Lipschitzian Regularity Conditions for the Minimizing Trajectories of Optimal Control Problems

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Summary

We survey some conditions for Lipschitzian regularity of minimizers in various problems of the calculus of variations and optimal control theory. Some recent results obtained by the authors are presented as well.

Introduction

First optimality conditions and first existence results in the calculus of variations have been separated in time by more than a century. The formalism based on Euler-Lagrange equation deals with a given (local) minimizer whose existence need not be established. Minimizers which appeared in the classical examples of the variational problems were either smooth or piecewise smooth and the question of validity of the optimality conditions for them has been solved rather easily.

Only at the end of the nineteenth century the problem of finding proper class of functions for setting the problems of the calculus of variations has been put. In 1915 Leonida Tonelli has established first general existence result for the basic problem of the calculus of variations in the class $W_{1,1}$ of absolutely continuous functions. It is not difficult to construct an example where minimum in the class of piecewise-smooth or even Lipschitzian functions is not attained – an absolutely continuous minimizer exists but has an unbounded derivative. Another problem arises here: necessary optimality conditions (different forms of the Euler-Lagrange equation) may cease to be valid for the minimizers with an unbounded derivative. These minimizers may exhibit other weird properties: for example it may happen that they cannot be approximated (by the value of the functional) by a sequence of piecewise smooth or Lipschitzian functions, since the infimum over the class of absolutely continuous functions is strictly less than the one over the class of Lipschitzian functions (Lavrentiev Phenomenon). An approach to overcoming these difficulties could be in finding the conditions (classes of integrands) for which respective minimizers are Lipschitzian. For them Euler-Lagrange equation is valid and often (under some additional assumptions) one can establish even more regularity, like piecewise, $C^1$ or $C^2$ differentiability.
Therefore obtaining conditions of Lipschitzian regularity is a pertinent problem. First regularity results for the basic problem of the calculus of variations belong to L. Tonelli. Various contributions have been done by C. Morrey and more recently by F. H. Clarke, R. B. Vinter and others. Less is done for the problem with high-order derivatives, and for the Lagrange problems of optimal control the results are scarce. Here we develop a new approach to establishing Lipschitzian regularity. It is based on a transformation of the initial problem into a time-optimal control problem and applying to the latter the Pontryagin Maximum Principle. This approach allows us to obtain conditions of Lipschitzian regularity for a broad class of Lagrange problems with a control-affine dynamics. When applying these results to the particular case of the basic problems of the calculus of variations or to the problems with high-order derivatives, we manage to obtain new conditions which are not covered by the previously known. The work on obtaining conditions for general optimal control problems is in progress and the results will appear elsewhere.

1 Problem (P) – Lagrange problem of optimal control with control-affine dynamics

We will be concerned with the Lagrange problem of optimal control with nonlinear control-affine dynamics. We look for an integrable control \( u(\cdot) \)

\[
u (\cdot) \in L_1 ([a, b] ; \mathbb{R}^m)
\]

and the corresponding absolutely continuous trajectory \( x(\cdot) \)

\[
x (\cdot) \in W_{1,1} ([a, b] ; \mathbb{R}^n)
\]

satisfying the differential equation

\[
\dot{x} (t) = \varphi (t, x(t), u(t)) := f (t, x(t)) + g(t, x(t)) u(t)
\]

the boundary conditions

\[
x(a) = x_a, \quad x(b) = x_b,
\]

such that they provide minimal value for the integral functional

\[
\int_a^b L (t, x(t), u(t)) \, dt.
\]

This problem is denoted by \( (P) \). We will assume that all data of the problem are \( C^1 \)–smooth:

\[
L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n, \quad g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}.
\]

Also there are no constraints on the control values: \( u \in \mathbb{R}^m \).

Two major issues related to the problem are existence of minimizers and minimality conditions.
2 Existence theorem for (P)

For our problem (P) we have the following existence theorem, which provides conditions under which the problem has a solution in the class of integrable controls:

If

- (coercivity) there exists a function $\theta : \mathbb{R}^+_0 \to \mathbb{R}$, bounded from below, such that
  \[ \lim_{r \to +\infty} \frac{\theta (r)}{r} = +\infty, \quad L (t, x, u) \geq \theta (\|u\|) \text{ for all } (t, x, u); \]
- (convexity) $L (t, x, u)$ is convex with respect to $u$ for every $(t, x)$;
- $\|f (t, x)\| \leq \phi (t) + c \|x\|$, $\|g (t, x)\| \leq \psi (t) + c \|x\|$,
  $\forall (t, x) \in [a, b] \times \mathbb{R}^n$, $\phi, \psi \geq 0$, $\phi \in L_1$, $\psi \in L_\infty$;

and provided that there exist at least one admissible pair $(x(\cdot), u(\cdot))$, then problem (P) has an absolute minimum in the space $u(\cdot) \in L_1$.

We call this result Tonelli-type theorem because in the particular case of the basic problem in the calculus of variations, that is when dynamics is only given by $\dot{x} = u$, we may choose $\phi(t) = 0$, $c = 0$, $\psi(t) = 1$ and obtain the classical Tonelli-existence theorem proved by Leonida Tonelli in 1915.

3 Necessary optimality condition for (P) – Pontryagin Maximum Principle

Now we formulate the necessary optimality condition for the problem (P). This is celebrated Pontryagin Maximum Principle – a fundamental result of optimal control theory:

If $(x(\cdot), u(\cdot))$ is a minimizer of our problem (P) and $u(\cdot) \in L_\infty$, then

\[ \exists (\psi_0, \psi(\cdot)) \neq 0, \text{ where } \psi(\cdot) \in W_{1,1} \text{ and } \psi_0 \text{ is constant less or equal than zero, such that } (x(\cdot), \psi_0, \psi(\cdot), u(\cdot)) \text{ satisfies:} \]

- the Hamiltonian system
  \[ \dot{x} = \frac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial H}{\partial x}, \]
  with the Hamiltonian
  \[ H = \psi_0 L (t, x, u) + \langle \psi, f (t, x) \rangle + \langle \psi, g (t, x) u \rangle \]
- the maximality condition
  \[ H (t, x (t), \psi_0, \psi (t), u (t)) = \sup_{u \in \mathbb{R}^m} H (t, x (t), \psi_0, \psi (t), u) \]
  for almost all $t \in [a, b]$.

Of course we should like to have necessary conditions at our disposal to identify minimizers predicted by existence theory. Here we are assuming that $u(\cdot)$ is not merely integrable, but measurable and bounded function.

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4 Gap between existence and optimality results

Analyzing the hypotheses of both necessary conditions and existence theorems, we come to the conclusion that there is a gap between ‘Existence Theory’ and ‘Necessary Optimality Conditions’:

- For minimizers predicted by existence theory, necessary optimality conditions may fail to be valid.
- In the class of controls where classical necessary optimality conditions are valid (measurable bounded controls) existence is not guaranteed.

One way to attack this problem is to postulate conditions which assure that all minimizing controls are bounded and therefore standard necessary optimality conditions are applicable to them. We shall call these conditions Lipschitzian regularity conditions.

But before making a survey of ‘Lipschitzian regularity conditions’, we will illustrate this mismatch between existence and necessary conditions, with an example.

5 Optimal trajectories can be non Lipschitzian

The following example has been constructed by J. Ball & V. Mizel in 1985:

\[
\int_0^1 \left( |x^3 - t^2|^2 |u|^{14} + \varepsilon |u|^2 \right) \, dt \rightarrow \min
\]

\[\dot{x}(t) = u(t)\]

\[x(0) = 0, \quad x(1) = k, \quad \varepsilon > 0.\]

It’s a rather simple exercise to see that all hypotheses of Tonelli’s existence theorem are satisfied. Nevertheless, its minimizer is an unbounded function. In fact, it has been proved by F. H. Clarke & R. B. Vinter, that for certain choices of constants \(k\) and \(\varepsilon\), this problem has a unique integrable optimal control

\[u(t) = kt^{-1/3}.\]

One can prove that Pontryagin Maximum Principle (Euler-Lagrange equation in integral form) is not satisfied since after substituting the minimizer into the right-hand side of the adjoint equation of the Pontryagin Maximum Principle we obtain

\[\dot{\psi}(t) = L_x(t, x(t), \dot{x}(t)) = ct^{-4/3}\]

which results in a divergent integral for calculation of \(\psi(\cdot)\).

Now we will present some ‘Lipschitzian regularity conditions’ which implies that minimizing controls are bounded. This will also provide validity of the Pontryagin Maximum Principle for these minimizers.
6 Brief survey of existing results on Lipschitzian regularity

6.1 Lipschitzian regularity conditions for the basic problem of the calculus of variations

Various conditions of Lipschitzian regularity are known for the basic problem of the calculus of variations

\[
\int_a^b L(t, x(t), u(t)) \, dt \rightarrow \min, \quad \dot{x}(t) = u(t).
\]

They start with a condition obtained by Leonida Tonelli at the beginning of the century. Some of recent contributions are due to Francis Clarke and Richard Vinter. Let us list some of these conditions

- **L. Tonelli - C. B. Morrey:** \( \|L_x\| + \|L_u\| \leq c |L| + r \) \((c > 0)\)
- **S. Bernstein, n = 1; F. H. Clarke & R. B. Vinter, n > 1:**
  \[ L \geq \gamma + \alpha \|u\|^{1+\beta}, \quad (\alpha, \beta > 0) \]
  \[ \|L^{-1}_{uu} (L_x - L_{ut} - L_{ux} u)\| \leq c \left(\|u\|^{2+\beta} + 1\right), \quad (L_{uu} > 0) \]
- **F. H. Clarke & R. B. Vinter:**
  * L autonomous: \( L = L(x, u) \)
  * \( |L_t| \leq c |L| + k(\cdot), \quad (k(\cdot) \in L_1) \)
  * \( \|L_x\| \leq c |L| + k(\cdot) \|L_u\| + m(\cdot), \quad (k(\cdot), m(\cdot) \in L_1) \)
- **R. B. Vinter:** \( L(t, \cdot, \cdot) \) convex for each \( t \).

Under each of these conditions the minimizing controls are bounded and satisfy the Pontryagin Maximum Principle.

6.2 Lipschitzian regularity for the problems of the calculus of variations with high-order derivatives

One may present problems of the calculus of variations with high-order derivatives in the following form

\[
\int_a^b L(t, x(t), u(t)) \, dt \rightarrow \min, \quad \dot{x}(t) = Ax(t) + Bu(t)
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]
The conditions of Lipschitzian regularity for this type of problems are more scarce. Principal contribution has been done by F. H. Clarke and R. B. Vinter (1990). They deduced a condition of the Tonelli–Morrey type

$$\|L_x\| \leq c \left( L + \|u\| \right) + \gamma(t) \ r(x)$$

$$\gamma(\cdot) \text{ integrable, } \ r(\cdot) \text{ locally bounded.}$$

Optimal controls for this class of problems exhibit some phenomena which do not occur in the basic problem of the calculus of variations. For example, the first of the authors has proved (answering to an open question posed by F. H. Clarke and R. B. Vinter) that autonomous (= time invariant) integrands may have minimizing controls which are unbounded. In fact it has been shown that Lavrentiev gap can be present for autonomous problems (see [1]).

6.3 Lipschitzian regularity of trajectories in optimal control problems

Lipschitzian regularity conditions for general Lagrange problem of optimal control are even a bigger rarity. We are only aware of a contribution due to F. H. Clarke and R. B. Vinter in 1990. They proved a Tonelli-Morrey type condition for the Lagrange problem of optimal control with linear autonomous dynamics:

$$\int_a^b L(t, x(t), u(t)) \ dt \to \min$$

$$\dot{x}(t) = A x(t) + B u(t) + d(t).$$

This result is obtained using a transformation of the problem into a problem of the calculus of variations with high-order derivatives and then using Tonelli–Morrey type condition from the previous section.

7 Lipschitzian regularity of trajectories for the problem (P)

Here we address the more general problem (P)

$$\int_a^b L(t, x(t), u(t)) \ dt \to \min$$

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)) \ u(t),$$

and present new results on Lipschitzian regularity of minimizing trajectories for this problem. To deal with it we employ a different method. The result will be formulated as some growth condition imposed on (the derivatives of) $L$ and $\varphi$, which implies boundedness of any minimizing control $\tilde{u}(\cdot)$ and Lipschitzian regularity of
the minimizing trajectory \( \tilde{x}(\cdot) \).

**Theorem:** Under the hypotheses:

- \( g(t, x) \) has rank \( m \) for all \( t \) and \( x \);
- (coercivity) \( \exists \theta : \mathbb{R}_0^+ \to \mathbb{R} \) and \( \zeta \in \mathbb{R} \) s.t. for all \( (t, x, u) \)
  \[ L(t, x, u) \geq \theta (\|u\|) \succ \zeta \quad \text{and} \quad \lim_{r \to +\infty} \frac{r}{\theta(r)} = 0; \]
- (growth condition) \( \exists \) constants \( \gamma, \beta, \eta \) and \( \mu \), with \( \gamma > 0, \beta < 2 \) and \( \mu \geq \max\{\beta - 2, -2\} \), s.t. for all \( (t, x, u) \)
  \[ \left(|L_t| + \|L_x\| + \|L \varphi_t - L_t \varphi\| + \|L \varphi_x - L_x \varphi\|\right) \|u\|^\mu \leq \gamma L^\beta + \eta; \]

all the minimizers \( u(\cdot) \) of the Lagrange problem \((P)\), which are not abnormal extremal controls, are essentially bounded on \([a, b]\).

**Corollary:** under the hypotheses of the theorem, all the minimizers of the problem satisfy (normal or abnormal form of) Pontryagin Maximum Principle.

**Sketch of the proof.**

Here only main ideas are represented. Details and complete proofs can be found in [2]. This approach will be summarized in four steps.

### 7.1 Reduction to a time-optimal problem

First we reduce the problem \((P)\) to an autonomous time-optimal control problem. Given the coercivity condition, Lagrangian \( L \) is bounded from below and since adding a constant to \( L \), in the problem \((P)\), does not change the minimizers, we may assume, without lost of generality, that \( L \) is strictly positive.

We introduce the new time variable \( \tau \)
\[
\tau(t) = \int_a^t L(\theta, x(\theta), u(\theta)) \, d\theta, \quad t \in [a, b],
\]
which is a strictly monotonous absolutely continuous function of \( t \). Then we consider \( t(\tau) \) (the inverse function of \( \tau(t) \)) and \( z(\tau) = x(t(\tau)) \) as components of the state trajectories, and \( v(\tau) = u(t(\tau)) \) as the control. The following time-optimal problem appears:

\[
T \to \min,
\]

\[
\begin{cases}
\dot{t}(\tau) = \frac{1}{L(t(\tau), z(\tau), v(\tau))}, & v : \mathbb{R} \to \mathbb{R}^m \\
\dot{z}(\tau) = \frac{\varphi(t(\tau), z(\tau), v(\tau))}{L(t(\tau), z(\tau), v(\tau))},
\end{cases}
\]
\[ t(0) = a, \ t(T) = b, \quad z(0) = x_a, \ z(T) = x_b. \]

So far, the new control variable continues to take its values in \( \mathbb{R}^m \) and, a priori, control \( v(\cdot) \) can be unbounded. In the next step, compactification of the space of admissible controls will be done.
7.2 Compactification of the control set

Following an idea developed by R. V. Gamkrelidze,\(^2\) we proceed with compactification of the set of control parameters. The point is that for each \(t\) and \(z\), the set of all velocities becomes compact if the point 0, which corresponds to the infinite value of the control \(v\), will be added.

We compactify the space \(\mathbb{R}^m\) of admissible controls, adding the infinity point: \(\mathbb{R}^m \approx S^m\). The one-to-one correspondence between the Euclidean space \(\mathbb{R}^m\) and the sphere \(S^m\) is established by means of the stereographic projection \(\pi\):

\[
\pi : S^m \rightarrow \mathbb{R}^m.
\]

The set \(\left\{ \left( \frac{1}{L(t, z, v)}, \frac{\varphi(t, z, v)}{L(t, z, v)} \right) : v \in \mathbb{R}^m \right\}\) is now compact and the following extended autonomous optimal control problem is defined properly:

\[
T \rightarrow \min, \quad
\begin{cases}
\dot{t}(\tau) = \frac{1}{L(t, z, \pi(w))}, & w \in S^m, \quad t(0) = a, \quad t(T) = b \\
\dot{z}(\tau) = \frac{\varphi(t, z, \pi(w))}{L(t, z, \pi(w))}, & z(0) = x_a, \quad z(T) = x_b.
\end{cases}
\]

It can be proved (see [2]) that this last problem is equivalent to the problem \((P)\), in the following sense: to every admissible pair \((x(\cdot), u(\cdot))\) of the original problem, there corresponds an admissible triple \((t(\cdot), z(\cdot), w(\cdot))\) of the extended system, such that \(w(\cdot)\) takes values different from the north pole (point at infinity) almost everywhere, \(t(0) = a, \ t(T) = b, \ z(0) = x_a, \ z(T) = x_b\), the transfer time \(T\) for this latter solution equals the value of the integral functional of the original problem associated with the pair \((x(\cdot), u(\cdot))\). Moreover, every solution of the extended/compactified problem, with the control different from the north pole almost everywhere, results from this correspondence.

To prove that \(u(\cdot)\) is bounded is the same as to prove that \(w(\cdot)\) does not take its values at the north pole.

7.3 Pontryagin Maximum Principle and Lipschitzian regularity

Assume for the moment that the Pontryagin Maximum Principle is applicable to the time-optimal control problem. The validity of this assumption will be discussed on the next step.

Writing down the maximality condition for the compactified problem, and from the fact that maximized Hamiltonian is constant along a minimizer of the autonomous problem, we conclude that there exists \(c \geq 0\) such that

\[
0 \leq c = \sup_{v \in \mathbb{R}^m} \frac{\psi_t + (\psi_z f(t, z) + g(t, z) v)}{L(t, z, v)}.
\]

If \( v \) tends to infinity then, by virtue of the coercivity condition, \( L \) has superlinear growth, and hence \( c \) must be zero:

\[
\lim_{\|v\| \to +\infty} \frac{\psi_t + (\psi_z, f(t, z) + g(t, z) v)}{L(t, z, v)} = 0 = c.
\]

It occurs (see [2, Proposition 1]) that this can only be an option if the minimizing control is an abnormal extremal control. Thus, for minimizers which are not abnormal extremal controls, there must be \( c > 0 \) and therefore \( v \) must be bounded.

### 7.4 Applicability of the Pontryagin Maximum Principle to the compactified problem

The growth conditions on \( L \) and \( \varphi \) which guarantee Lipschitzian regularity, will arise from the applicability of the Pontryagin Maximum Principle.

Given the coercivity condition, the right-hand side of the compactified problem equals zero when \( w \) coincides with the north pole. So the right-hand side is continuous on the entire sphere and to apply the Pontryagin Maximum Principle to the compactified problem, we only need to assure that the right-hand side is continuously differentiable with respect to the state variables \( t \) and \( z \). The only problem is the continuous differentiability at the north pole. For it the fulfillment of the following growth condition is sufficient:

\[
\exists \gamma > 0, \beta < 2, \mu \geq \max \{ \beta - 2, -2 \} \quad \text{and} \quad \eta \in \mathbb{R}, \text{ such that}
\]

\[
(|L_t| + \|L_x\| + \|L \varphi_t - L_t \varphi\| + \|L \varphi_x - L_x \varphi\|) \|u\|^\mu \leq \gamma L^\beta + \eta.
\]

### 8 Corollary for the basic problem of the calculus of variations

In this section we apply the main theorem to the basic problem of the calculus of variations and to the problem with high-order derivatives. It is straightforward to see that the dynamics corresponding to those problems is controllable and therefore there are no abnormal extremals.

For the basic problem of the calculus of variations

\[
\int_a^b L(t, x, \dot{x}) dt \to \min, \quad x(\cdot) \in W_{1,1},
\]

or in the optimal control notation,

\[
\int_a^b L(t, x, u) dt \to \min, \quad \dot{x} = u, \quad u(\cdot) \in L_1,
\]

the growth condition takes form: \( \exists \gamma > 0, \beta < 2, \mu \geq \max \{ \beta - 1, -1 \} \), such that
\[
(\|L_t\| + \|L_x\|) \|\dot{x}\|^\mu \leq \gamma L^\beta + \eta \quad \Leftrightarrow \quad (\|L_t\| + \|L_x\|) \|u\|^\mu \leq \gamma L^\beta + \eta.
\]

This implies that all the minimizers \(x(\cdot)\), predicted by Tonelli’s existence theorem, are Lipschitzian:

\[
u(\cdot) \in L_\infty \Rightarrow x(\cdot) \in W_{1,\infty}.
\]

9 Corollary for the problem of the calculus of variations with high-order derivatives

The problems of the calculus of variations where the Lagrangian \(L\) depends on derivatives up to order \(m, m \geq 1\),

\[
\int_a^b L(t, x, \dot{x}, \ldots, x^{(m)}) dt \rightarrow \min \quad x(\cdot) \in W_{m,1}
\]

can be formulated as in Section 6.2

\[
\int_a^b L(t, \xi, u) dt \rightarrow \min \quad \dot{\xi} = A\xi + B u \quad \xi(\cdot) \in W_{1,1}, u(\cdot) \in L_1.
\]

The following growth condition \((\gamma > 0, \beta < 2, \mu \geq \max \{\beta - 1, -1\})\)

\[
(\|L_t\| + \|L_{x^{(i)}}\|) \|x^{(m)}\|^\mu \leq \gamma L^\beta + \eta \quad \Leftrightarrow \quad (\|L_t\| + \|L_{\xi}\|) \|u\|^\mu \leq \gamma L^\beta + \eta
\]

implies that all the minimizers \(x(\cdot)\) belong to \(W_{m,\infty}\):

\[
u(\cdot) \in L_\infty \Rightarrow x(\cdot) \in W_{m,\infty}.
\]

Final remarks

The approach we suggest seems to be more effective in dealing with the Lagrange problems of optimal control – it covers a much broader class of problems. But even for the simplest case – the basic problem of the calculus of variations – these conditions turn out to be new (as far as we know). For example the following minimization problem:

\[
\int_0^1 \left( (\dot{x}^4 + 1)^3 e^{(\dot{x}^4+1)} (t+\frac{2}{\pi} \arctan x) \right) dt \rightarrow \min,
\]

\footnote{We use the notation \(W_{m,p}, m = 1, \ldots, 1 \leq p \leq \infty\), to represent the class of functions which are absolutely continuous together with their derivatives up to order \(m-1\) and have \(m\)-th derivative belonging to \(L_p\).}
with boundary conditions \( x(0) = x_0, \ x(1) = x_1 \), satisfies the hypotheses of Tonelli’s existence theorem and the condition of the Section 8 allow us to conclude that minimizer is Lipschitzian while the regularity conditions of the Section 6.1 fail.

References

