ON COPS AND ROBBERS ON G^{Ξ} AND COP-EDGE CRITICAL GRAPHS.

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Abstract. Cop Robber game is a two player game played on an undirected graph. In this game cops try to capture a robber moving on the vertices of the graph. The cop number of a graph is the least number of cops needed to guarantee that the robber will be caught. In this paper we present results concerning games on G^{Ξ}, that is the graph obtained by connecting the corresponding vertices in G and its complement $\overline{G}$. In particular we show that for planar graphs $c(G^{\Xi}) \leq 3$. Furthermore we investigate the cop edge-critical graphs, i.e. graphs that for any edge e in G we have either $c(G-e) < c(G)$ or $c(G-e) > c(G)$. We show a couple of examples of cop edge-critical graphs having cop number equal to 3.

1. Introduction

In a graph G, a set S of vertices is a *dominating set* if every vertex not in S has a neighbour in S. The minimum cardinality of a dominating set is the *domination number* of G, denoted by $\gamma(G)$.

The cop number of a graph is a graph parameter related with the domination number $\gamma(G)$ of a graph. It can be seen like the game of Cops and Robber. This is a vertex pursuit game played on a graph G. There are two players, a set of k cops (or searchers) C, where $k > 0$ is a fixed integer, and the robber R. At the beginning of the game cops place themselves on a set of k vertices (more than one cop is allowed to occupy a single vertex), next the robber choose one vertex. The game proceed as follows: first the cops moves, that is, switch from their vertices to adjacent ones, or pass, that is, remain on their current vertex, then the robber moves in the same way as cops (remaining at a vertex or moving to an adjacent one). The

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game continues with alternate moves by cops and robber, which we will call rounds or steps. Cops win and the game ends if at the end of their round one of the cops may occupy the same vertex as the robber; otherwise R wins. Players knows each others actual position (that is, the game is played with complete information). The minimum number of cops required to catch the robber (regardless of robber's strategy) is called the *cop number* of G, and is denoted $c(G)$. This parameter is well studied for several types of graphs (see [2, 3, 7, 8, 17]). Since the existence of loops or parallel edges has no influence in the results of this paper, throughout the text we consider only simple graphs which are herein called graphs.

We call a graph G as *cop win* if a single cop win the Cop and Robber game on G. A graph G is said to be *cop vertex-critical* if for any vertex v in G either $c(G-v) < c(G)$ or $c(G-v) > c(G)$. Similarly a graph G is *cop edge-critical* if for any edge e in $E(G)$ either $c(G-e) < c(G)$ or $c(G-e) > c(G)$. It is immediate that every totally disconnected graph, i.e., a graph where all its vertices are isolated, of order $n > 1$ is cop vertex-critical. On the other hand, every graph where each connected component is K_2 is cop edge-critical. Another trivial example of a cop edge-critical graph is a tree of order $n > 1$.

If a graph has cop number k and is cop edge-critical we call it k-cop edge-critical. S. L. Fitzpatrick [11] characterized edge critical planar graphs with cop number 2 whose cop number decreases after removal of any edge. A more general study of 2-cop edge-critical graphs is due to N.E. Clarke et.al [10]. So far, according to our knowledge, the only known example of a 3-cop edge-critical graph is the Petersen graph and it is due to W. Baird et al [4]. In this paper we present new examples of cop edge-critical graphs with cop number equal to 3.

In the next section we present some preliminary results on Cops and Robbers game on graphs and its complements. In the third section a particular attention is given to the games on graphs G^{Ξ}, that is, graphs obtained by connecting the corresponding vertices in G and its complement $\overline{G}$. Those graphs were already been considered in a different contexts, for example in [1]. We show that for planar graphs $c(G^{\Xi}) \leq 3$. In the last section the cop edge-critical graphs are investigated. Among them we find examples of graphs created by taking the G^{Ξ}. For instance, the Petersen graph is the graph C_5^{Ξ}. We conclude the paper with a few conjectures and remarks. We believe that in general there are many cop edge-critical graphs among graphs of the form G^{Ξ}.

2. Preliminary results

The complement of a graph G is a graph $\overline{G}$ on the same set of vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. In this section we focus on analysing the cop number of the graph and its complement $\overline{G}$. Notice that placing a cop on each element of a dominating set ensures that the cops win in at most two rounds, thus $c(G) \leq \gamma(G)$ (see [8]). For the domination number itself, the following bounds have been proved (see [13, 14]).

Proposition 2.1. [13, 14] *Considering a graph G of order n, we may conclude the following upper bounds on the domination number.*

$$\begin{aligned} \gamma(G)+\gamma(\overline{G}) &\leq n+1. \\ \gamma(G)+\gamma(\overline{G}) &\leq \frac{n}{2}+2, \text{ if } \delta(G), \delta(\overline{G}) \geq 1, \end{aligned}$$

where $\delta(G)$ is the minimum degree of G.

Regarding the cop number, as immediate consequence of this proposition, we have the following upper bounds.

Proposition 2.2. *Let G be a graph of order $n \geq 4$.*

a) *$c(G)+c(\overline{G}) \leq n+1$ and the equality holds if and only if either G or $\overline{G}$ is totally disconnected (that is, all vertices are isolated).*
b) *$c(G)+c(\overline{G}) \leq \frac{n}{2}+2$ if $\delta(G), \delta(\overline{G}) \geq 1$ and the upper bound is attained if G is C_4.*

Proof. Let us prove each of the cases as follows.

a) Since $c(G) \leq \gamma(G)$, from Proposition 2.1, we may conclude the inequalities
$$c(G)+c(\overline{G}) \leq \gamma(G)+\gamma(\overline{G}) \leq n+1.$$
Furthermore, it is immediate that if G is totally disconnected, then $c(G)=n$ and $c(\overline{G})=1$ and thus the equality holds. Conversely, let us assume that G and $\overline{G}$ are not totaly disconnected and the equality holds.

Case-1: Consider the case that either G or $\overline{G}$ contains at least one isolated vertex v. Assume that $d_{\overline{G}}(v)=n-1$, then $c(\overline{G})=1$ and since G has at least one edge we have $c(G) \leq n-1$. It follows that $c(G)+c(\overline{G}) \leq n-1+1=n$, which is a contradiction.

Case-2: Now, assume that there exists no isolated vertex either in G or in $\overline{G}$. This implies that $\delta(G), \delta(\overline{G}) \geq 1$ and from Proposition 2.1,
$$c(G)+c(\overline{G}) \leq \gamma(G)+\gamma(\overline{G}) \leq \frac{n}{2}+2 \leq n,$$
which is contradiction.

b) This part is direct consequence of Proposition 2.1.

□

Proposition 2.3. *If G is a disconnected graph, then $c(\overline{G}) \leq 2$.*

Proof. The domination number $\gamma(\overline{G})$ is equal to 2 (since we can consider two vertices from different components of G as a dominating set). □

By $diam(G)$ we denote the diameter of G which is the greatest distance between any pair of vertices in G.

Lemma 2.4. *If G is a graph with $diam(G) \geq 3$, then $c(\overline{G}) \leq 2$.*

Proof. Take two vertices w, v such that the length of shortest path connecting this two vertices is ≥ 3. We know that $N[w] \cap N[v] = \emptyset$ so in the complement v is adjacent to every vertex in $N[w]$ and the same is true for w and $N[v]$. All the other vertices in $\overline{G}$ are adjacent to both w and v. Thus the domination number of $\overline{G}$ is equal to 2, implying that its cop number is bounded by 2. □

3. The cops and Robber on G^{Ξ}

We define a graph G^{Ξ} as the graph obtained by the disjoint union of G with its complement $\overline{G}$ and adding a perfect matching between the corresponding vertices of G and $\overline{G}$. That is, considering the graph G and its complement $\overline{G}$, every vertex v in G is adjacent to its copy (herein also called mirror vertex) v' in $\overline{G}$. From now on, we denote each vertex of G by a letter and the corresponding mirror vertex in $\overline{G}$ by the same letter with apostrophe. Recall that, as it was already mentioned, the Petersen graph is isomorphic to C_5^{Ξ}.

Lemma 3.1. *For every graph G, the graph G^{Ξ} is connected.*

Proof. It is well-known that at least one of G, $\overline{G}$ is connected, and this immediately implies that G^{Ξ} is connected. □

Considering a graph G of order n, a vertex ordering $(v_1, \ldots, v_n)$ is a cop-win ordering (or dismantling ordering) if for each $i < n$, there is $j > i$ such that $N_i[v_i] \subseteq N_i[v_j]$, where $N_i[v_j]$ is the closed neighborhood of the vertex v_j in the subgraph of G induced by the vertices in $(v_i, \ldots, v_n)$ (see [9, 16]). A graph is cop-win if and only if it has a cop-win ordering [16]. According to Bandelt and Prisner [5], the cop-win graphs were introduced by Poston [18] and Quilliot [19] under the name *dismantlable graphs* defined recursively as follows: the trivial graph with just one vertex is dismantlable and a graph G with at least two vertices is dismantlable if there exists two vertices x and y such that $N[x] \subseteq N[y]$ and $G - \{x\}$ is dismantlable.

An induced subgraph H of a graph G is called a retract of a graph G if there is a homomorphism ϕ from $V(G)$ onto $V(H)$ such that $\phi(v) = v$ for every $v \in V(H)$; that is ϕ is an identity function on $V(H)$ (see [8]). If G is a connected graph and H is a retract of G, then $c(H) \leq c(G)$ (see [6]).

Proposition 3.2. *Let G be any connected graph of order $n \geq 2$. Then G^{Ξ} is cop win if and only if either G or $\overline{G}$ is a complete graph.*

Proof. If G is the complete graph K_n, then it is immediate that G^{Ξ} is cop win. Let us prove the only if implication by contraposition, assuming that G is not complete but (wlog) it is connected. Consider the partition of the vertex set of G^{Ξ} into the four vertex subsets $K_G \cup K_{\overline{G}} \cup V_G \cup V_{\overline{G}} = V(G^{\Xi})$, where K_G is the set of vertices of G with degree equal to $|V(G)|$ in G^{Ξ}, $K_{\overline{G}}$ is the set of vertices of degree 1 in G^{Ξ}, $V_G = V(G) - K_G$ and $V_{\overline{G}} = V(\overline{G}) - K_{\overline{G}}$. Retracting all the vertices of the set $K_G \cup K_{\overline{G}}$ into just one vertex z, we obtain a new graph H such that $V(H) = V_G \cup V_{\overline{G}} \cup \{z\}$. Notice that since G is not complete, $|V_G \cup V_{\overline{G}}| \geq 4$ and then H has at least five vertices. Now we show that in H no pair of vertices dominate each other and this implies that H is not dismantlable. In fact, we have that:

- z cannot dominate vertices from $V_{\overline{G}}$ because it does't have neighbours in $V_{\overline{G}}$ and cannot dominate vertices from V_G because every vertex in V_G has one neighbour in $V_{\overline{G}}$,
- each pair of adjacent vertices $x \in V_G$ and $x' \in V_{\overline{G}}$ cannot dominate each other because x has neighbors in $V_G \cup \{z\}$ and x' has neighbors in $V_{\overline{G}}$,
- each pair of adjacent vertices $x, y \in V_G$ $(x, y \in V_{\overline{G}})$ cannot dominate each other because $\exists x' \in N(x) \setminus N(y)$ $(\exists x \in N(x') \setminus N(y'))$ and $\exists y' \in N(y) \setminus N(x)$ $(\exists y \in N(y') \setminus N(x'))$.

Therefore, H is not dismantlable and thus G is also not dismantlable which is equivalent to say that G is not cop win. □

Proposition 3.3. *Let T be a tree of order $n \geq 3$. Then $c(T^{\Xi}) = 2$.*

Proof. Let v be a leaf of T. Consider the vertex v' corresponding to v in T^{Ξ}. Place the first cop on the vertex v'. The cop c_1 can guard vertices in $N[v']$, the closed neighborhood of v'. Remaining vertices in $T^{\Xi} - N[v']$ forms a tree. Place the second cop on this tree. We know from [8] that every tree is cop win. Thus, two cops are always sufficient to defeat a robber in this graph. Moreover, by Proposition 3.2, $c(T^{\Xi}) \geq 2$. □

Proposition 3.4. *Let C_n be the cycle of order $n \geq 5$. Then $c(C_n^{\Xi}) = 3$.*

Proof. Let v' be any vertex among the vertices of the $\overline{C_n}$ part of C_n^{Ξ}. Place the first cop c_1 on the vertex v'. Now the cop c_1 can guard the vertices in $N[v']$, the closed neighborhood of v'. Notice that remaining vertices in $C_n^{\Xi} - N[v']$ forms a cycle C_i of order $i \geq 5$ and, since $c(C_i) = 2$, it follows that $c(C_n^{\Xi}) \leq 3$.

For the reverse inequality let us consider the following two cases.

(1) The cycle C_n is such that $n = 5$. Then C_5^{Ξ} is the Petersen graph and we know that its cop number is 3 (see [4]).
(2) The cycle C_n is such that $n \geq 6$ and that we have just two cops in C_n^{Ξ}. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices in C_n and $V' = \{v'_1, v'_2, \ldots, v'_n\}$ be the vertices in $\overline{C}_n$. If both cops starts on vertices inside $\overline{C}_n$ we can place the robber on any vertex of C_n not adjacent to cops positions. Assume that c_2 place himself at v'_i , $1 \leq i \leq n$. Now we place the robber R either on v'_{i+1} or v'_{i-1} by using the following strategy: If c_1 is at v_{i+1}, then place the robber on v'_{i-1}. If c_1 is at v_{i-1}, then place the robber on v'_{i+1}. Let $M = V - \{v_{i+1}, v_{i-1}\}$. If

c_1 is at any vertex from M, then we put the robber on v'_{i+1} or v'_{i-1} arbitrarily. If both cops moves (or place themselves) inside $\overline{C_n}$, then the robber is forced to move (place himself) on a vertex from C_n. As long as cops stays in $\overline{C_n}$ robber can move through the cycle C_n (or maintain his position) without being caught. If both c_1 and c_2 moves to vertices of C_n, robber immediately moves back to adjacent vertex from $\overline{C_n}$. If robber is on C_n while c_1 is at some v_j and c_2 is at some v'_i he can move along the cycle C_n up to the point when he can move to one of the vertices v'_{i+1} or v'_{i-1}. Notice that any of those operations take the robber from a "safe" position to another "safe" position.

This implies that there is no configuration of locations for the cops and the robber in which the robber cannot escape in his next step. Thus we have $c(C_n^{\Xi}) > 2$.

□

It is an easy exercise to show that $c(C_3^{\Xi}) = 1$ (notice that C_3 is a complete graph) and that $c(C_4^{\Xi}) = 2$.

Proposition 3.5. *Let G be a graph which is not cop win (that is, $c(G) \geq 2$), then*

$$c(G^{\Xi}) \leq \max\left\{c(G), c(\overline{G})\right\} + 1$$

and the bound is attained if G is C_5.

Proof. We want to prove that in any case $\max\left\{c(G), c(\overline{G})\right\}+1$ cops is suffice to catch the robber on G^{Ξ}.

Our strategy will be as follows: first we force the robber to go to the $\overline{G}$ part of the graph and then using one cop we prevent robber from going back to G. In the last step $\max\left\{c(G), c(\overline{G})\right\}$ cops will catch the robber on $\overline{G}$.

To force robber to go to $\overline{G}$, at the first round of the game we place all cops besides one (let us denote him by c_0) inside G. Assume that the robber starts in the vertex $v \in G$. Since $\max\left\{c(G), c(\overline{G})\right\} \geq c(G)$ after some number of rounds he will be forced to go to $\overline{G}$ (or lose). Let us assume that the robber goes from the vertex $r \in V(G)$ to the mirror vertex r' and also that c_0 is in a vertex of $\overline{G}$ not adjacent to r' (since otherwise the robber is caught). Then, after switching the two groups of cops (all cops from G goes to $\overline{G}$ and c_0 goes from $\overline{G}$ to G), the cop c_0 is located in a vertex adjacent to r. This implies that in the next round robber cannot go back to the mirror vertex r.

We claim that either there is an easy strategy for cop c_0 to prevent the robber from moving back onto graph G or $\gamma(\overline{G}) = 2$.

Assume that the robber R moves from vertex r' to t' in $\overline{G}$ and that cop c_0 stays at vertex c. If $(c, t) \in E(G)$, then cop c_0 can stay at his position and

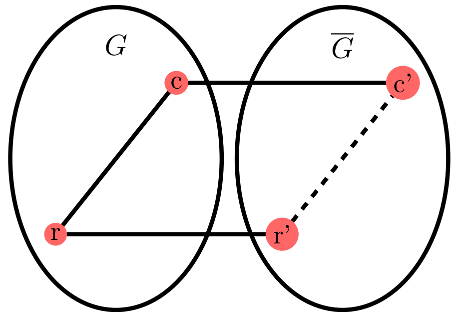

FIGURE 1. Cop c_0 goes from c' to c when the robber appears in r'.

block robber from going back to G. In the second case there is a vertex g connected to current position of the cop such that $(g, t) \in E(G)$ (see Figure 2). In the third case assume that cop c_0 is on vertex c and there is no vertex g that suffice our conditions, then every vertex in $\overline{G}$ is connected either to c' or t'. It means that the domination number of $\overline{G}$ is equal 2 (i.e. t' and c' dominate whole part $\overline{G}$).

a) In the first two cases, as c_0 will prevent the robber from going back to G and $\max\left\{c(G), c(\overline{G})\right\} \geq c(\overline{G})$, the number of cops inside $\overline{G}$ part is enough to catch the robber.
b) In the third case (when $\gamma(\overline{G}) = 2$), we have at least 3 cops in the game. Let us denote them by c_0, c_1 and c_2. When the robber moves, he either end up in a vertex adjacent to some cop and the game is finished or he moves to a vertex which is not adjacent to the vertices occupied by cops. Thus, in the next round, we move cop c_0 back to the graph $\overline{G}$ and move c_1 onto G. The cop c_1 forbids then the robber to move to graph G. Next, we move cop c_2 towards one of the dominating vertices in $\overline{G}$. In the next round we do the same but with c_2 going onto graph G, c_1 onto graph $\overline{G}$ and c_0 towards one of the vertices from dominating set (other than c_2). It is easy to see that after few rounds we will obtain a state when both c_0 and c_2 would be positioned onto vertices of dominating set in $\overline{G}$ and since in every step we forbid robber to move back onto graph G he will stay in $\overline{G}$. Finally we have reached the state where c_1 stop robber from being able to move to the graph G, and c_0 and c_2 dominates all vertices of $\overline{G}$. Thus in next step cops will win the game.

□

Proposition 3.6. *For graphs G and $\overline{G}$ with $c(G) \neq c(\overline{G})$, and $c(\overline{G}), c(G) \geq 2$ we have*

$$c(G^{\Xi}) \leq \max\{c(G), c(\overline{G})\}.$$

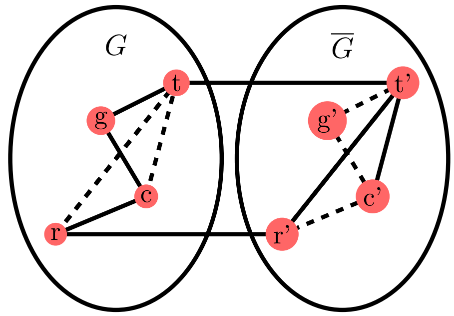

FIGURE 2. Cop c_0 goes from c to g when the robber goes form r' to t'.

Proof. Our goal is to prove that in any case $\max\left\{c(G), c(\overline{G})\right\}$ cops are suffice to catch the robber on G^{Ξ}. W.l.o.g. assume that $c(G) > c(\overline{G})$.

We use almost the same strategy as described in the previous proof. First (as in the previous proof) we force the robber to go to the $\overline{G}$ part of the graph and then (using a single cop) we prevent robber from going back to the G part. In order to force the robber to go to $\overline{G}$, at the first round of the game, we place all cops on the graph G. If the robber put himself on a graph $\overline{G}$, then we move all cops on $\overline{G}$ part of the graph and try to catch him there. Since $c(G)$ cops are sufficient to catch the robber on graph G, at some point he will be forced to move to the graph $\overline{G}$ (he can also make the transition any time earlier when he is not at any danger from cops). Let r' be the vertex in which the robber stays at the moment. We take any cop from the vertex $u \in V(G)$ and move him onto its adjacent vertex u'. Let us denote this cop as c_0. The rest of the cops follows the optimal strategy to catch the robber as if he would have stayed at vertex r. There are two possible options for the robber:

a) The robber came back onto G. Then cops continue the pursuit on this graph. Notice that from the point of view of cops robber stayed at the same position for the two rounds and all apart from cop c_0 are one step closer to catching the robber following the optimal strategy on the graph G. As there are at least 3 cops in play we can alter them in a way that finally they reach a game state where the robber can not get back to the graph G without being caught. I.e. we move $c(G) - 1$ cops towards optimal strategy and treat the last one as if he maintain his position. If robber repeats his behavior of going on and back from $\overline{G}$, we alternate vertices such that after $c(G)$ rounds every one is one step closer to catch the robber on G. At this stage only b) remains.
b) The robber moves from r' onto another vertex $t' \in V(\overline{G})$. Then either $t'u' \in E(\overline{G})$ and the game is over or we move the cop c_0, bringing him to the vertex $u \in G$. Notice that when c_0 jumps to $u \in G$ he is in a vertex adjacent to t. This implies that in the next

round the robber cannot go back to the graph G. At this stage, we bring all the other $c(G)-1$ cops onto the graph $\overline{G}$ and then we have the following two possibilities (this analysis is similar to the one used in the previous proposition):

(i) The diameter of G is equal to 2. It means that we can move the cop c_0 in such a way that he will block the robber from going back to the graph G. In this case as $c(\overline{G}) < c(G)$ the rest of the cops can catch the robber on the graph $\overline{G}$ and thus winning the game.

(ii) The diameter of G is greater or equal 3. From Lemma 2.4, it follows that $c(\overline{G}) \leq 2$ and, since we have at least 3 cops in play we are able to win.

□

Proposition 3.7. *Let G be a graph with connected components $\{G_1, G_2, \ldots, G_n\}$, $\max\{c(G_1), \ldots, c(G_n)\} \neq c(\overline{G})$, and $c(\overline{G}) \geq 2$. Then*

$$c(G^{\Xi}) \leq \max\{c(G_1), c(G_2), \ldots, c(G_n), c(\overline{G})\}.$$

Proof. We can follow the strategy from previous proposition. Assume first that $\max\{c(G_1), ..., c(G_n)\} > c(\overline{G})$. Moreover assume that the robber starts at some component G_i of the graph G. We then start to move all the cops through graph $\overline{G}$ onto the component G_i and try to capture the robber over there. Notice that the robber can not move on other component of G without going through the graph $\overline{G}$. As soon as the robber moves to the graph $\overline{G}$ and makes a move within this graph we play on the $\overline{G}$ with the same strategy as in previous proof. Previous arguments shows that eventually the cops will win the game.

Let us now consider the case when $\max\{c(G_1), \ldots, c(G_n)\} < c(\overline{G})$. We start the game on graph $\overline{G}$ forcing the robber to move onto G. Assume that the robber goes to some component G_i of G (or is there in the first step). Then after moving some cop, say c_0, to one vertex $v' \in V(\overline{G})$ which is a copy of a vertex $v \in V(G_j)$ such that $j \neq i$, the robber can not return to $\overline{G}$. Therefore, moving the remaining cops to G_i they are enough to catch the robber after a finite number of steps. □

Notice that as stated earlier either G or $\overline{G}$ is connected, thus the above proposition cover all possible cases (we can not have multiple components in both G and $\overline{G}$). Now, using the above arguments together with Proposition 3.5, we get to the following corollary.

Corollary 3.8. *Let G be a graph with connected components $\{G_1, G_2, \ldots, G_n\}$ and $\max\{c(G_1), \ldots, c(G_n), c(\overline{G})\} \geq 2$, then we have that*

$$c(G^{\Xi}) \leq \max\{c(G_1), c(G_2), \ldots, c(G_n), c(\overline{G})\} + 1.$$

Furthermore we can bound the cop number of the graphs G^{Ξ} in terms of the minimum degree of G.

Proposition 3.9. *Let G be a simple graph with $c(G) = k$ and $\delta(G) = m$. Then $c(G^{\Xi}) \leq \max\{k, m+1\} + 1$.*

Proof. Consider $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(\overline{G}) = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$. Assume that v_1 is a vertex in G with minimum degree m and neighbors $v_2, \ldots v_{m+1}$. Now place a cop on $v_1^{'}$. This cop can guard the vertices in $N[v_1^{'}] = \{v_1, v_1^{'}, v_{m+2}^{'}, \ldots, v_n^{'}\}$. Now in $\overline{G}$ we have m vertices which remains unguarded. We can then put m cops to guard them. Thus the cop number of $\overline{G}$ is at most $m+1$. Hence the total number of cops we need to win in this game is at most $\max\{m+1, k\} + 1$. □

Before to proceed, it is worth to introduce the following proposition.

Proposition 3.10. *Let G be a planar graph and $\overline{G}$ its complement, with components $\{\overline{G_1}, \overline{G_2}, \ldots, \overline{G_n}\}$. Then $\max\{c(\overline{G_1}), \ldots, c(\overline{G_n})\} \leq 3$.*

Proof. Since G is a planar graph, it has a vertex u of degree less or equal to five. Let us us assume that $u \in V(G)$ is such that $N_G(u) = \{v_1, v_2, \ldots, v_t\}$, with $t \leq 5$ and let us consider the graph $\overline{G}$ which is the complement of G. Then, the vertex u' is adjacent to each vertex of $\overline{G}$ apart itself and the the vertices $v_1', v_2', \ldots, v_t'$. If we put the first cop at the vertex u', then it prevents the robber from moving onto any vertex of $\overline{G}$ apart from $v_1', v_2', \ldots, v_t'$. Since we know that the Petersen graph is the smallest graph with cop number equal to three (see [4]) two cops are enough to win the game on every connected graph build on the remaining t vertices. Therefore, together with a cop placed on vertex u it implies that 3 cops can protect any connected component of the graph $\overline{G}$. □

It is now worth to recall the bound obtained in [8] for planar graphs which will be used in the proof of Proposition 3.12 below and in a couple of additional results throughout the paper.

Proposition 3.11. [8] *If G is a planar graph, then $c(G) \leq 3$.*

Proposition 3.12. *If G is planar, then $c(G^{\Xi}) \leq 3$.*

Proof. Assuming that G is a planar graph, according to [8], $c(G) \leq 3$ and we have two cases.

(1) The diameter of G is greater or equal to 3. Therefore, $\overline{G}$ has domination number 2 and hence $c(\overline{G}) \leq 2$. Thus,
   (a) if $c(G) = 3$, by Proposition 3.7, $c(G^{\Xi}) \leq 3$ when $c(\overline{G}) = 2$ and hence it is immediate that this inequality also holds when $c(\overline{G}) = 1$;
   (b) if $\max\{c(G), c(\overline{G})\} = 2$, by Corollary 3.8, the inequality holds;
   (c) Finally, if $\max\{c(G), c(\overline{G})\} = 1$, the inequality is immediate.

(2) The diameter of G is less than three. We may apply the result of Goddard and Henning [12] which states that every planar graph of diameter 2 has domination number at most 2 except for the graph F of Figure 3 which has domination number 3. Since, as it is easy

to check, the graph F has cop number equal to 2, it follows that all planar graphs of diameter two have the cop number at most two. Moreover, by Proposition 3.10 all the components in the complement of G have cop number at most 3, thus again, due to Proposition 3.7 and Corollary 3.8, we have $c(G^{\Xi}) \leq 3$.

$\square$

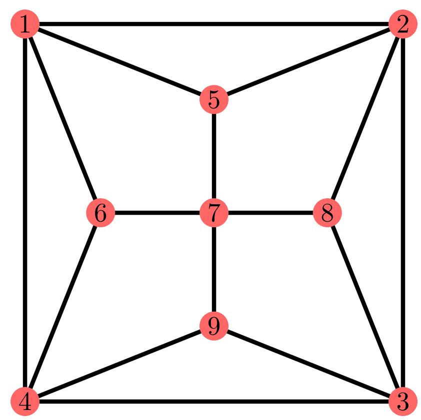

FIGURE 3. The uniqe planar graph F of diameter 2 with $\gamma(F) = 3$ and $c(F) = 2$.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$, where two vertices (u, v) and (x, y) are adjacent in $G \square H$ if, and only if, $u = x$ and v is adjacent to y in H or $v = y$ and u is adjacent to x in G.

A *two dimensional grid* graph is the graph obtained by the cartesian product $P_n \square P_m$, where m and n are integers.

Proposition 3.13. *Let P_n be the path of order $n \geq 3$ and let $G = P_n \square K_2$. Then $c(G^{\Xi}) \leq 3$.*

Proof. Let $G = P_n \square K_2$. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n in G and $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ be the vertices in the second copy of P_n in G (let denote it by $\widetilde{P_n}$). Also let $v_1^{'}, v_2^{'}, \ldots v_n^{'}$ be the copies of the vertices of P_n in $\overline{G}$ and $\overline{v}_1^{'}, \overline{v}_2^{'}, \ldots, \overline{v}_n^{'}$ be the copies of the vertices of $\widetilde{P_n}$ in $\overline{G}$.

Consider the vertex $v_1^{'}$ and place the first cop c_1 on the vertex $v_1^{'}$. This cop can guard the vertices in $N[v_1^{'}] = \{v_1, v_1^{'}, v_3^{'}, v_4^{'}, \ldots, v_n^{'}, \overline{v}_2^{'}, \ldots, \overline{v}_n^{'}\}$. The remaining vertices in G^{Ξ} create a graph with cop number equal to two. $\square$

4. COP EDGE-CRITICAL GRAPHS

As it was already referred, according to [8], every tree is cop win. Furthermore, taking into account that a wheel graph W_n is a graph with n

vertices ($n \geq 4$), formed by connecting a single vertex to all vertices of an $(n-1)$-cycle, we have the following result.

Lemma 4.1. [8] *For every integer $n \geq 4$ we have that $c(W_n) = c(P_n) = c(K_n) = 1$, and $c(C_n) = 2$.*

There are graphs which are both cop vertex and cop edge-critical, as it is the case of C_n, with $n \geq 4$. Notice that $c(C_n) = 2$ and both $C_n - \{e\}$ and $C_n - \{v\}$ are paths, which implies that $c(C_n - \{e\}) = c(C_n - \{v\}) = 1$. Furthermore, it is immediate that the cop number of a unicyclic graph with a cycle C_n such that $n \geq 4$ is 2.

Proposition 4.2. *Let G be a unicyclic graph and C_n a cycle in G. If $n \geq 4$, then G is cop edge-critical.*

Proof. Let G be a unicyclic graph and e one of its edges. If we remove e, then we have one of the following cases:

(1) $G - e$ is a tree (when edge $e \in C_n$) and then $c(G - e) = c(T) = 1$.
(2) $G - e$ is the sum of a unicyclic graph G_1 and a tree (when $e \notin C_n$). Therefore, $c(G - e) = c(G_1 \cup T) = c(G_1) + c(T) = 2 + 1 = 3$.

□

Lemma 4.3. *Removing a single edge or single vertex from a graph G can decrease its cop number by at most one.*

Proof. If we put a single cop onto a removed vertex v (or on the end-vertex of a removed edge e) we get $c(G) \leq c(G - v) + 1$ $(c(G) \leq c(G - e) + 1)$. □

As mentioned earlier S. L. Fitzpatrick [11], characterized edge-critical planar graphs whose cop number changes from 2 to 1. A more general study of this problem is due to N.E. Clarke et al. [10], they examined when the cop number of the graph grow from 1 to 2, after addition, deletion, subdivision, or contraction of edges. W. Baird et al. [4] proved that the Petersen graph, C_5^{Ξ}, is the minimum order 3-cop edge-critical graph. The following proposition states that the next two graphs of the family, C_n^{Ξ}, with $n \geq 5$, have the same property but since $|V(C_n^{\Xi})| = 2n$ they have two and four more vertices, respectively.

Proposition 4.4. *The graphs C_6^{Ξ} and C_7^{Ξ} are 3-cop edge-critical.*

Proof. By Proposition 3.4, $c(C_7^{\Xi}) = c(C_6^{\Xi}) = 3$. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices in C_n and $V' = \{v_1', v_2', \ldots, v_n'\}$ be the vertices in $\overline{C}_n$ as usually. We will prove that for any $e \in E(C_n^{\Xi})$, $c(C_n^{\Xi} - e) = 2$, when $n \in \{6, 7\}$.

We consider the following three cases for the deleted edge e:

a) $e = v_{i-1}v_i$ (an edge connecting two vertices in C_n),
b) $e = v_i'v_i$ (an edge between C_n and $\overline{C}_n$),
c) $e = v_i'v_j'$ (an edge connecting two vertices in $\overline{C}_n$).

W.l.o.g. assume that $i = 2$. Consider two cops. In both cases a) and b) we start the game by putting these cops on the vertices v_1' and v_2'.

a) The vertices v_1' and v_2' dominate all vertices in $\overline{C}_n$, so the robber has to start the game in C_n. In particular, he has to choose one of the vertices in $\{v_3, v_4, \ldots, v_n\}$. Let us assume that the robber starts the game at the vertex v_k, $3 \leq k \leq n$. Then we move the cops on the vertices v_k' and v_{k-1}' and this forces the robber to move to the vertex v_{k+1}. We move the cop from v_{k-1}' to v_{k+1}' and so on, until the robber will reach the vertex v_1, where he will be caught (as he can not move to v_2).

b) We start the game similarly to the case a) and force the robber to move to the vertex v_2, while cops stay at v_n' and v_1'. Then we move the cops on the vertices v_1' and v_3' and this forces the robber to stay on the vertex v_2. Next we move the cop on v_1' to the vertex v_1 and the cop on v_3' to the vertex v_3 and then the robber will be caught.

c) If we delete an edge of this type from C_6^{Ξ} (C_7^{Ξ}), then we can find two vertices v_k' and v_{k+1}' of degree three (four) in the $\overline{C}_6$ ($\overline{C}_6$) part whose mirror vertices are adjacent in C_6 (C_7) and such that v_k' and v_{k+1}' dominates all vertices in $\overline{C}_6$ ($\overline{C}_7$). Those vertices would be the starting points for cops. We begin the game similarly to the case a) and b) by placing the cops on the vertices v_k' and v_{k+1}' and force the robber to move to the vertex v_2. Once the robber moves to the vertex v_2 we get a state where cops stay at v_3' and v_1'. Now the robber has two choices (1) remains at his position or (2) move to the vertex v_2'.

   (1) Suppose that the robber remains on the vertex v_2. Then we move one cop, say c_1, to the vertex v_1 and the other cop, c_2, will stay on the vertex v_3'. Now the robber must move to the vertex v_2' and we move c_2 to prevent $N[v_2']$ (jointly with c_1). Thus, the robber will be captured in the next round.

   (2) Suppose the robber moves to the vertex v_2'. Then we move one cop, say c_1, to the vertex v_1 and the other cop, c_2, is moved in order to prevent $N[v_2']$ (jointly with c_1). Thus, the robber will be captured in the next round.

□

Notice that a similar result to the above proposition is not valid for the graphs C_n^{Ξ}, such that $n \geq 8$. In fact, deleting the edge between the vertices $v_1', v_{\lfloor n/2 \rfloor+1}' \in V(\overline{C}_n)$, the cop number of C_n^{Ξ} does not decrease, when $n \geq 8$. Hence those graphs are no longer cop edge-critical. Although removing any edge of C_n or any edge between a vertex in C_n and a vertex in $\overline{C}_n$ still do decrease the cop number of C_n^{Ξ} to 2.

The girth of a graph is the length of a shortest cycle contained in the graph. The minimum degree of G is denoted by $\delta(G)$. In [2] the following elementary but useful result is presented.

Proposition 4.5. [2] *If G has girth at least 5, then $c(G) \geq \delta(G)$.*

Proposition 4.6. *Let G be the dodecahedron graph. Then G is 3-cop edge-critical.*

Proof. Let G be the dodecahedron graph. Then G is planar 3-regular graph with girth 5. Therefore, from Proposition 4.5 combined with Proposition 3.11, it follows that $c(G) = 3$. Suppose we delete an edge e from G (see, for instance, the graph depicted in Figure 4). By symmetry playing the game on RHS is equivalent to playing the game on LHS. Moreover, since the dodecahedron is edge transitive, if we remove any edge e from it, then we would get the same figure as depicted in Figure 4 (consider the dodecahedron embedded in the sphere).

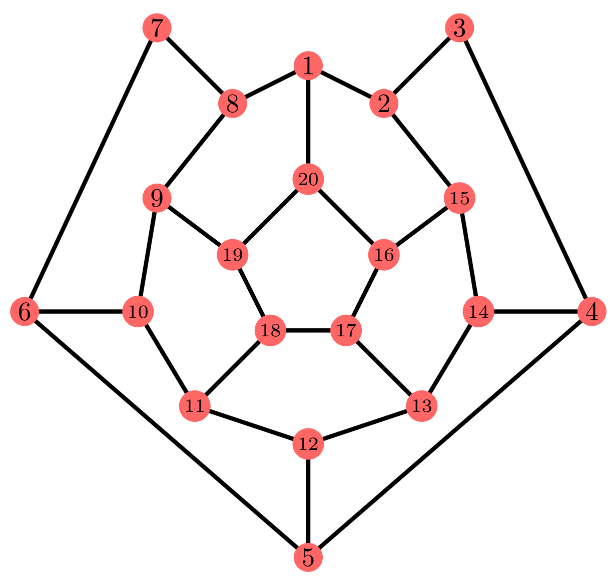

FIGURE 4. Dodecahedron minus one edge.

Claim: $c(G - e) = 2$ for any edge $e \in G$.

We always start the game by placing the cop c_1 on the vertex 12 and the cop c_2 on the vertex 20. These two cops dominates the vertices 1,16,19,11,13,5,20 and 12. Now the remaining vertices on the right hand side of the axis of symmetry are 17,2,15,14,3 and 4. We play the game by placing the robber at each of these vertices and we get the following play steps for each of the following cases:

Case-I : Assume that the robber is on the vertex 17. The cops can catch the robber in at most 6 rounds.

R1 : The cop c_1 moves to the vertex 11 and the cop c_2 moves to the vertex 16. The robber must move to the vertex 13.

R2 : The cop c_1 moves to the vertex 12 and the robber must move to the vertex 14.

R3 : The cop c_2 moves to the vertex 15 and the robber must move to the vertex 4.

R4 : The cop c_1 moves to the vertex 5 and the robber must move to the vertex 3.

R5 : The cop c_1 moves to the vertex 4 and the cop c_2 moves to the vertex 2. These two vertices forms a dominating set of the robber's neighborhood, so the cops win.

Case-II : Assume that the robber is on the vertex 2. Now the cops can catch the robber in at most 5 rounds.

R1 : The cop c_1 moves to the vertex 13 and the cop c_2 moves to the vertex 1. Now the robber can moves to the vertices 3 and 15. Suppose he moves to the vertex 15.

R2 : The cop c_1 moves to the vertex 14 and the cop c_2 moves to the vertex 20. Now the robber must move to the vertex 2.

R3 : The cop c_2 moves to the vertex 1 and the robber must move to the vertex 3.

R4 : The cop c_1 moves to vertex 4 and the cop c_2 moves to the vertex 2 and the game finish in the next step.

Now suppose that after the first round the robber moves to the vertex 3

R2 : The cop c_1 moves to the vertex 14 and the cop c_2 moves to the vertex 2 and the game finish in the next step.

Case-III : Assume that the robber is on the vertex 15. Now the cops can catch the robber in at most 4 rounds.

R1 : The cop c_1 moves to the vertex 13 and then the robber either stays on the vertex 15 or he can move to the vertex 2. Suppose he stay on the vertex 15.

R2 : The cop c_1 moves to the vertex 14 and the robber must move to the vertex 2.

R3 : The cop c_2 moves to the vertex 1 and the robber must move to the vertex 3.

R4 : The cop c_1 moves to the vertex 4 and the cop c_2 moves to the vertex 2. Then the game finish in the next step.

Assume that after the first round the robber moves to the vertex 2.

R2 : The cop c_1 moves to the vertex 14 and c_2 moves to the vertex 1. Now the robber must move to the vertex 3.

R3 : The cop c_1 moves to the vertex 4 and c_2 moves to the vertex 2. Then the game finish in the next step.

Case-IV : Assume that the robber is on the vertex 14. Now the cops can catch the robber in at most 4 rounds.

R1 : The cop c_2 moves to the vertex 16 and the robber either stays on the vertex 14 or he can move to the vertex 4. Suppose he stays on the vertex 14.

R2 : The cop c_2 moves to the vertex 15 and the robber must move to the vertex 4.

R3 : The cop c_1 moves to the vertex 5 and the robber must move to the vertex 3.

R4 : The cop c_1 moves to vertex 4 and the cop c_2 moves to the vertex 2. Then the game finish in the next round.

Assume that after the first round, the robber moves to the vertex 4.

R2 : The cop c_1 moves to the vertex 5, and the cop c_2 moves to the vertex 15 and then the robber must move to vertex 3.

R3 : The cop c_1 moves to the vertex 4 and the c_2 moves to the vertex 2. Then the game finish in the next step.

Case-V : Assume that the robber is on the vertex 3. Now the cops can catch the robber in at most 2 rounds.

R1 : The cop c_1 moves to the vertex 5 and the cop c_2 moves to the vertex 1. Now the robber only can stay on the vertex 3.

R2 : The cop c_1 moves to the vertex 4 and the cop c_2 moves to the vertex 2. Then the game finish in the next step.

Case-VI : Assume that the robber is on the vertex 4. Now the cops can catch the robber in at most 4 rounds.

R1 : The cop c_2 moves to the vertex 16 and the robber either stays on the vertex 4 or he can move to the vertices 3 or 14.

R2 : The cop c_2 moves to the vertex 15 and the robber must be in one of the vertices 4 or 3.

R3 : The cop c_1 moves to the vertex 5 and then the robber must be in the vertex 3.

R4 : The cop c_2 moves to the vertex 2 and the cop c_1 moves to the vertex 4. Then the game finish in the next step.

Thus we need only two cops to win the game on the graph depicted in Figure 4. □

The Heawood graph is the point/line incidence graph on the Fano plane. It has 14 vertices, 21 edges, it is cubic, and all cycles in this graph have six or more edges.

Proposition 4.7. *Let G be the Heawood graph. Then G is 3-cop edge-critical.*

Proof. Let G be the Heawood graph. Then G has girth 6. Therefore from 4.5 we have $c(G) \geq 3$. If we show that $c(G - e) = 2$, from Lemma 4.3 we get conclusion that $c(G) = 3$. Suppose we delete an edge e from G see, for instance, the graph depicted in Figure 5. By symmetry playing the game on

LHS is equivalent to playing the game on RHS. Moreover as the Heawood graph is edge transitive, therefore if we remove any edge e from it, then we get the same figure as depicted in Figure 5.

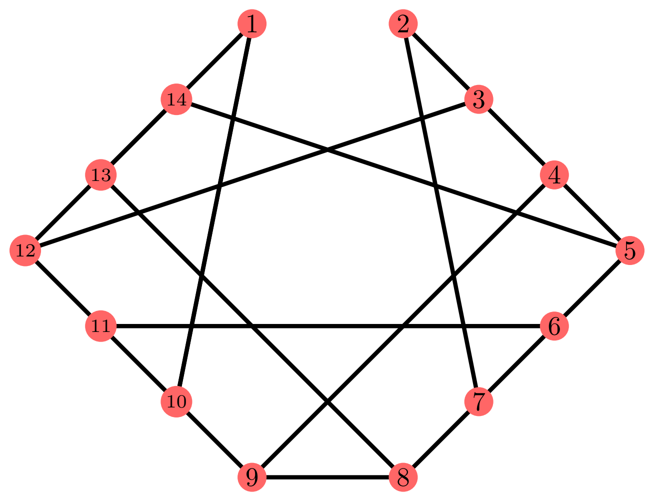

FIGURE 5. Heawood graph minus one edge.

Claim: $c(G-e) = 2$ for any edge $e \in G$.

We always start the game by placing the cop c_1 on the vertex 10 and the cop c_2 on the vertex 7. These two cops dominates the vertices 1,2,6,7,8,9,10 and 11. Now the remaining vertices on the right hand side of the axis of symmetry are 3,4 and 5. We play the game by placing the robber at each of these vertices and we get play steps for each of the following cases:

Case-I : Assume that the robber is on the vertex 5. Now the cops can catch the robber in at most 5 rounds.

- R1 : The cop c_1 moves to the vertex 1 and the cop c_2 moves to the vertex 6. Now the robber must move to the vertex 4.
- R2 : Now the cop c_1 moves to the vertex 10 and the cop c_2 moves to the vertex 5 and the robber must move to the vertex 3.
- R3 : The cop c_1 moves to the vertex 11 and the cop c_2 moves to the vertex 4, again the robber must move to the vertex 2.
- R4 : The cop c_1 moves to the vertex 6 and the cop c_2 moves to the vertex 3. In this case we can observe that there is no chance for the robber to escape. If he moved he will be caught by the cop c_1 and when he stays in his position he will be caught by the cop c_2. Then the game finish in the next step.

Case-II : Assume that the robber is on the vertex 4. Now the cops can catch the robber in at most 5 rounds.

R1 : The cop c_2 moves to the vertex 6. Now the robber can stay in the vertex 4 (then after this round we get situation described in CASE-I round-2) or move to the vertex 3 (we will investigate this case).

R2 : Now the cop c_1 moves to the vertex 11 and the cop c_2 moves to the vertex 5 and the robber can stay in the vertex 3 (then after this round we get situation described in CASE-I round-3) or he moves to the vertex 2 (we investigate this case).

R3 : The cop c_1 moves to the vertex 6 and the cop c_2 moves to the vertex 4, again the robber has to stay in the vertex 2 (then after this round we get situation described in CASE-I round-4) or, otherwise the cops win in the next round. Thus, in both cases the cops win the game.

Case-III : Assume that the robber is on the vertex 3. Now the cops can catch the robber in at most 5 rounds.

R1 : The cop c_1 moves to the vertex 11 and the cop c_2 moves to the vertex 6. Now the robber can stay in the vertex 3 (we investigate this as a subcase a) or he moves to the vertex 2 (we investigate this case as a subcase b) or he moves to the vertex 4 (then after this round we get situation described in CASE-I round-2).

R2a : Now the cop c_2 moves to the vertex 5 and the robber stays in the vertex 3 or moves to the vertex 2 (both possibilities are described in CASE-II round-2).

R2b : Now the cop c_1 moves to the vertex 12 and c_2 moves to the vertex 7 and the robber stays in the vertex 2 (in next round he will be caught by the cop c_2) or he moves to the vertex 3 (then in the next round he will be caught by the cop c_1).

Thus, we need only two cops to win the game on the graph depicted in Figure 5.

□

5. Open problems

There are several interesting questions regarding families of 3-cop-edge critical graphs. The most general one is the following.

Problem 5.1. *Give an example of an infinite family of* 3*-cop edge-critical graphs or prove that such family does not exists.*

We believe that such family might be ,,hidden" among the graphs G^{Ξ}.

Given a positive integer s, we say that v is an *s-trap* if one can place s cops on the vertices of $G-v$ such that v and all neighbors of v are adjacent to the vertices occupied by the cops. The next conjecture follows naturally from the given examples of 3-cop-edge critical graphs (since all of them preserve this property).

Conjecture. *If G is a graph such that $c(G) = 3$ and removing any edge would create a* 2*-trap, then G is* 3*-cop edge-critical.*

Problem 5.2. *Does there exist a graph G with diameter 2 and $c(G) \geq 3$ such that $c(\overline{G}) = c(G)$? If it is the case, is G^{Ξ} cop edge-critical?*

Notice that the graphs of order n and diameter 2 can have a cop number of order $O(\sqrt{n})$ (see [15, 20]).

Problem 5.3. *Characterize graphs G for which $c(\overline{G}) = c(G)$.*

Problem 5.4. *Characterize graphs G for which $c(G^{\Xi}) = c(G)$.*

In particular, we know that there are graphs G such that $c(\overline{G}) = k = c(G)$, with $k \leq 2$, as it is the case of P_4 and C_5, and on the other hand $c(C_4^{\Xi}) = c(C_4) = 2$. However, the above problems remain open.

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References

1. Aida Abiad, Phil DeOrsey, Leslie Hogben, Kirsten Hogenson, Franklin Kenter, Jephian C.-H. Lin, Sarah Loeb, Heather Smith and Michael Young, *Zero forcing number on the counterprism of graphs*, manuscript.
2. M. Aigner, M. Fromme, *A game of cops and robbers*, Discrete Applied Mathematics, **8** (1984) 1–11.
3. W. D. Baird, *Cops and robbers on graphs and hypergraphs*, Theses and dissertations Ryerson University. Paper 821, 2011.
4. William Baird, Andrew Beveridge, Paolo Codenotti, Aaron Maurer, John McCauley, Silviya Valeva, *On the minimum order of k-cop-win graphs*, Contributions to Discrete Mathematics, **9** (2014) 70–84.
5. H.-J. Bandelt, E. Prisner, Clique graphs and Helly graphs, J. Combin. Theory B, **51** (1991) 34–45.
6. A. Berarducci, B. Intrigrila, *On the cop number of a graph*, Advances in Applied Mathematics, **14** (1993) 389-403.
7. A. Beveridge, A. Maurer, J. McCauley and S. Valeva, *Cops and Robbers on Planar Graphs*, https://apps.carleton.edu/curricular/math/assets/PursuitEvasionReport3.pdf, report.
8. A. Bonato and R. J. Nowakowski, *The Game of Cops and Robbers on Graphs*, Student Mathematical Library, American Mathematical Society, **61**, 2011.
9. A. Brandstädt, V. B. Le, J. P. Spinrad, *Graph Classes : a survey*, SIAM Monographs on Discrete Mathematics and Applications, 1999.
10. N.E. Clarke, S.L. Fitzpatrick, A. Hill and R.J. Nowakowski, *Edge critical cops and robber*, Discrete Mathematics **310** (2010) 2299–2309.
11. S. L. Fitzpatrick, *Edge-critical cops and robber in planar graphs*, Discrete Mathematics, **329** (2014) 1–11.
12. W. Goddard, M. A. Henning, *Domination in planar graphs with small diameter*, J. Graph Theory, **40**, (2002), 1–25.
13. F. Jaeger and C. Payan, *Relations du type Nordhais-Gaddum pour le nombre d'absorption d'un graphe simple*, C. R. Acad. Sci. Paris S'er. A **274**, (1972), 728–730.
14. J. P. Joseph and S. Arumugam, *Domination in graphs*, Internat. J. Management Systems, **11**, (1995), 177–182.
15. G. MacGillivray and K. Seyffarth, *Domination numbers of planar graphs*, J. Graph Theory, **22**, (1996), 213–229.

16. R. Nowakowski and P. Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Mathematics **43**, (1983), 235–239.
17. P. Pralat, *When does a random graph have constant cop number?*, Australian Journal of Cominatorics. **46**, (2010), 285–296.
18. T. Poston, *Fuzzy geometry*, Ph. D. Thesis, University of Warwick, 1971.
19. A. Quilliot, *Homomorphisms, points fixes, rétractions et jeux de poursuite dans les graphs, les ensembles ordonnés et les espaces métriques*, Ph. D. Thesis, Université de Paris VI, 1983.
20. Z. A. Wagner, *Cops and Robbers on diameter two graphs*, arXiv:1312.0755v3, (2014).

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