called \( \Omega \)-surjective if \( f_\alpha \) is surjective for every ordinal \( \alpha \).

Hence, 1-surjective are just surjective maps, while our observation above means that a biquotient map in \( \text{Top} \) is a 2-surjective map.

### 5. 3-surjective Maps

The continuous maps \( f : X \to Y \) between topological spaces such that \( f_2 \) (and then also \( f_0 \) and \( f_1 \)) is surjective are very well-known: they are exactly the effective descent maps in \( \text{Top} \), as we show below. For that we will make use of Reiterman-Tholen characterization (Theorem 2.3). We first start showing that the data they used may be easily interpreted using the functor \( \text{Ult} \).

**Lemma 5.1.** If \( Y \) is a topological space and \( \mathcal{F}_Y = \{\{I, u, (f_i), (y_i), y) \mid u \text{ ultrafilter on } I, f_i \to y_i \text{ and } y_i \overset{u}{\to} y\} \), there are maps \( \Phi : Y_2 \to \mathcal{F}_Y \) and \( \Psi : \mathcal{F}_Y \to Y_2 \) such that \( \Psi \cdot \Phi = 1_{Y_2} \).

**Proof.** For any \((\mathcal{B}, (b, y))\) in \( Y_2 \), \( \mathcal{B} \) is an ultrafilter on \( Y_1 \) and, for \( \phi = p : Y_1 \to Y \),
\[
\Phi(\mathcal{B}, (b, y)) := (Y_1, \mathcal{B}, (f_i)(f_i', y_i')(y_i')_{(i, y') \in Y_1}, y)
\]
belongs to \( \mathcal{F}_Y \) since \((f, y') \in Y_1 \), that is \( f \to y' \), and \( y' \overset{\mathcal{B}}{\to} y \) by the definition of the ultrarelational structure on \( Y_1 \).

On the other hand, if \((I, u, (f_i), (y_i), y) \in \mathcal{F}_Y \), with \( \phi : I \to Y \) inducing \( y_i \overset{u}{\to} y \), for \( \psi : I \to Y_1 \) defined by \( \psi(i) = (f_i, y_i) \), we may define
\[
\Psi(I, u, (f_i), (y_i), y) := (\psi(u), (\phi(u), y)),
\]
and it is easy to check that $\Psi \cdot \Phi = 1_{Y_2}$.  

**Theorem 5.2.** A topological continuous map $f : X \to Y$ is effective descent if and only if it is 3-surjective, that is:

$$
\begin{array}{c}
\xymatrix{
X \ar[d]^f & \mathfrak{A} \ar[r] & a \ar[r] & x \\
Y & \mathfrak{B} \ar[r] & b & y
}\end{array}
$$

*Proof.* Assume first that $f_2$ is surjective and let $I$ be an index set, $b_i (i \in I)$ be a family of ultrafilters on $Y$ converging to $y_i$ and $y_i \buildrel a \over \to y$ with $u$ ultrafilter on $I$. Considering its corresponding element $(\mathfrak{B}, (b, y))$ in $Y_2$, since $f_2$ is surjective there exist an element $x \in f^{-1}(y)$ and ultrafilters $a$ on $X$ and $\mathfrak{A}$ on $X_1$ such that

$$
\begin{array}{c}
\xymatrix{
X \ar[d]^f & \mathfrak{A} \ar[r] & a \ar[r] & x \\
Y & \mathfrak{B} \ar[r] & b & y
}\end{array}
$$

Hence we have, for each $U \in u$,

$$
\begin{align*}
\bigcup_{i \in U} (f^{-1}(y_i) \cap \text{adh}(f^{-1}(b_i))) &= p_X(\text{Ult}(f)^{-1}(\psi(U))) \in a.
\end{align*}
$$

Assume now that $f$ is effective descent. Let $(\mathfrak{B}, (b, y)) \in Y_2$. For its corresponding data $(f, u, (f_i), y_i, y)$, since $f$ is effective descent, there exist an ultrafilter $a$ on $X$ and an element $x \in f^{-1}(y)$ such that $a \to x$ and, for each $B \in \mathfrak{B}$,

$$
\begin{align*}
p_X(\text{Ult}(f)^{-1}(B)) &= \bigcup_{(\mathfrak{B}', y') \in B} (f^{-1}(y') \cap \text{adh}(f^{-1}(\mathfrak{B}'))) \in a.
\end{align*}
$$

Hence $\text{Ult}(f)^{-1}(\mathfrak{B}) \cup p_X^{-1}(a)$ induces a filter on $X_1$ which can be refined to an ultrafilter $\mathfrak{A}$, that clearly satisfies the conditions $\mathfrak{A} \to (a, x)$ and $f_1(\mathfrak{A}) = \mathfrak{B}$.  

¿From an argument like the one used in [9] for the ultrafilter monad, one can prove the following:

**Lemma 5.3.** For each ultrarelational continuous map $f : (X, r) \to (Y, s)$, the diagram

$$
\begin{array}{c}
\xymatrix{
\text{Ult}(f) \ar[r] & R(Y, s) \\
\text{Ult}(f) \ar[u]^{\mu(X, r)} \ar[r] & \text{Ult}(f) \ar[u]^{\mu(Y, s)}
}\end{array}
$$

commutes.

Now the “Key Lemma 4.1” of [13] is an obvious consequence of the result above:

**Corollary 5.4.** Let $f : (X, r) \to (Y, s)$ be a continuous map such that $f_2$ is surjective. If $(X, r)$ is transitive, then so is $(Y, s)$.  

To prove the “Key Lemma 4.2” of [13] using our techniques, one uses the following

**Proposition 5.5.** For every pullback diagram in URS

$$
\begin{align*}
\xymatrix{
X \times Z \ar[r]^\rho \ar[d]^-\pi & Y \\
X \ar[r]^-f & Z
}\end{align*}
$$

the canonical map $k : (X \times Z)_1 \to X_1 \times_{Z_1} Y_1$ is a perfect surjection.
Proof. Consider the diagram

and let \( \mathcal{A} \) be a filter on \((X \times Z, Y)\) such that \( k(\mathcal{A}) \to ((a, x), (b, y)) \) in \( X_1 \times Z_1, Y_1 \). Hence, \( \pi_1(\mathcal{A}) \to (a, x) \) and \( \rho_1(\mathcal{A}) \to (b, y) \), therefore, for \( c := p_{X \times Z} Y(\mathcal{A}), \epsilon \to (x, y) \) in \( X \times Z \), since \( \pi(\epsilon) = p_X(\pi_1(\mathcal{A})) = a \to x \) and \( \rho(\epsilon) = p_Y(\rho_1(\mathcal{A})) = b \to y \). This means that \( \mathcal{A} \to (\epsilon, (x, y)) \). Since \( k((x, y)) = ((a, x), (b, y)) \), and \( (\epsilon, (x, y)) \) is the only possible choice, \( k \) is perfect as claimed. The surjectivity of \( k \) follows from the fact that each pullback diagram satisfies Beck-Chevalley condition (see [12]).

Now, observing that, given a pullback diagram (1), in the diagrams below

\[ \rho_1 = \rho' \cdot k, \quad \rho_2 = \rho'' \cdot k' \cdot k_1, \] where \( k, k' \) are perfect surjections, \( k_1 = \text{Ult}(k) \) is surjective and \( \rho' \) (\( \rho'' \)) is the pullback of \( f_1 \) (\( f_2 \)) along \( g_1 \) (\( g_2 \)), we conclude that:

**Corollary 5.6.** 3-surjective maps are pullback-stable.

6. \( \Omega \)-surjective maps

We are now going to characterize topological triquotient maps inside \( \text{URS} \) as the \( \Omega \)-surjective maps. First we state an auxiliary result.

**Lemma 6.1.** Let \( X \) be a weak reflexive ultrarelational space. Then, for each ordinal \( \alpha \), \( X_\alpha \) is weak reflexive and \( (p^\alpha_0)X \) is a surjection for each \( \beta \leq \alpha \).

**Proof.** It follows immediately from the preservation of weak reflexivity by \( \text{Ult} \) and from the construction of \( X_\lambda \) for every limit ordinal \( \lambda \). \( \square \)

**Proposition 6.2.** Let \( f : X \to Y \) be a topological continuous map together with a map \((.,^f) : \mathcal{O}X \to \mathcal{O}Y \) satisfying (T1) and (T4)\(^1\). Then, for each ordinal \( \alpha \) and each \( U \in \mathcal{O}X \), \( (p^\alpha_0)^{-1}(U^f) \subseteq f_\alpha ((p^\alpha_0)^{-1}(U)) \).

**Proof.** For \( \alpha = 0 \), the assertion follows from the fact that \( U^2 \subseteq f(U) \) for each \( U \in \mathcal{O}X \). For \( \alpha > 0 \) assume that the condition above holds for each \( \beta \in \alpha \). Let \( U \in \mathcal{O}X \), \( y \in U^1 \) and \( ((b_\beta)_{\beta \in \alpha}, y) \in Y_\alpha \). We define

\[ \Sigma = \{ S \in \mathcal{O}X \mid \exists \beta \in \alpha : f_\beta ((p^\beta_0)^{-1}(S)) \not\subseteq b_\beta \}. \]

\(^1\)See Definition 2.1.
Σ is directed since all \( b_\beta (\beta \in \alpha) \) are ultrafilters and all \((p_0^\beta)_{X}(\gamma \leq \beta < \alpha)\) are surjective. We are now going to show that \( y \notin S^\sharp \) for each \( S \in \Sigma \). Assume that \( y \in S^\sharp \) for some \( S \in \Sigma \). Then we have \( S^\sharp \in b_0 \) and therefore, for all \( \beta \in \alpha \), \((p_0^\beta)^{-1}(S^\sharp) \in b_\beta \). But this is impossible since, by induction hypothesis, we have
\[
(p_0^\beta)^{-1}(S^\sharp) \subseteq f_\beta((p_0^\beta)^{-1}(S)) \notin b_\beta.
\]
By (T4), there exists \( x \in f^{-1}(y) \cap U \) such that, for all \( S \in \Sigma \), \( x \notin S \). Hence for each \( V \in \mathcal{O}(x) \) and each \( \beta \in \alpha \) we have \( f_\beta((p_0^\beta)^{-1}(V)) \in b_\beta \) and therefore \((p_0^\beta)^{-1}(\mathcal{O}(x)) \cup f_\beta^{-1}(b_\beta) \) induces a filter \( f_\beta \) on \( X_\beta \). For each \( \beta \in \alpha \) we put
\[
\mathcal{M}_\beta = \{ a \in \mathcal{U}(X_\beta) \mid a \supseteq f_\beta \}.
\]
Each \( \mathcal{M}_\beta (\beta \in \alpha) \) is non-empty and Zariski-closed, hence, since a codirected limit of non-empty compact Hausdorff spaces is non-empty (cf. [2]), there exists \( (a_\beta)_{\beta \in \alpha} \) with \( a_\beta \) ultrafilter on \( X_\beta \) such that \((p_0^\beta)_{X}(a_\beta) = a_\beta \) for all \( \beta \leq \beta < \alpha \). We have by definition \( a_0 \supseteq \mathcal{O}(x) \) and \( f_\beta(a_\beta) = b_\beta \) for all \( \beta \in \alpha \), hence \((a_\beta)_{\beta \in \alpha}, x \) \( X_\alpha \) and \( f_\alpha((a_\beta)_{\beta \in \alpha}, x) = ((b_\beta)_{\beta \in \alpha}, y) \).

Since this shows in particular that every triquotient map (between topological spaces) is \( \Omega \)-surjective, we conclude immediately that triquotient maps are effective descent.

For a set \( Y \), let \( \lambda_Y \) be the least regular cardinal larger than the cardinal of \( Y \).

**Proposition 6.3.** Let \( f : X \to Y \) be a topological continuous map. Then, for each \( U \in \mathcal{O}X \), the set
\[
(2) \quad U^\sharp = \{ y \in Y \mid (\forall \alpha \in \lambda_Y) (p_0^\alpha)^{-1}(y) \subseteq f_\alpha((p_0^\alpha)^{-1}(U)) \}
\]
is open and the map \((\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y\) satisfies (T1) and (T4).

**Proof.** First we show that \( U^\sharp \) is open for each \( U \in \mathcal{O}X \). For that, let \( y_0 \in \text{cl}(Y \setminus U^\sharp) \). There exists an ultrafilter \( b_0 \) on \( Y \) converging to \( y_0 \) such that \( Y \setminus U^\sharp \in b_0 \). For each \( y \in Y \setminus U^\sharp \) there exist \( \alpha_y \in \lambda_Y \) and \((b_\beta)_{\beta \in \alpha_y}, y \) \( Y \) such that \( \hat{h}((a_\beta)_{\beta \in \alpha_y}, x) \in X_{\alpha_y} : (x \in U \land f_\alpha((a_\beta)_{\beta \in \alpha_y}, x) = ((b_\beta)_{\beta \in \alpha_y}, y) \),
by definition of \( U^\sharp \). For \( \alpha := \sup_{y \in (Y \setminus U^\sharp)} \alpha_y \), we have \( \alpha \in \lambda_Y \). Considering \( B = (p_0^\alpha)^{-1}(Y \setminus U^\sharp) = f_\alpha((p_0^\alpha)^{-1}(U)) \), one has \((p_0^\alpha)^{-1}(B) = Y \setminus U^\sharp \in b_0 \). Let \( b_\alpha \) be any ultrafilter containing \((B) \cup (p_0^\alpha)^{-1}(b_0) \). Assume that there exist an ultrafilter \( a_\alpha \) on \( X_\alpha \) and an element \( x_0 \in f^{-1}(y_0) \) such that \( a_\alpha = (p_0^\alpha)^{-1}(a_\alpha) \to x_0 \), \( x_0 \in U \) and \( f_\alpha(a_\alpha) = b_\alpha \). Then we have \( U \in a_\alpha \) and therefore \((p_0^\alpha)^{-1}(U) \in a_\alpha \). Hence \( f_\alpha((p_0^\alpha)^{-1}(U)) \cap B \not= \emptyset \) which contradicts the definition of \( B \). Therefore, we have proved that \( y_0 \in Y \setminus U^\sharp \).

This means that (2) defines a map \((\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y\), that satisfies obviously (T1). So it remains to show that it also satisfies (T4). Let \( U \in \mathcal{O}X \), \( y \in U^\sharp \) and \( \Sigma \subseteq \mathcal{O}X \) directed such that, for each \( S \in \Sigma \), \( y \notin S^\sharp \). There exists an ordinal \( \alpha \in \lambda_Y \) such that, for each \( S \in \Sigma \), the set \( \mathcal{B}_S = (p_0^\alpha)^{-1}(y) - f_\alpha((p_0^\alpha)^{-1}(S)) \) is non-empty. Since \( \Sigma \) is filtered, \((\mathcal{B}_S \mid S \in \Sigma) \) forms a filter base. Let \( b_\alpha \) be any ultrafilter containing \((\mathcal{B}_S \mid S \in \Sigma) \cup (p_0^\alpha)^{-1}(\eta(y)) \). Since \( y \in U^\sharp \), there exist an ultrafilter \( a_\alpha \) on \( X_\alpha \) and an element \( x \in U \) such that \( a_\alpha = (p_0^\alpha)_{X}(a_\alpha) \to x \), \( f(x) = y \) and \( f_\alpha(a_\alpha) = b_\alpha \). Let \( V \in \mathcal{O}(x) \). Then \( V \in a_\alpha \) and therefore \((p_0^\alpha)^{-1}(V) \in a_\alpha \). Hence \( f_\alpha((p_0^\alpha)^{-1}(V)) \cap \mathcal{B}_S \not= \emptyset \) and therefore \( V \not= S \), for each \( S \in \Sigma \). Hence, \( \Sigma \) does not cover \( f^{-1}(y) \cap U \), and (T4) follows.

\( \square \)
This proposition gives a general way of defining a map \((\_)^2\) – in fact, the largest possible one –, as required in the definition of triquotient map, but that in general does not satisfy (T2): \(X^2 = Y\). By the definition of \((\_)^2\) it is clear what this condition means: \(f\) is \(\lambda Y\)-surjective; that is:

\[
\begin{array}{cccccccc}
X & \cdots & \rightarrow & a_{n+1} & \rightarrow & a_n & \cdots & \rightarrow & a_1 & \rightarrow & x \\
f & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \cdots & \rightarrow & b_{n+1} & \rightarrow & b_n & \cdots & \rightarrow & b_1 & \rightarrow & y
\end{array}
\]

Hence, we may now state the characterization of topological triquotient maps.

**Theorem 6.4.** Let \(f : X \rightarrow Y\) be a continuous map between topological spaces. The following conditions are equivalent:

(i) \(f\) is a triquotient map;

(ii) \(f\) is \(\Omega\)-surjective;

(iii) \(f\) is \(\lambda Y\)-surjective.

In the finite case, since all ultrafilters are fixed, \(X_n\) may be described as the set of all \((n+1)\)-chains \(x_n \rightarrow \cdots \rightarrow x_0\) of elements of \(X\). The ultrarelational structure is then described by

\[
(x_n, \cdots, x_1, x_0) \rightarrow (x'_n, \cdots, x'_1, x'_0) : \iff (x_{n-1}, \cdots, x_0) = (x'_n, \cdots, x'_1).
\]

From Theorem 6.4, we know that, if \(X\) and \(Y\) are finite, then \(f : X \rightarrow Y\) is a triquotient map if and only if \(f_n\) is surjective for every \(n \in \mathbb{N}\), which is exactly Theorem I.

### References


**Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal**

E-mail address: mmc@mat.uc.pt, dirk@mat.uc.pt