Chapter 5
On a Class of Integral Equations Involving Kernels of Cosine and Sine Type*

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5.1 Introduction

Integral equations involving kernels of cosine and sine type play a significant role in the modelling of different kinds of applied problems. This is the case e.g. when using information-bearing entities like signals which are usually represented by functions of one or more independent variables [8]. The possibilities of application are very diverse, but it is not our intention to present here their description. For this we just refer the interested reader e.g. to the monograph [8]. Anyway, we would like to observe that this occurs also in the larger class of the so-called *linear canonical transforms*, where several kernels and integral transforms can be agglutinated into a single integral operator, and so into a consequent integral equation, opening the possibility of defining very flexible classes of integral equations which are quite useful in applications.

* Accepted author's manuscript (AAM) published in [Castro L.P., Guerra R.C., Tuan N.M. (2017) On a Class of Integral Equations Involving Kernels of Cosine and Sine Type. In: Constanda C., Dalla Riva M., Lamberti P., Musolino P. (eds) Integral Methods in Science and Engineering, Volume 1. Birkhäuser, Cham]. The final publication is available at Springer via `https://doi.org/10.1007/978-3-319-59384-5_5`

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This paper is divided into three sections. After this introduction, the second section is concerned with the proof of a polynomial identity, for the cosine and sine type integral operator T, defined below by (5.2), and with the solvability of the integral equations of the type (5.1), generated by T. A key step in this section is the construction of certain projection operators, by which our initial integral equation (5.1) may be equivalently transformed into a new integral equation (see (5.8)), governed by special projections. The third section is concentrated on the analysis of properties of the operator T. In particular, we will characterize its spectrum, its invertibility property, derive a formula for its inverse, and obtain a corresponding Parseval-type identity. The 3-order involution property of the operator T is also presented, which is remarkably different from that one of some well-known integral operators such as the Fourier, Cauchy, Hankel and Hilbert integral operators. Moreover, a new convolution and a consequent factorization identity are obtained – allowing, therefore, further studies for associated new convolution type integral equations, as well as eventual new applications; cf. [1, 3, 4, 6] and the references cited there.

5.2 Integral equations generated by an integral operator with cosine and sine kernels

Within the framework of $L^2(\mathbb{R}^n)$, we will consider integral equations of the type

$$\alpha\varphi + \beta T\varphi + \gamma T^2\varphi = g, \tag{5.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, with $|\alpha| + |\beta| + |\gamma| \neq 0$, and $g \in L^2(\mathbb{R}^n)$ are the given data, and the operator T is defined, in $L^2(\mathbb{R}^n)$, by

$$T := aI + bT_s + cT_c^2, \quad a, b, c \in \mathbb{C}, \quad b \neq \pm c, \quad bc \neq 0, \tag{5.2}$$

with $(T_s f)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sin(xy) f(y) dy$ and $(T_c f)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \cos(xy) f(y) dy$.

We recall the concept of algebraic operators – which will be an important tool in what follows. Let X be a linear space over the complex field $\mathbb{C}$, and let $L(X)$ be the set of all linear operators with domain and range in X.

Definition 1 (see [9, 10]). An operator $K \in L(X)$ is said to be *algebraic* if there exists a normed (non-zero) polynomial $P(t) = t^m + \alpha_1 t^{m-1} + \cdots + \alpha_{m-1} t + \alpha_m$, $\alpha_j \in \mathbb{C}$, $j = 1, 2, \ldots, m$ such that $P(K) = 0$ on X.

We say that an algebraic operator $K \in L(X)$ is of order m if there does not exist a normed polynomial $Q(t)$ of degree $k < m$ such that $Q(K) = 0$ on X. In this case, $P(t)$ is called the *characteristic polynomial* of K, and the roots of this polynomial are called the *characteristic roots* of K. In the sequel, the characteristic polynomial of an algebraic operator K will be denoted by $P_K(t)$. Algebraic operators with a characteristic polynomial $t^m - 1$ or $t^m + 1$ $(m \geq 2)$ are called *involutions* or *anti-involutions of order m*, respectively. An involution (or anti-involution) of order 2 is called, in brief, *involution* (or *anti-involution*). In appropriate spaces, most of the important integral operators are algebraic operators. For instance, the Hankel operator J, the Cauchy singular integral operator S on a closed curve, and the Hartley operator $\mathscr{H}$ are involutions involutions of order 2, i.e., $P_J(t) = P_S(t) = P_{\mathscr{H}}(t) = t^2 - 1$ and the Fourier operator F is an involution of order 4, with $P_F(t) = t^4 - 1$. On another hand, the Hilbert operator H is an anti-involution, as $P_H(t) = t^2 + 1$ (see [2, 5, 7]). Algebraic operators possess some properties that are very useful for solving equations somehow characterized by these operators. Several kinds of integral, ordinary and partial differential equations with transformed arguments can be identified in such a class of operators (see [9, 10]).

Lemma 1 is useful for proving Theorem 1.

Lemma 1 ([12, Theorem 12]). *The formula* $\lim_{\lambda\to\infty} \frac{1}{\pi}\int_{\mathbb{R}} f(t)[\sin\lambda(x-t)]/(x-t)dt = (f(x+0)+f(x-0))/2$ *holds if* $f(x)/(1+|x|)$ *belongs to* $L^1(\mathbb{R})$.

We start with a theorem on the characteristic polynomial of the operator T.

Theorem 1. *The operator T (presented in (5.2)) is an algebraic operator, whose characteristic polynomial is given by*

$$P_T(t) := t^3 - (3a+c)t^2 + (3a^2 - b^2 + 2ac)t - (a^2 - b^2)(a+c). \tag{5.3}$$

Thus, T fulfills the operator polynomial identity:

$$T^3 - (3a+c)T^2 + (3a^2 - b^2 + 2ac)T - (a^2 - b^2)(a+c)I = 0. \tag{5.4}$$

Proof. Firstly, we shall prove that $T_c^2 + T_s^2 = I$, $T_c^3 = T_c$ and $T_s^3 = T_s$ (see [9]). Indeed, we have that $(T_c f)(x) = (T_c f)(-x)$, $(T_s f)(x) = -(T_s f)(-x)$, and $T_c T_s = T_s T_c = 0$. For $\lambda > 0$, let $B(0,\lambda) := \{y = (y_1,\dots,y_n) \in \mathbb{R}^n : |y_k| \le \lambda, \forall k = 1,\dots,n\}$ be the n-dimensional box in $\mathbb{R}^n$.

In the sequel, we will denote by $\mathscr{S}$ the Schwartz space. Let $f \in \mathscr{S}$ be given. We can prove inductively on n that

$$\int_{B(0,\lambda)} \cos(y(x-t))dy = \frac{2^n \sin(\lambda(x_1-t_1))\cdots\sin(\lambda(x_n-t_n))}{(x_1-t_1)\cdots(x_n-t_n)}. \tag{5.5}$$

Using (5.5) and Lemma 1, we have

$$\begin{aligned}
(T_c^2 f)(x)dx &= \lim_{\lambda\to\infty} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(t)\int_{B(0,\lambda)} \cos(xy)\cos(yt)\,dy\,dt \\
&= \lim_{\lambda\to\infty} \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} f(t)\int_{B(0,\lambda)} [\cos(y(x-t)) + \cos(y(x+t))]\,dy\,dt \\
&= \frac{1}{2(2\pi)^n}\lim_{\lambda\to\infty}\int_{\mathbb{R}^n} f(t)\left[\frac{2^n \sin(\lambda(x_1-t_1))\cdots\sin(\lambda(x_n-t_n))}{(x_1-t_1)\cdots(x_n-t_n)}\right. \\
&\quad \left. + \frac{2^n \sin(\lambda(x_1+t_1))\cdots\sin(\lambda(x_n+t_n))}{(x_1+t_1)\cdots(x_n+t_n)}\right] dt = \frac{f(x)+f(-x)}{2}.
\end{aligned}$$

Hence, $(T_c^2 f)(x) = (f(x) + f(-x))/2$. By the same way, we are able to prove that $(T_s^2 f)(x) = (f(x) - f(-x))/2$. Therefore, $(T_c^2 f)(x) + (T_s^2 f)(x) = f(x)$, which proves the first identity.

For the second one, we have $(T_c^3 f)(x) = T_c(T_c^2 f)(x) = ((T_c f)(x) + (T_c f)(-x))/2 = (T_c f)(x)$. So, $T_c^3 = T_c$. Similarly, we have that $T_s^3 = T_s$. The identities are proved for any $f \in \mathscr{S}$. Note that the space $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, and the operators T_c, T_s can be extended into the Hilbert space $L^2(\mathbb{R}^n)$. So, we will consider that the operators T_c and T_s are defined in this space and the above identities also hold true for all $f \in L^2(\mathbb{R}^n)$. Thus, T_c, Ts are algebraic operators in $L^2(\mathbb{R}^n)$ with characteristic polynomials: $P_{T_c}(t) = P_{T_s}(t) = t^3 - t$ (see also [9, 10]).

We now define three projections associated with T_c,

$$Q_1 = I - T_c^2, \quad Q_2 = (T_c^2 - T_c)/2 \quad \text{and} \quad Q_3 = (T_c^2 + T_c)/2, \tag{5.6}$$

which satisfy the identities $Q_j Q_k = \delta_{jk} Q_k$, for $j,k = 1,2,3$, $Q_1 + Q_2 + Q_3 = I$, $T_c = -Q_2 + Q_3$, and three projections corresponding to T_s,

$$R_1 = I - T_s^2, \quad R_2 = (T_s^2 - T_s)/2, \quad \text{and} \quad R_3 = (T_s^2 + T_s)/2, \tag{5.7}$$

which satisfy $R_j R_k = \delta_{jk} R_k$, for $j,k = 1,2,3$, $R_1 + R_2 + R_3 = I$, $T_s = -R_2 + R_3$, where δ_{jk} is denoting the Kronecker delta.

Therefore, we are able to rewrite T in terms of orthogonal projection operators:

$$\begin{aligned} T &= a(Q_2 + Q_3 + R_2 + R_3) + b(-R_2 + R_3) + c(Q_2 + Q_3) \\ &= [0 \cdot Q_1 + (a+c)Q_2 + (a+c)Q_3] + [0 \cdot R_1 + (a-b)R_2 + (a+b)R_3] \\ &=: [0; a+c; a+c; 0; a-b; a+b]. \end{aligned}$$

By the above-mentioned identities and some computations, we have that

$$\begin{aligned} T^3 - (3a+c)T^2 &+ (3a^2 - b^2 + 2ac)T - (a^2 - b^2)(a+c)I \\ &= \left[0; (a+c)^3; (a+c)^3; 0; (a-b)^3; (a+b)^3\right] \\ &\quad -(3a+c)\left[0; (a+c)^2; (a+c)^2; 0; (a-b)^2; (a+b)^2\right] \\ &\quad + \left(3a^2 - b^2 + 2ac\right)\left[0; (a+c); (a+c); 0; (a-b); (a+b)\right] \\ &\quad -(a^2 - b^2)(a+c)\,[0; 1; 1; 0; 1; 1] \\ &= [0; 0; 0; 0; 0; 0] = 0. \end{aligned}$$

It remains to be proven that there does not exist any polynomial Q with $\deg(Q) < 3$, and $Q(T) = 0$. For that, suppose that there is a such polynomial $Q(t) = t^2 + pt + q$ such that $Q(T) = 0$. This is equivalent to

$$\begin{cases} (a+c)^2 + p(a+c) + q = 0 \\ (a-b)^2 + p(a-b) + q = 0 \\ (a+b)^2 + p(a+b) + q = 0, \end{cases}$$

whose solutions are $b = c = 0$ or $b = 0$ and $c \neq 0$ or $b = \pm c$ and $b \neq 0$. Any solution is not under the conditions $b \neq \pm c$ and $bc \neq 0$.

The polynomial (5.3) has the single roots $t_1 := a - b$, $t_2 := a + b$ and $t_3 := a + c$. Having this in mind, we are able to built projections, induced by T, in the sense of the Lagrange interpolation formula. Namely:

$$\begin{aligned}
P_1 &:= \frac{(T - t_2 I)(T - t_3 I)}{(t_1 - t_2)(t_1 - t_3)} = \frac{T^2 - (t_2 + t_3)T + t_2 t_3}{(t_1 - t_2)(t_1 - t_3)};\\
P_2 &:= \frac{(T - t_1 I)(T - t_3 I)}{(t_2 - t_1)(t_2 - t_3)} = \frac{T^2 - (t_1 + t_3)T + t_1 t_3}{(t_2 - t_1)(t_2 - t_3)};\\
P_3 &:= \frac{(T - t_1 I)(T - t_2 I)}{(t_3 - t_1)(t_3 - t_2)} = \frac{T^2 - (t_1 + t_2)T + t_1 t_2}{(t_3 - t_1)(t_3 - t_2)}.
\end{aligned}$$

Then, we have $P_j P_k = \delta_{jk} P_k$, and $T^\ell = t_1^\ell P_1 + t_2^\ell P_2 + t_3^\ell P_3$, for any $j, k = 1, 2, 3$, and $\ell = 0, 1, 2$.

The construction of the just presented projections has the profit of allowing to rewrite (5.1) in the following equivalent way:

$$m_1 P_1 \varphi + m_2 P_2 \varphi + m_3 P_3 \varphi = g, \quad \text{with } m_j = \alpha + \beta t_j + \gamma t_j^2, \ j = 1, 2, 3. \tag{5.8}$$

Let $\varphi_k(x)$ denote the multi-dimensional Hermite functions (see [11]). The Hermite functions are essential in several applications, and are also somehow associated with our operator T. Namely, we have (see [13, 14])

$$(T_c \varphi_k)(x) = \begin{cases} \varphi_k(x), & \text{if } |k| \equiv 0 \pmod 4\\ 0, & \text{if } |k| \equiv 1, 3 \pmod 4\\ -\varphi_k(x), & \text{if } |k| \equiv 2 \pmod 4 \end{cases} \tag{5.9}$$

and

$$(T_s \varphi_k)(x) = \begin{cases} 0, & \text{if } |k| \equiv 0, 2 \pmod 4\\ \varphi_k(x), & \text{if } |k| \equiv 1 \pmod 4\\ -\varphi_k(x), & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{5.10}$$

By (5.9)–(5.10), we obtain

$$(T \varphi_k)(x) = \begin{cases} (a + c)\, \varphi_k(x), & \text{if } \ |k| \equiv 0, 2 \pmod 4\\ (a + b)\, \varphi_k(x), & \text{if } \ |k| \equiv 1 \pmod 4\\ (a - b)\, \varphi_k(x), & \text{if } \ |k| \equiv 3 \pmod 4. \end{cases} \tag{5.11}$$

Therefore, the Hermite functions are eigenfunctions of T with the eigenvalues $a \pm b, a + c$.

Theorem 2. (i) *The integral equation* (5.8) *(or* (5.1)*) has a unique solution if and only if* $m_1 m_2 m_3 \neq 0$.

(ii) *If* $m_1m_2m_3 \neq 0$*, then the unique solution of* (5.8) *is given by*

$$\varphi = m_1^{-1}P_1g + m_2^{-1}P_2g + m_3^{-1}P_3g. \tag{5.12}$$

(iii) *If* $m_j = 0$*, for some* $j = 1,2,3$*, then the equation* (5.8) *has solution if and only if* $P_jg = 0$.

(iv) *If* $P_jg = 0$*, for some* $j = 1,2,3$*, then the equation* (5.8) *has an infinite number of solutions given by*

$$\varphi = \sum_{\substack{j\leq 3\\ m_j\neq 0}} m_j^{-1}P_jg + z, \quad \textit{where} \quad z \in \ker(\sum_{\substack{j\leq 3\\ m_j\neq 0}} P_j). \tag{5.13}$$

Proof. Suppose that the equation (5.1) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (5.8), we obtain a system of three equations $m_jP_j\varphi = P_jg$, $j = 1,2,3$. In this way, if $m_1\, m_2\, m_3 \neq 0$, then we have the following equivalent system of equations

$$\begin{cases} P_1\varphi = m_1^{-1}P_1g \\ P_2\varphi = m_2^{-1}P_2g \\ P_3\varphi = m_3^{-1}P_3g. \end{cases}$$

Using the identity $P_1 + P_2 + P_3 = I$, we obtain (5.12). Conversely, we can directly verify, by substitution, that φ given by (5.12) fulfills (5.8).

If $m_1\, m_2\, m_3 = 0$, then $m_j = 0$, for some $j \in \{1,2,3\}$. Therefore, it follows that $P_jg = 0$. Then, we have

$$\sum_{\substack{j\leq 3\\ m_j\neq 0}} P_j\varphi = \sum_{\substack{j\leq 3\\ m_j\neq 0}} m_j^{-1}P_jg.$$

Using the fact that $P_jP_k = \delta_{jk}P_k$, we get $(\sum_{\substack{j\leq 3\\ m_j\neq 0}} P_j)\varphi = (\sum_{\substack{j\leq 3\\ m_j\neq 0}} P_j)[\sum_{\substack{j\leq 3\\ m_j\neq 0}} m_j^{-1}P_jg]$ or, equivalently, $(\sum_{\substack{j\leq 3\\ m_j\neq 0}} P_j)[\varphi - \sum_{\substack{j\leq 3\\ m_j\neq 0}} m_j^{-1}P_jg] = 0$. By this, we obtain the solution (5.13).

Conversely, we can directly verify that all the elements φ, with the form of (5.13), fulfill (5.8). As the Hermite functions are eigenfunctions of T (cf. (5.9)–(5.10)), we conclude that the cardinality of all elements φ in (5.13) is infinite.

5.3 Operator properties

In this section, we will present some properties of the operator T, defined by (5.2). Namely, we will analyse its invertibility, Parseval-type identity, point spectrum, and the 3-order involution property of T.

5.3.1 Invertibility and spectrum

In this subsection, we determine the spectrum and characterize the invertibility of T. Moreover, we will explicitly determine the inverse of T (when it exists).

Proposition 1. *The spectrum of the operator T is given by*

$$\sigma(T) = \{a-b, a+b, a+c\}.$$

Proof. By using (5.11), we deduce that $\{a-b, a+b, a+c\} \subset \sigma(T)$. On the other side, for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} &t^3 - (3a+c)t^2 + \left(3a^2 - b^2 + 2ac\right)t - \left(a^2 - b^2\right)(a+c) \\ &\quad = (t-\lambda)\left[t^2 + (\lambda - 3a - c)t + \left(\lambda^2 - (3a+c)\lambda + 3a^2 - b^2 + 2ac\right)\right] + P_T(\lambda), \end{aligned}$$

cf. (5.3).

Suppose that $\lambda \notin \{a-b, a+b, a+c\}$. This implies that

$$P_T(\lambda) = \lambda^3 - (3a+c)\lambda^2 + \left(3a^2 - b^2 + 2ac\right)\lambda - \left(a^2 - b^2\right)(a+c) \neq 0.$$

Then, the operator $T - \lambda I$ is invertible, and the inverse operator is defined by

$$\begin{aligned} (T-\lambda I)^{-1} = &-\frac{1}{P_T(\lambda)}\left[T^2 + (\lambda - (3a+c))T\right. \\ &\left.+\left(\lambda^2 - (3a+c)\lambda + 3a^2 - b^2 + 2ac\right)I\right]. \end{aligned}$$

Thus, we have $\sigma(T) = \{a-b, a+b, a+c\}$.

Theorem 3 (Inversion theorem). *The operator T, presented in (5.2), is an invertible operator if and only if*

$$a-b \neq 0, \quad a+b \neq 0 \quad \textit{and} \quad a+c \neq 0. \tag{5.14}$$

In case (5.14) *holds, the inverse operator is defined by*

$$T^{-1} := \frac{1}{(a^2-b^2)(a+c)}\left[T^2 - (3a+c)T + (3a^2 - b^2 + 2ac)I\right]. \tag{5.15}$$

Proof. If T is invertible, then it is, at least, injective. Taking into account the Hermite functions φ_k, we have already observed that (5.9)–(5.10) holds true. Indeed, by (5.11), we see that the Hermite functions are eigenfunctions of T with eigenvalues $a \pm b$ and $a+c$. So, we deduce that $(a^2-b^2)(a+c) \neq 0$, which is equivalent to (5.14).

Conversely, suppose that $(a^2-b^2)(a+c) \neq 0$. Hence, it is possible to consider the operator defined in (5.15) and, by a straightforward computation, verify that this is, indeed, the inverse of T.

5.3.2 Parseval type identity and unitary properties

This subsection is dedicated to obtain conditions which characterize when the operator T is unitary, and to derive a Parseval type identity associated with T.

Let $\langle\cdot,\cdot\rangle_2$ denote the usual inner product of the Hilbert space $L^2(\mathbb{R}^n)$.

Theorem 4 (Parseval-type identity). *For any $f,g \in L^2(\mathbb{R}^n)$, the identity*

$$\begin{aligned}\langle Tf,Tg\rangle_2 = |a|^2\langle f,g\rangle_2 + 2Re\left\{a\bar{b}\right\}\langle f,T_s g\rangle_2 \\ + \left[2Re\{a\bar{c}\} + |c|^2\right]\langle f,T_c^2 g\rangle_2 + |b|^2\langle f,T_s^2 g\rangle_2\end{aligned} \tag{5.16}$$

holds.

Proof. For any $f,g \in L^2(\mathbb{R}^n)$, a direct computation yields $\langle T_s f,g\rangle = \langle f,T_s g\rangle$, $\langle T_c f,g\rangle = \langle f,T_c g\rangle$, $\langle T_s f,T_c^2 g\rangle = \langle f,T_s T_c^2 g\rangle = 0$, $\langle T_c^2 f,T_s g\rangle = \langle f,T_c^2 T_s g\rangle = 0$. Thus, (5.16) directly appears simply by using the definition of T and the just presented identities.

Theorem 5 (Unitary property). *Let $a := a_1 + ia_2$, $b := b_1 + ib_2$, $c := c_1 + ic_2$, with $a_1,a_2,b_1,b_2,c_1,c_2 \in \mathbb{R}$. T is a unitary operator if and only if one of the following conditions holds:*

(i) $a = 0, |b| = 1$ *and* $|c| = 1$*;*
(ii) $a \in \mathbb{R}\setminus\{0\}$, $b_1 = 0$, $a_1^2 + b_2^2 = 1$ *and* $(a_1 + c_1)^2 + c_2^2 = 1$*;*
(iii) $a_2 \neq 0$, $b_1 = \pm\frac{a_2\sqrt{1-|a|^2}}{|a|}$, $b_2 = -\frac{a_1 b_1}{a_2}$ *and* $(a_1 + c_1)^2 + (a_2 + c_2)^2 = 1$.

Proof. Let T^* be the adjoint operator of T, this is, the operator satisfying $\langle Tf,g\rangle_2 = \langle f,T^*g\rangle_2$, for any $f,g \in L^2(\mathbb{R}^n)$. From the last identity, we obtain $T^* = \bar{a}I + \bar{b}T_s + \bar{c}T_c^2$ and we know that T is a unitary operator if T is bijective and $T^* = T^{-1}$. In this way, taking into account (5.15), we have $\bar{a} = \frac{a}{a^2-b^2}$, $\bar{b} = -\frac{b}{a^2-b^2}$ and $\bar{c} = -\frac{b^2+ac}{(a^2-b^2)(a+c)}$, whose solutions are (i)–(iii).

5.3.3 Involution

In this subsection, we present conditions under which T is an involution of order 3. This is totally different from the operator structure of other integral operators exhibiting involutions of either 2 or 4 orders.

Theorem 6. *Consider the operator T defined in* (5.2). *We have $T^3 = I$ for*

(i) $a = \frac{1}{2}e^{i\frac{\pi}{3}}$, $b = \frac{\sqrt{3}}{2}e^{i\frac{5\pi}{6}}$ *and* $c = \frac{3}{2}e^{i\frac{4\pi}{3}}$*;*
(ii) $a = \frac{1}{2}e^{i\frac{\pi}{3}}$, $b = \frac{\sqrt{3}}{2}e^{-i\frac{\pi}{6}}$ *and* $c = \frac{3}{2}e^{i\frac{4\pi}{3}}$*;*
(iii) $a = -\frac{1}{2}$, $b = -i\frac{\sqrt{3}}{2}$ *and* $c = \frac{3}{2}$*;*
(iv) $a = -\frac{1}{2}$, $b = i\frac{\sqrt{3}}{2}$ *and* $c = \frac{3}{2}$*;*

(v) $a = \frac{1}{2}e^{-i\frac{\pi}{3}}$, $b = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{6}}$ *and* $c = \frac{3}{2}e^{i\frac{2\pi}{3}}$;

(vi) $a = \frac{1}{2}e^{-i\frac{\pi}{3}}$, $b = \frac{\sqrt{3}}{2}e^{-i\frac{5\pi}{6}}$ *and* $c = \frac{3}{2}e^{i\frac{2\pi}{3}}$.

This means that the operator T is an involution of order 3.

Proof. If we consider the operator T and its characteristic polynomial (5.3), we have (5.4), which is equivalent to $T^3 - (3a+c)T^2 + (3a^2 - b^2 + 2ac)T = (a^2 - b^2)(a+c)I$. If we consider $-(3a+c) = 0$, $3a^2 - b^2 + 2ac = 0$ and $(a^2 - b^2)(a+c) = 1$, we obtain the solutions (i)–(vi), which means that, for these cases of the parameters, we have $T^3 = I$.

Remark 1. Note that the last property help us to realize that if the coefficients a, b, c are those of the Theorem 6, then our main integral equation, presented before in (5.1), is the most general integral equation that can be generated by the operator T and its powers.

5.3.4 New convolution

In this subsection we will focus on obtaining a new convolution $\circledast$ for the operator T in the case of $a = 0$. This means that we are introducing a new multiplication operation that have a corresponding multiplicative factorization property for the operator T, in the sense as it occurs for the usual convolution and the Fourier transform: $T(f \circledast g) = (Tf)(Tg)$.

Definition 2. Considering $b, c \in \mathbb{C}$ with $b \neq \pm c$ and $bc \neq 0$, we define the new multiplication (convolution) for any two elements $f, g \in L^2(\mathbb{R}^n)$:

$$\begin{aligned} f \circledast g := -\frac{1}{b^2 c} \big\{ & b^4 \left[T_s^2 \left[(T_s f)(T_s g) \right] - (T_s f)(T_c^2 g) - (T_c^2 f)(T_s g) \right] \\ & + b^3 c \left[T_s^2 \left[(T_s f)(T_c^2 g) \right] + T_s^2 \left[(T_c^2 f)(T_s g) \right] - T_s \left[(T_s f)(T_s g) \right] \right] \\ & + b^2 c^2 \left[T_s^2 \left[(T_c^2 f)(T_c^2 g) \right] - (T_c^2 f)(T_c^2 g) \right] - bc^3\, T_s \left[(T_c^2 f)(T_c^2 g) \right] \big\} . \end{aligned} \tag{5.17}$$

Theorem 7. *For the operator $T_0 := bT_s + cT_c^2$, with $b, c \in \mathbb{C}$, $b \neq \pm c$ and $bc \neq 0$, and $f, g \in L^2(\mathbb{R}^n)$, we have the following T_0–factorization:*

$$T_0 \,(f \circledast g) = (T_0 f)(T_0 g). \tag{5.18}$$

Proof. Using the definition of T_0 and a direct (but long) computation, we obtain the equivalence between (5.17) and $f \circledast g = -\left[T_0^2 - cT_0 - b^2 I\right]\left[(T_0 f)(T_0 g)\right]/(b^2 c)$. Thus, having in mind (5.15), for $a = 0$, which provides the formula for the inverse of T_0, we identify the last identity with $f \circledast g = T_0^{-1}\left[(T_0 f)(T_0 g)\right]$, which is equivalent to (5.18), as desired.

We would like to point out that we can also obtain an even more general convolution (in comparison with (5.17)–(5.18)), for T with $a \neq 0$, but due to the lack of space we are not presenting it in here. We notice that these convolutions, and consequent factorization identities, allow the consideration of new convolution type equations.

Acknowledgement

This work was supported in part by Portuguese funds through the CIDMA – *Center for Research and Development in Mathematics and Applications*, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), within project UID/MAT/04106/2013. The second named author was also supported by FCT through the Ph.D. scholarship PD/BD/114187/2016. The third named author was supported partially by the Viet Nam National Foundation for Science and Technology Developments (NAFOSTED).

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