A simplex like approach based on star sets for recognizing convex-QP adverse graphs

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Abstract

A graph G with convex-QP stability number (or simply a convex-QP graph) is a graph for which the stability number is equal to the optimal value of a convex quadratic program, say $P(G)$. There are polynomial-time procedures to recognize convex-QP graphs, except when the graph G is adverse or contains an adverse subgraph (that is, a non complete graph, without isolated vertices, such that the least eigenvalue of its adjacency matrix and the optimal value of $P(G)$ are both integer and none of them changes when the neighborhood of any vertex of G is deleted). In this paper, from a characterization of convex-QP graphs based on star sets associated to the least eigenvalue of its adjacency matrix, a simplex-like algorithm for the recognition of convex-QP adverse graphs is introduced.

Keywords: convex quadratic programming in graphs, star sets, graphs with convex-QP stability number, simplex-like approach.

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with at least one edge, where $V(G) = \{1, 2, \ldots, n\}$ and $E(G)$ denote respectively the vertex and the edge sets. We will write $ij \in E(G)$ to represent the edge linking nodes i and j of $V(G)$. For each node $i \in V(G)$, $N_G(i)$ will denote the set of vertices of G which are adjacent to i and $|N_G(i)|$ will be the vertex i degree (in general, given a finite set S, the number of elements of S will be denoted by $|S|$). Given $S \subseteq V$, the subgraph of G induced by S is defined as $G[S] = (S, E(S))$, where $E(S) = \{ij \in E : i, j \in S\}$.

The adjacency matrix $A_G = [a_{ij}]$ of G is the symmetric matrix such that $a_{ij} = 1$ if $ij \in E(G)$ and 0 otherwise. The eigenvalues of A_G are usually called the eigenvalues of G; note that they are real because A_G is symmetric. The minimum (maximum) eigenvalue of A_G will be denoted by $\lambda_{\min}(G)$ (resp. $\lambda_{\max}(G)$). It is well known that if G has at least one edge then $\lambda_{\min}(G) \leq -1$. Actually, $\lambda_{\min}(G) = 0$ iff G has no edges, $\lambda_{\min}(G) = -1$ iff G has at least one edge and every component is complete, and $\lambda_{\min}(G) \leq -\sqrt{2}$ otherwise

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[5]. We will use the notation $\sigma(G)$ for representing the multiset of eigenvalues of A_G. Then, for $\lambda \in \sigma(G)$, $\mathcal{E}(\lambda) = \ker(A_G - \lambda I_n)$ is the eigenspace of A_G associated to the eigenvalue λ (in general, $\ker(M)$ will denote the null space of a matrix M).

A stable set (or independent set) of G is a subset of nodes of $V(G)$ whose elements are pairwise nonadjacent. The stability number (or independence number) of G is defined as the cardinality of a largest stable set and is usually denoted by $\alpha(G)$. The stable set problem on G is to find a maximum stable set in G, i.e., a stable set with $\alpha(G)$ nodes.

For any graph G, the following convex quadratic program $P(G)$ introduced in [7] allows to obtain the upper bound $\upsilon(G)$ on $\alpha(G)$, i.e.,

$$(P(G)) \qquad \alpha(G) \leq \upsilon(G) = \max\{2e^T x - x^T\,(H + I)\,x : x \geq 0\}, \tag{1}$$

where e is the $n \times 1$ all-ones vector, T stands for the transposition operation, I is the identity matrix of order n and $H = A_G/\tau$ with $\tau = -\lambda_{\min}(G)$. Since G has at least one edge (i.e., $E(G) \neq \emptyset$), A_G is indefinite since its trace is zero. Hence $\lambda_{\min}(H) = -1$ and this guarantees the convexity of $P(G)$ because $H + I$ is positive semidefinite. Consequently, $\upsilon(G)$ can be computed in polynomial time.

The graphs that satisfy $\alpha(G) = \upsilon(G)$ were introduced in [7] and subsequently studied in [9, 3]. They are currently known as graphs with convex-QP stability number (or simply convex-QP graphs, where QP means quadratic programming), a denomination introduced in [3].

The upper bound $\upsilon(G)$ defined in (1) constitutes a quadratic programming approach to the stable set problem. It should be mentioned that we may find in the literature several different quadratic programming approaches to combinatorial problems in graphs, as it is the case, for instance, in [1, 2, 6, 8, 10].

This paper is devoted to the recognition of convex-QP graphs. Specifically, we use the theory of star complements (see [4, p. 136]) for giving a new characterization of graphs with convex-QP stability number. Applying this characterization to the so called adverse graphs (see below), we show how a simplex algorithm can be used for deciding if a given adverse graph is a convex-QP graph.

The remaining sections of this paper are organized as follows. In section 2 we will review and extend some facts about the convex-QP graphs to be used in the sequel. Next, in section 3, some results which are fundamental to the theory of star complements are recalled. Also, the above mentioned characterization of convex-QP graphs is proved, introducing the concept of star solutions of $P(G)$. In section 4, the notion of adverse graph is precisely defined and some issues related to the recognition of convex-QP graphs are recalled. The open question of recognizing convex-QP adverse graphs, which is studied in the next sections, is stated. In section 5, several properties of adverse graphs are established and the equivalence between a star solution and a basic feasible solution of a linear problem with a set of constraints defined by a particular star set for $\lambda_{\min}(G)$ is proved. In section 6, a simplex algorithm for recognizing if a given adverse graph is a convex-QP graph is given as well as some computational experiments. Finally, some conclusions are presented in section 7.

2 Convex-QP Graphs

If x is an optimal solution of convex program $P(G)$ given in (1), the Karush-Kuhn-Tucker optimality conditions guarantee the existence of vector $y \geq 0$ such that

$$(H+I)x = e + y \text{ and } x^T y = 0. \tag{2}$$

Such vector y is called the *complementary solution* associated to x and is unique as it was proved in [9].

Theorem 1 [9] *Let G be a graph with at least one edge. If x_1^* and x_2^* are optimal solutions for program $P(G)$, the difference $x_1^* - x_2^*$ belongs to $\mathcal{E}(\lambda_{\min}(G))$. Additionally, $y_1^* = y_2^*$, where y_1^* and y_2^* are the complementary solutions associated to x_1^* and x_2^*, respectively.*

When $y = 0$ in (2), x is a critical point for the objective function of $P(G)$ and then all the optimal solutions are also critical points for that objective function. In such case, a nonnegative real vector x is an optimal solution of $P(G)$ if and only if it is a solution of the system

$$(A_G + \tau I)\; x = \tau e. \tag{3}$$

Based on the Karush-Khun-Tucker conditions (2) the following necessary and sufficient optimality conditions for $P(G)$ can be obtained.

Theorem 2 [3] *Consider a graph G with n vertices and at least one edge and let a_G^i be the i-th row of the matrix A_G. Then the n-tuple of real numbers x is an optimal solution $P(G)$ if and only if*

$$\forall i \in V(G), \quad x_i = \max\left\{0, 1 - \frac{a_G^i\; x}{\tau}\right\},$$

where x_i is the i-th entry of x.

According to Theorem 2, considering a graph G of order n with at least one edge, it is immediate that an optimal solution x and the optimal value $\upsilon(G)$ of $P(G)$, have the following properties:

$$\forall i \in V(G):\; 0 \leq x_i \leq 1 \text{ and } 1 \leq \upsilon(G) \leq n.$$

Furthermore,

$$\forall U \subset V(G), \quad \upsilon(G-U) \leq \upsilon(G).$$

Another consequence of Theorem 2 is the following:

Corollary 2.1 *Let G be a graph with at least one edge and x a 0-1 optimal solution of $P(G)$. Then the vertex subset $S = \{i \in V(G) : x_i = 1\}$ is a maximum stable set and $\alpha(G) = \upsilon(G)$.*

Proof. According to the optimality conditions of Theorem 2, S is a stable set (since every vertex u adjacent to a vertex $v \in S$ is such that $x_u = 0$ and thus $u \notin S$). Since x is the characteristic vector of S, $\upsilon(G) = |S|$. Therefore, from $|S| = \upsilon(G) \geq \alpha(G) \geq |S|$ the result follows. ∎

From now on, the class of convex-QP graphs is denoted by $\mathcal{Q}$ and the graphs belonging to $\mathcal{Q}$ will also be called $\mathcal{Q}$-graphs.

3 A Characterization of $\mathcal{Q}$-graphs Based on Star Sets

In [7] a characterization of $\mathcal{Q}$-graphs was given. This section presents a new characterization of $\mathcal{Q}$-graphs based on star sets. With this aim we recall some basic concepts of the theory of star complements (see [4, pp. 136-140]).

Considering a graph G with n vertices and an eigenvalue $\lambda \in \sigma(G)$, let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\lambda)$ with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Then the set of vectors Pe_j $(j = 1, \ldots, n)$ spans $\mathcal{E}(\lambda)$ and therefore there exists $X \subseteq V(G)$ such that the vectors Pe_j $(j \in X)$ form a basis for $\mathcal{E}(\lambda)$. Such a set X is called a star set for λ in G. If X is a star set for the eigenvalue λ then $\bar{X} = V(G) \setminus X$ is said a co-star set while $G - X = G\left[\bar{X}\right]$ is called a star complement for λ in G.

The next result, which can be seen in [4], gives other two ways for characterizing star sets.

Theorem 3 [4, Proposition 5.1.1] *Let G be a graph with $\lambda \in \sigma(G)$ as an eigenvalue of multiplicity $k > 0$. The following conditions on a subset X of $V(G)$ are equivalent:*

1. *X is a star set for λ;*
2. *$\mathbb{R}^n = \mathcal{E}_G(\lambda) \oplus \mathcal{V}$, where $\mathcal{V} = \left\langle e_i : i \in \bar{X} \right\rangle$;*
3. *$|X| = k$ and λ is not an eigenvalue of $G - X$.*

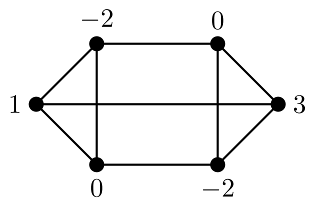

Figure 1: Graph whose vertices of star sets are labelled with the eigenvalues.

In Figure 1, the vertices having the same label form a star set for the eigenvalue that coincides with the label. For instance, -2 is an eigenvalue of multiplicity 2 and if we eliminate the two vertices labelled -2, we obtain a graph that does not have -2 as eigenvalue.

We recall another result, known as the Reconstruction Theorem, that states another characterization of star sets needed in the sequel.

Theorem 4 [4, pp. 140] *Let $X \subset V(G)$ be a set of vertices of graph G, $\bar{X} = V(G) \backslash X$ and assume that G has adjacency matrix $A_G = \begin{bmatrix} A_X & N^T \\ N & C_{\bar{X}} \end{bmatrix}$, where A_X and $C_{\bar{X}}$ are the adjacency matrices of the subgraphs induced by X and $\bar{X}$, respectively. Then X is a star set for λ in G if and only if λ is not an eigenvalue of $C_{\bar{X}}$ and*

$$A_X - \lambda I_X = N^T (C_{\bar{X}} - \lambda I_{\bar{X}})^{-1} N,$$

where I_X and $I_{\bar{X}}$ are respectively the identity matrices of orders $|X|$ and $|\bar{X}|$.

Furthermore, $\mathcal{E}(\lambda)$ is spanned by the vectors

$$\left[\begin{array}{c} x \\ -\left(C_{\bar{X}} - \lambda I\right)^{-1} N x \end{array}\right], \tag{4}$$

where $x \in \mathbb{R}^{|X|}$.

Returning to problem $P(G)$, we now introduce the star solution concept.

Definition 1 *An optimal solution x of $P(G)$ is called a star solution of $P(G)$ if there exists a star set X for $\lambda_{\min}(A_G)$ such that $x_i = 0$, for all $i \in X$. The star set X is said to be associated with the star solution x and vice-versa.*

For example, denoting the Petersen graph depicted in Figure 2 by G, an optimal solution x of $P(G)$ associated to the star set $X = \{2, 4, 8, 10\}$ for $\lambda_{\min}(G) = -2$ is $x_2 = x_4 = x_8 = x_{10} = 0$, $x_3 = x_5 = x_6 = x_7 = 1$ and $x_1 = x_9 = 0$.

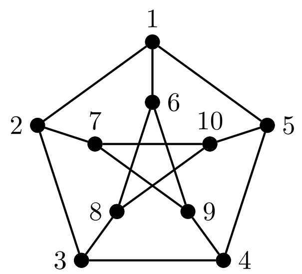

Figure 2: The Petersen graph

The main result of this section follows from the next two theorems. To prove the first one it is worth to recall a lemma of [4]. The second theorem was proved in [3] and hence it will be presented without proof.

Lemma 1 [4, pp. 139] *If the column space of the symmetric matrix $\left[\begin{array}{cc} C & D^T \\ D & E \end{array}\right]$ has the columns of $\left[\begin{array}{c} C \\ D \end{array}\right]$ as a basis, then the columns of C are linearly independent.*

This lemma allows to prove:

Theorem 5 *Let G be a graph with vertices $1, 2, \ldots, n$ and at least one edge. If x is a 0-1 optimal solution of $P(G)$ and $S = \{i \in V(G) : x_i = 1\}$, then there is a star complement for $\lambda_{\min}(G)$ containing the subgraph of G induced by S. Consequently, x is a 0-1 star solution of $P(G)$.*

Proof. Since $P(G)$ has a 0-1 optimal solution x, from Corollary 2.1 we may conclude that the vertex set $S = \{i \in V(G) : x_i = 1\}$ is a maximum stable set and G is a convex-QP graph. Let us assume, without loss of generality, that $S = \{1, \ldots, \alpha(G)\}$. Then, considering the matrix $A_G - \lambda_{\min}(G)I$, it is immediate that the first $\alpha(G)$ columns are linearly independent. Denote by $c_1, \ldots, c_n$ the columns of matrix $A - \lambda_{\min}(G)I$ and let $\{c_j : j \in Y\}$ be the basis of the column space of $A - \lambda_{\min}(G)I$, obtained by deleting each column which is a linear combination of the previous ones. It is immediate that $\{1, \ldots, \alpha(G)\} \subseteq Y$. By Lemma 1, we conclude that the principal submatrix of $A - \lambda_{\min}(G)I$ determined by Y is invertible, hence the subgraph induced by Y does not have $\lambda_{\min}(G)$ as an eigenvalue. Since $|Y| = n - \dim(\mathcal{E}(\lambda_{\min}(G)))$, $\bar{Y} = V(G)\backslash Y$ is a star set for $\lambda_{\min}(G)$ and the subgraph of G induced by Y is a star complement for $\lambda_{\min}(G)$ containing the subgraph induced by S. Finally, since $x_i = 0$, for all $i \in \bar{Y}$, x is a star solution of $P(G)$. ■

Theorem 6 [3] *Let G be a graph such that, for any subset of vertices $U \subseteq V(G)$, $G - U$ has at least one edge and $\upsilon(G - U) = \upsilon(G)$. If $\lambda_{\min}(G) < \lambda_{\min}(G - U)$, then G is a $\mathcal{Q}$-graph.*

We may now present the main result of this section.

Theorem 7 *Let G be a graph with at least one edge. Then, G is a $\mathcal{Q}$-graph if and only if there is a star set X associated to the eigenvalue $\lambda_{\min}(G)$ such that $\upsilon(G - X) = \upsilon(G)$.*

Proof. We will start by proving the sufficient condition. If $G - X$ has no edges, we have that $\alpha(G - X) = \upsilon(G - X) = |V(G - X)|$ and also

$$\upsilon(G) \geq \alpha(G) \geq \alpha(G - X) = \upsilon(G - X) = \upsilon(G).$$

Consequently, $\alpha(G) = \upsilon(G)$. Otherwise, if $E(G-X) \neq \emptyset$, and X is a star set for $\lambda_{\min}(G)$, then $\lambda_{\min}(G)$ is not an eigenvalue for $G - X$ and hence $\lambda_{\min}(G) < \lambda_{\min}(G - X)$. Since we have assumed that $\upsilon(G - X) = \upsilon(G)$, Theorem 6 implies that $G \in \mathcal{Q}$.

Conversely, in order to prove the necessary condition, suppose that $\alpha(G) = \upsilon(G)$. If x is the characteristic vector of a maximum stable set of G, then x is a 0-1 optimal solution for $P(G)$. Theorem 5 assures that x is a star solution for $P(G)$. Calling X to the star set for $\lambda_{\min}(G)$ associated to x, we have that $\alpha(G - X) = \alpha(G)$ and hence

$$\upsilon(G - X) \geq \alpha(G - X) = \alpha(G) = \upsilon(G);$$

as a consequence, taking into account that $\upsilon(G - X) \leq \upsilon(G)$, we have $\upsilon(G - X) = \upsilon(G)$. ■

4 Recognizing $\mathcal{Q}$-graphs: The Open Question

This section is concerned with the recognition problem which consists on being able to know whether or not a given graph belongs to $\mathcal{Q}$. This problem has resisted to be completely solved and is currently an interesting open question. This difficulty has its origin in the recognition of the so called adverse graphs which are defined as follows:

Definition 2 *A graph G without isolated vertices is called an adverse graph if:*

- *$\upsilon(G)$ and $\lambda_{\min}(A_G)$ are integers;*
- *For any vertex $i \in V(G)$, $\upsilon(G - N_G(i)) = \upsilon(G)$;*
- *For any vertex $i \in V(G)$, $\lambda_{\min}(G - N_G(i)) = \lambda_{\min}(G)$.*

Note that if G is an adverse graph the equalities $\upsilon(G-i) = \upsilon(G)$ and $\lambda_{\min}(G-i) = \lambda_{\min}(G)$ are also valid for any vertex $i \in V(G)$. In fact, since G has no isolated vertices, for any vertex $i \in V(G)$, there exists a vertex j such that $i \in N_G(j)$ and then, taking into account the above definition and the inequalities

$$\upsilon(G - N_G(j)) \leq \upsilon(G - i) \leq \upsilon(G),$$

the equality $\upsilon(G-i) = \upsilon(G)$ holds. By similar arguments we may conclude that $\lambda_{\min}(G-i) = \lambda_{\min}(G)$ for any vertex $i \in V(G)$.

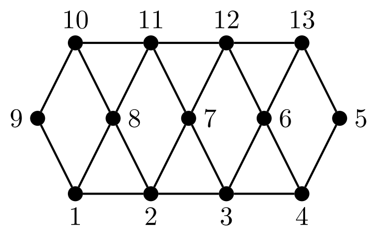

Figure 3: An adverse graph G such that $\lambda_{\min}(G) = -2$ and $\alpha(G) = \upsilon(G) = 5$.

Among others, the Petersen graph (Figure 2) and the graph depicted in Figure 3 are two examples of adverse graphs.

Presently, we do not know how to recognize in polynomial-time whether or not an adverse graph belongs to $\mathcal{Q}$. This open question would be solved by a positive answer to the following conjecture: "All adverse graphs belong to $\mathcal{Q}$". Despite all performed computational tests with adverse graphs support this conjecture, it remains unsettled until now. Meanwhile, the Algorithm 1 recalled below, known as the Recognition Algorithm, recognizes in polynomial-time if a given graph belongs or not to $\mathcal{Q}$ or, if none of these conclusions is possible, identifies an adverse subgraph of that graph. This algorithm is based on Theorem 6 presented in the above section as well as on several results proved in [3], which are recalled next.

Theorem 8 [3] *Let G be a graph with at least one edge.*

(a) *G belongs to $\mathcal{Q}$ if and only if each of its components belongs to $\mathcal{Q}$.*

(b) *If $\exists i \in V(G)$ such that $\upsilon(G) \neq \max\{\upsilon(G-i), \upsilon(G - N_G(i))\}$, then $G \notin \mathcal{Q}$.*

(c) *Consider that $\exists i \in V(G)$ such that $\upsilon(G-i) \neq \upsilon(G - N_G(i))$.*

(c.1) *If $\upsilon(G) = \upsilon(G-i)$ then $G \in \mathcal{Q}$ if and only if $G - i \in \mathcal{Q}$;*

(c.2) *If $\upsilon(G) = \upsilon(G - N_G(i))$ then $G \in \mathcal{Q}$ if and only if $G - N_G(i) \in \mathcal{Q}$.*

In the next section we give some properties of the adverse graphs having in mind to design a simplex recognition algorithm for those graphs.

Algorithm 1 Recognition Algorithm

Require: A graph G with vertex set V and at least one edge.
Ensure: Whether or not $G \in \mathcal{Q}$ or find out an adverse subgraph of G.

1. Let $Iso(G)$ be the set of isolated vertices of G; set $G := G - Iso(G)$
2. **if** $\exists v \in V$ such that $\lambda_{\min}(G) < \lambda_{\min}(G - N_G(v))$ **then**
3.   **If** $\upsilon(G - v) = \upsilon(G - N_G(v))$ **then** $G \in \mathcal{Q} \to$ STOP **endif**
4. **endif**
5. **if** $\exists v \in V$ such that $\upsilon(G - v) \neq \upsilon(G - N_G(v))$ **then**
6.   **if** $\upsilon(G) \notin \{\upsilon(G - v), \upsilon(G - N_G(v))\}$ **then**
7. $G \notin \mathcal{Q} \to$ STOP
8.   **else**
9.   **if** $\upsilon(G) = \upsilon(G - v)$ **then**
10. $G := G - v$
11.   **else**
12. $G := G - N_G(v)$
13.   **endif**
14.   **goto** step 1.
15.   **endif**
16. **else**
17.   **if** $\exists v \in V$ such that $\upsilon(G) \neq \upsilon(G - (v))$ **then**
18. $G \notin \mathcal{Q} \to$ STOP
19.   **else**
20. G is adverse $\to$ STOP
21.   **endif**
22. **endif**

5 Properties of Adverse Graphs

Proposition 1 *If G is an adverse graph, then x^* is an optimal solution of $P(G)$ if and only if $x^* \geq 0$ and it is a critical point of the objective function, i.e., x^* is a nonnegative solution of the system (3).*

Proof. If x^* is nonnegative and a solution of (3), then x^* is a critical point for the objective function of $P(G)$, hence it is an optimal solution of $P(G)$.

Conversely, let us assume that x^* is an optimal solution of $P(G)$. Since $x^* \geq 0$, it remains to see that x^* satisfies system (3). Let i be a vertex of G and $\bar{x}^*$ be an optimal solution of $P(G - N_G(i))$. Thus $\bar{x}^*_i = 1$ and, since G is adverse, $\upsilon(G) = \upsilon(G - N_G(i))$ and $\lambda_{\min}(A_G) = \lambda_{\min}(A_{G-N_G(i)})$; then vector $\tilde{x}^*$ defined as

$$\tilde{x}^*_j = \begin{cases} \bar{x}^*_j & \text{if } j \text{ is a vertex of } G - N_G(i) \\ 0 & \text{if } j \in N_G(i) \end{cases},$$

is an optimal solution of $P(G)$. Therefore, the coordinate of the complementary solution associated to $\tilde{x}^*_i$ is null and by Theorem 1, the same is true for the coordinate i of any optimal solution of $P(G)$. Performing the same reasoning to any other vertex of G, we conclude that the complementary solution associated to any optimal solution of $P(G)$ is null, hence x^* satisfies system (3), as required. ■

The next results are stated for graphs G whose optimal solutions of $P(G)$ are critical points of the objective function. After Proposition 1 they can also be viewed as adverse graph properties.

Theorem 9 *Let G be a graph with n vertices and at least one edge whose optimal solutions of $P(G)$ are critical points of the objective function.*

(a) *If x^* is an optimal solution of $P(G)$, any nonnegative vector $\hat{x}$ is an optimal solution of $P(G)$ if and only if it has the form*

$$\hat{x} = x^* + \sum_{i=1}^{k} \beta_i u_i, \tag{5}$$

where the set $\{u_1, \ldots, u_k\} \subset \mathbb{R}^n$ is a basis for $\mathcal{E}(\lambda_{\min}(G))$ and $\beta_i \in \mathbb{R},\ i = 1, \ldots, k$.

(b) *The following equality holds,*

$$\upsilon(G) = \max\{e^T x : x \geq 0 \text{ and satisfies (3)}\}, \tag{6}$$

and the optimal solutions of this linear programming problem and of $P(G)$ coincide.

Proof. Note first that the necessary condition of (a) follows from Theorem 1. To prove the sufficient condition of (a), suppose that the nonnegative vector $\hat{x}$ is obtained in (5). Since x^* is a critical point of the objective function of $P(G)$, considering x^* in (3) and multiplying on the left this equation by u_i $(i = 1, \ldots, k)$, we obtain

$$u_i^T (A_G + \tau I) x^* = \tau u_i^T e \Leftrightarrow u_i^T e = 0,\ i = 1, \ldots, k, \tag{7}$$

taking into account that $\mathcal{E}(\lambda_{\min}(G)) = \ker(A_G + \tau I)$. Hence the eigenspace $\mathcal{E}(\lambda_{\min}(G))$ is orthogonal to the all-ones vector $e \in \mathbb{R}^n$, and then $e^T \hat{x}^* = e^T x^* = \upsilon(G)$, i.e., $\hat{x}^*$ is an optimal solution of $P(G)$.

To prove (b), suppose that $\hat{x}$ is an optimal solution of (6). Then $\hat{x}$ is a critical point for the objective function of $P(G)$, hence it is an optimal solution of $P(G)$. Conversely, if $\hat{x}$ is an optimal solution of $P(G)$, and $\hat{x}$ has the form (5) by (a), hence it is a feasible solution of (6). As the objective function of (6) is constant over all vectors defined in (5), $\hat{x}$ is also an optimal solution of (6). As in both cases $\upsilon(G) = e^T \hat{x}$, the equality (6) follows. ■

We recall now one more result concerning problem $P(G)$.

Theorem 10 [7] *Let G be a graph with at least one edge. Then $P(G)$ has an optimal solution x such that $x_i = 0$, for some vertex $i \in V(G)$.*

Below, we generalize this result for graphs whose optimal solutions are critical points of the objective function of $P(G)$.

Theorem 11 *Let G be a graph with n vertices and at least one edge. If the optimal solutions of $P(G)$ are critical points of the objective function, then there exists an optimal solution x^* with at least k null coordinates, where $k = \dim \mathcal{E}(\lambda_{\min}(G))$.*
Furthermore, there exists a basis of $\mathcal{E}(\lambda_{\min}(G))$ formed by k vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ such that the submatrix of $U = [u_1 \ \cdots \ u_k]$ indexed by the rows corresponding to the k null components of x^* coincides with the identity matrix of order k.

Proof. Let us assume that the optimal solutions of $P(G)$ are critical points of the objective function and x^* is one of these optimal solutions. Consider a basis $\mathcal{B} = \{u_1, \ldots, u_k\}$ for $\mathcal{E}(\lambda_{\min}(G))$ and apply the following procedure, where u_{ij} will denote the i-th coordinate of u_j, $j = 1, \ldots, k$.

The nullifying components procedure

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1. **set** $j := 1$;
2. **while** $j \leq k$ **do**
3.   **compute** $\frac{x_r^*}{u_{rj}} := \min_{i=1,\ldots,n} \left\{ \frac{x_i^*}{u_{ij}} : u_{ij} > 0 \right\}$;
4.   **set** $\tilde{x}^* := x^* - \frac{x_r^*}{u_{rj}} u_j$;
5.   **while** $1 \leq q \leq k$ and $q \neq j$ **do**
6. $u_q := u_q - \frac{u_{rq}}{u_{rj}} u_j$;
7.   **end while**
8.   **set** $u_j := \frac{1}{u_{rj}} u_j$;
9.   **set** $x^* := \tilde{x}^*$ and $j := j + 1$;
10. **end while**

Note that the existence of a positive coordinate of u_j in step 3 is guaranteed (otherwise, the equalities (7) are not fulfilled). Then, it is immediate that, after the application of this procedure, x^* and the eigenvectors $u_1, \ldots, u_k$ are in the required conditions. ■

The next result shows that the above procedure (in the proof of Theorem 11) outputs a star solution.

Theorem 12 *Let G be a graph with n vertices and at least one edge. Assume also that the optimal value of $P(G)$ is attained at the critical points of the objective function. Then, the optimal solution of $P(G)$ obtained after the application of the nullifying components procedure (used in the proof of Theorem 11) is a star solution of $P(G)$.*

Proof. Suppose that $\lambda_{\min}(G)$ has multiplicity k and consider the k-null coordinates optimal solution x^* of $P(G)$ and the corresponding $(k-1)$-null coordinates basis $\mathcal{B}$ of $\mathcal{E}(\lambda_{\min}(G))$ determined by the nullifying components procedure of Theorem 11. Let $X \subset V(G)$ be the vertex subset of G, corresponding to the k coordinates of x^* nullified by this procedure. It suffices to prove that X is a star set for $\lambda_{\min}(G)$ in G.

Without loss of generality, let us assume that the first k coordinates of x^* correspond to the vertices in X. Thus, the matrix U, whose columns are the eigenvectors in $\mathcal{B}$, takes the form

$$U = \begin{bmatrix} I_k \\ M \end{bmatrix},$$

where I_k is the identity matrix of order k and M is a $(n-k) \times k$-matrix.

According to Theorem 3-condition 2, X is a star set associated to $\lambda_{\min}(G)$ if and only if $\mathbb{R}^n = \mathcal{E}(\lambda) \oplus \mathcal{V}$, with $\mathcal{V} = \langle e_j : j \in V(G)\backslash X \rangle$. Juxtaposing on the right of the matrix $U = \begin{bmatrix} I_k \\ M \end{bmatrix}$ the unitary vectors e_j spanning $\mathcal{V}$, the full rank square matrix $\begin{bmatrix} I_k & O \\ M & I_{n-k} \end{bmatrix}$ is obtained. As its columns form a basis of $\mathbb{R}^n$, we have $\mathbb{R}^n = \mathcal{E}(\lambda_{\min}(G)) + \mathcal{V}$. Furthermore, since $\dim \mathcal{E}(\lambda_{\min}(G)) = k$ and $\dim \mathcal{V} = n - k$, then $\mathcal{E}(\lambda_{\min}(G)) \cap \mathcal{V} = \{0\}$. ■

From this theorem and comparing the eigenvectors in Theorem 4 with the ones stated in Theorem 11 we have the following result.

Theorem 13 *Let G be a graph with n vertices and at least one edge. Assume that the optimal value of $P(G)$ is attained at the critical points of the objective function and that $\lambda_{\min}(G)$ has multiplicity k. If $X \subset V(G)$ is the star set for $\lambda_{\min}(G)$ associated to the star solution of $P(G)$ stated by Theorem 11, then the $(k-1)$-null coordinates eigenvectors u_j determined by the nullifying components procedure coincide with those obtained from (4) replacing λ by $\lambda_{\min}(G)$ and x by $\tilde{e}_j$, i.e.,*

$$u_j = \begin{bmatrix} \tilde{e}_j \\ -(C_{\bar{X}} + \tau I_{\bar{X}})^{-1} N \ \tilde{e}_j \end{bmatrix}, \quad j = 1, \ldots, k,$$

where $\tilde{e}_j$ denotes the j-th vector of the canonical basis of $\mathbb{R}^k$.

Proof. Note first that the vectors u_j, $j = 1, \ldots, k$, produced by the nullifying components procedure can be written in the form $\begin{bmatrix} \tilde{e}_j \\ \tilde{u}_j \end{bmatrix}$, where $\tilde{u}_j \in \mathbb{R}^{n-k}$. Let us suppose that, for some $j \in \{1, \ldots, k\}$,

$$\begin{bmatrix} \tilde{e}_j \\ \tilde{u}_j \end{bmatrix} \neq \begin{bmatrix} \tilde{e}_j \\ -(C_{\bar{X}} + \tau I_{n-k})^{-1} N \tilde{e}_j \end{bmatrix}. \tag{8}$$

Since the eigenvalue equation $\begin{bmatrix} A_X & N^T \\ N & C_{\bar{X}} \end{bmatrix} u = \lambda_{\min}(G)\ u$ implies

$$N\tilde{e}_j + C_{\bar{X}}\ \tilde{u}_j = \lambda_{\min}(G)\tilde{u}_j, \tag{9}$$

$$N\tilde{e}_j - C_{\bar{X}}(C_{\bar{X}} + \tau I_{\bar{X}})^{-1} N\ \tilde{e}_j = -\lambda_{\min}(G)(C_{\bar{X}} + \tau I_{\bar{X}})^{-1} N\tilde{e}_j, \tag{10}$$

subtracting (9) to (10) it follows

$$C_{\bar{X}}\ \left[-(C_{\bar{X}} + \tau I_{n-k})^{-1} N\ \tilde{e}_j - \tilde{u}_j\right] = \lambda_{\min}(G) \left[-(C_{\bar{X}} + \tau I_{n-k})^{-1} N\ \tilde{e}_j - \tilde{u}_j\right].$$

Therefore, taking into account the inequality (8), $-(C_{\bar{X}} + \tau I_{n-k})^{-1} N\ \tilde{e}_j - \tilde{u}_j$ is an eigenvector of $C_{\bar{X}}$ associated to the eigenvalue $\lambda_{\min}(G)$, a contradiction. ■

If G is an adverse graph, from Proposition 1, the optimal solutions of $P(G)$ are critical points for the objective function, that is, they are the nonnegative solutions of the system (3). If X is a star set for $\lambda_{\min}(G)$, using the notation introduced in Theorem 4, we can write that system in the form

$$\begin{bmatrix} A_X + \tau I_X & N^T \\ N & C_{\bar{X}} + \tau I_{\bar{X}} \end{bmatrix} x = \tau \begin{bmatrix} e_X \\ e_{\bar{X}} \end{bmatrix}, \tag{11}$$

where $e_{\bar{X}}$ and e_X are the all-ones vectors of lengths $|\bar{X}|$ and $|X|$, respectively.

The kernel of the coefficient matrix of system (11) is the subspace $\mathcal{E}(\lambda_{\min}(G))$. Therefore, since $\lambda_{\min}(G)$ is not an eigenvalue of $C_{\bar{X}}$, the submatrix $C_{\bar{X}} + \tau I_{\bar{X}}$ is nonsingular and the row space of the coefficient matrix of system (11) is spanned by the last $|\bar{X}|$ rows. Thus if system (11) is consistent, it is equivalent to the following:

$$\left[\begin{array}{cc} N & C_{\bar{X}} + \tau I_{\bar{X}} \end{array} \right] x = \tau e_{\bar{X}}. \tag{12}$$

Theorem 14 *Let G be a graph with n vertices and at least one edge. Assume that the optimal value of $P(G)$ is attained at the critical points of the objective function. If $X \subset V(G)$ is a star set for $\lambda_{\min}(G)$ and the critical points of the objective function of $P(G)$ are the nonnegative solutions of system (11), then x is a star solution of $P(G)$ if and only if x is a basic feasible solution of system (12).*

Proof. If x is a star solution of $P(G)$ associated to the star set $X \subset V(G)$, then $x = \left[\begin{array}{c} x_N \\ x_B \end{array} \right]$, with $x_N = 0$, where x_N denotes the subvector of x with indices in X and thus the columns of the submatrix N are indexed by the vertices in X, and x_B is the subvector of x with indices in $V(G) \setminus X$ which is also the index set of the columns of the matrix $B = C_{\bar{X}} + \tau I_{\bar{X}}$. Now, multiplying both sides of (12) by $(C_{\bar{X}} + \tau I_{\bar{X}})^{-1}$ we obtain

$$\left[\begin{array}{cc} (C_{\bar{X}} + \tau I_{\bar{X}})^{-1} N & I_{\bar{X}} \end{array} \right] \left[\begin{array}{c} x_N \\ x_B \end{array} \right] = \tau \left(C_{\bar{X}} + \tau I_{\bar{X}} \right)^{-1} e_{\bar{X}}.$$

Since $x_N = 0$, then it is immediate that $\left[\begin{array}{c} x_N \\ x_B \end{array} \right] = \left[\begin{array}{c} 0 \\ \tau \left(C_{\bar{X}} + \tau I_m \right)^{-1} e_{\bar{X}} \end{array} \right]$ is a basic feasible solution of the system (12) corresponding to the reduced simplex tableau:

	x_N	
x_B	$\left(C_{\bar{X}} + \tau I_m\right)^{-1} N$	$\tau \left(C_{\bar{X}} + \tau I_m \right)^{-1} e_{\bar{X}}$

If x' is a star solution of $P(G)$, associated to a star set X' of $\lambda_{\min}(G) = -\tau$ such that $X' \neq X$, then using the same arguments as above $x' = \left[\begin{array}{c} x_{N'} \\ x_{B'} \end{array} \right]$ is a basic feasible solution of the system

$$\left[\begin{array}{cc} N' & C_{\bar{X}'} + \tau I_{\bar{X}'} \end{array} \right] x = \tau e_{\bar{X}'},$$

such that $x_{N'} = 0$ and $x_{B'} = \tau \left(C_{\bar{X}'} + \tau I_{\bar{X}'} \right)^{-1} e_{\bar{X}'}$.
It is immediate that x' is also a solution of the system (12). Furthermore, since the rows of the coefficient matrix of the system (12) span the row space of the coefficient matrix of the system (11), there is a $|\bar{X}'| \times |\bar{X}'|$ square matrix L and a permutation matrix P of order n such that

$$\left[\begin{array}{cc} N' & C_{\bar{X}'} + \tau I_{\bar{X}'} \end{array} \right] = L \left[\begin{array}{cc} N & C_{\bar{X}} + \tau I_{\bar{X}} \end{array} \right] P. \tag{13}$$

The right hand side of the equation (13) is a linear combination of the rows of the matrix $\left[\begin{array}{cc} N & C_{\bar{X}} + \tau I_{\bar{X}} \end{array} \right]$ and a permutation of its columns. Furthermore, the restriction of this linear combination to the columns of the submatrix defined by indices in $\bar{X}'$ is equal to the invertible matrix $C_{\bar{X}'} + \tau I_{\bar{X}'}$. Thus, this submatrix is also invertible and consequently x' is a basic feasible solution for the system (12).

Conversely, let $x = \begin{bmatrix} x_N \\ x_B \end{bmatrix}$ be a basic feasible solution of the system (12) and let us assume that the corresponding reduced simplex tableau has the form

	x_N	
x_B	$B^{-1}N$	$\tau B^{-1} e_{\bar{X}}$

From this simplex tableau, we obtain the following linear independent vectors which spans the kernel of the coefficient matrix of the system (12),

$$\begin{bmatrix} e_j \\ -B^{-1}Ne_j \end{bmatrix}, \text{ for } j \in J_N,$$

where $B = C_{\bar{X}} + \tau I_{\bar{X}}$ and J_N denotes the set of nonbasic indices. It is clear that all these vectors belong also to the kernel of $A_G + \tau I_n$ and then they span the eigenspace $\mathcal{E}(\lambda_{\min}(G))$ (see (4) in Theorem 4). Therefore, supposing that the cardinality of the set of nonbasic indices J_N is equal to k, juxtaposing on the right of the matrix $U = \begin{bmatrix} I_k \\ -B^{-1}N \end{bmatrix}$, the unitary vectors e_j, with $j \in V(G) \setminus J_N$, a full rank square matrix

$$\begin{bmatrix} I_k & O \\ -B^{-1}N & I_{n-k} \end{bmatrix}$$

of order n is obtained. Therefore, $\mathbb{R}^n = \mathcal{E}_G(\lambda) \oplus \mathcal{V}$, with $\mathcal{V} = \langle e_j : j \in V(G) \setminus J_N \rangle$. Finally, applying Theorem 3-condition 2, it follows that J_N is a star set associated to $\lambda_{\min}(G)$ and then x is a star solution for $P(G)$. ■

6 A Simplex Algorithm for Recognizing Adverse Graphs

From the proof of Theorem 7, we may conclude that when $G \in \mathcal{Q}$ every characteristic vector of a maximum stable set of G is a star solution of $P(G)$. If the graph G is adverse, we conclude by (6) in Theorem 9-(b) that

$$\upsilon(G) = \max\{e^T x : x \geq 0 \text{ and satisfies (12)}\}.$$

Therefore, by Theorem 14, we may apply the fractional dual algorithm for ILP with Gomory cuts (see for example [11]) for deciding if there exists (or not) a 0-1 star solution of $P(G)$. This algorithm can be started with the basic feasible solution obtained by the nullifying components procedure of Theorem 11. Thus the following can be asserted:

Theorem 15 *If G is an adverse $\mathcal{Q}$-graph, the fractional dual algorithm for ILP with Gomory cuts yields a 0-1 star solution of $P(G)$ in a finite number of iterations.*

We applied the fractional dual algorithm initialized with the basic feasible solution determined by the nullifying components procedure to adverse graphs whose names are shown in the first column of Table 1. The subsequent columns present the order n, the least eigenvalue $\lambda_{\min}(G)$ and its multiplicity k. In the next columns, for each graph G, the values of $\upsilon(G)$ are recorded as well as the conclusion obtained by the algorithm relatively to the question "$G \in \mathcal{Q}$?" , the number of cutting planes used (ncp) and the times (in seconds) spent by the overall algorithm.

It should also be noted that the first eight graphs presented in Table 1 are DIMACS clique benchmark graphs or complements of these graphs (preceded by "c-"). LD13 is the graph of Figure 3, LD16=$\mathcal{L}$(LD13-{4,5,13}), LD38=$\mathcal{L}$(LD16), LD152=$\mathcal{L}$(LD38) and LD1112=$\mathcal{L}$(LD152) (here, $\mathcal{L}$ stands for linegraph).

The tests were carried out on a computer using an Intel(R) Core(TM) i7-2630QM / 2.0GHz processor with 6.0 Gb RAM and Windows 7 Home Premium as the operating system. The interactive matrix language MATLAB (version 7.6) was used for implementing the fractional dual algorithm.

Adverse Graph G	n	$\lambda_{\min}(G)$	k	$\upsilon(G)$	$G \in Q$?	ncp	time (sec.)
`johnson8-2-4`	28	-5	7	7	Yes	1	0.02
`johnson16-2-4`	120	-13	15	15	Yes	0	0.02
`c-johnson8-2-4`	28	-2	20	4	Yes	0	0.02
`c-johnson8-4-4`	70	-4	14	14	Yes	0	0.00
`c-johnson16-2-4`	120	-2	104	8	Yes	0	0.03
`c-johnson32-2-4`	496	-2	464	16	Yes	0	0.53
`c-hamming6-2`	64	-6	1	32	Yes	0	0.05
`c-hamming8-2`	256	-8	1	128	Yes	0	0.05
`Petersen`	10	-2	4	4	Yes	0	0.02
`LD13`	13	-2	4	5	Yes	0	0.00
`LD16`	16	-2	6	5	Yes	1	0.03
`LD38`	38	-2	22	8	Yes	0	0.00
`LD152`	152	-2	114	19	Yes	1	0.11
`LD1112`	1112	-2	960	76	Yes	6	35.6

Table 1: Computational results

From the results presented in Table 1 some conclusions are in order. For almost all tested adverse graphs, the basic feasible solution obtained by the nullifying components procedure of Theorem 11 to initialize the fractional dual algorithm is integer. Only in four cases it was necessary to proceed with the Gomory cuts in order to obtain an integer solution. On the other hand, it should be noted that the times spent by our implementation of fractional dual algorithm are very reduced.

7 Conclusions

In section 4 it was recalled an efficient procedure that recognizes convex-QP graphs (i.e., $\mathcal{Q}$-graphs), except when the graph is adverse or contains an adverse subgraph. We show in this paper how a simplex algorithm can be used for recognizing adverse $\mathcal{Q}$-graphs. The designing of the algorithm follows from a new characterization of $\mathcal{Q}$-graphs based on star sets given in section 3 as well as from a set of properties of adverse graphs proved in section 5.

With the simplex like algorithm presented in this paper we now have a finite procedure for recognizing $\mathcal{Q}$-graphs, even though it is not a polynomial algorithm. As a matter of fact, it remains as an open question to know how to recognize $\mathcal{Q}$-graphs in polynomial-time. As referred in section 4, this open question would be solved by a positive answer to the following conjecture: "All adverse graphs belong to $\mathcal{Q}$". It should be noted that this conjecture has been supported by several performed computational tests, including those whose results are presented in Table 1. This conjecture remains as an unsolved open question which requires additional work to be answered.

Another issue that can be discussed related with the graphs of DIMACS collection used in our tests is the following: one can ask if there are many adverse graphs in that collection or even if some particular family just includes adverse graphs. Relatively to the first question we can say that most DIMACS graphs used in our experiments over time are not $\mathcal{Q}$-graphs. This is well illustrated by the computational results for graphs of DIMACS collection presented in [9]. On the other hand, we found out that some (but not all) hamming graphs are adverse and that all johnson graphs we have tested are also adverse. However we don't know if this is true for all johnson graphs and it may be another good research question for the near future.

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