A heat conduction problem of 2D unbounded composites with imperfect contact conditions[1]

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Abstract: We consider a steady-state heat conduction problem in 2D unbounded doubly periodic composite materials with temperature independent conductivities of their components. Imperfect contact conditions are assumed on the boundaries between the matrix and inclusions. By introducing complex potentials, the corresponding boundary value problem for the Laplace equation is transformed into a special R-linear boundary value problem for doubly periodic analytic functions. The method of functional equations is used for obtaining a solution. Thus, the R-linear boundary value problem is transformed into a system of functional equations which is analysed afterwards. A new improved algorithm for solving this system is proposed. It allows to compute the average property and reconstruct the temperature and the flux at an arbitrary point of the composite. Computational examples are presented.

Key words: 2D composite material; steady-state conductivity problem; effective conductivity; imperfect contact conditions; functional equations

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1 Introduction

This work is devoted to analytical investigations of a heat conduction problem of 2D unbounded doubly periodic composite materials with an arbitrary number of inclusions and imperfect contact conditions on the boundary of the components.

The problem of determining the temperature and flux fields and also the effective conductivity of composites with imperfect contact conditions on boundaries between the matrix and inclusions has its history, and some essential results have been obtained. Benveniste and Miloh [2] presented a general framework to determine the effective conductivity of two-phase composites with imperfect interfaces between the matrix and inclusions. Hasselman and Johnson [10] obtained the effective thermal conductivity of composites with interfacial thermal barrier resistance by modifying Rayleigh and Maxwell theories. Lipton and Vernescu [13] extended the Hashin-Shtrikman variational principles to anisotropic two-phase heat conducting composites with thermal contact resistance and derived the bounds of the effective thermal conductivity. A new way for

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the estimation of the bounds of the effective thermal conductivity of composites with imperfect interfaces was suggested by Wu [17] where the transition layer for each spherical inclusion was introduced for construction of the temperature field for the upper bound and the heat flux field for the lower bound. Based on the principles of minimum potential energy and minimum complementary energy, the upper and lower bounds were rigorously derived. The effects of the distribution and size of spherical inclusions on the effective thermal conductivity of composites were analyzed. On the base of the two-scale asymptotic homogenization method, the effect of the interface properties on the effective conductivity and temperature/flux field of a granular composite with a simple cubic array of spherical inclusions was studied by Andrianov et al. in [1]. The effective transverse conductivity of a fibrous composite with imperfect contact between the matrix and parallel elliptic fibers was quite recently found using Maxwell homogenization scheme by Kushch et al. in [12]. Using numerical technique - the finite element method, Graham and McDowell [9] calculated thermal conductivity of a random fiber composite with imperfect interface for the unit cell model.

Analytical investigation of the thermal conductivity of a composite material consisting of periodic body–centred and face–centred cubic array of spheres with interfacial resistance belongs to Cheng and Torquato [6] who extended and applied Rayleigh's method. The effect of imperfect interface phenomenon on the effective conductivity has been studied analytically by Goncales and Kołodziej in [8], and explicit formula for the effective thermal conductivity of composite with regular fiber distribution was obtained. Recently, Riva and Musolino in [15] studied a behavior of the effective conductivity of composites with thermal resistance at the interface for the case when a small parameter ε proportional to a diameter of inclusions tends to 0.

According to classification of the contact conditions done by Benveniste and Miloh in [3], we consider "spring type" interface conditions which characterize a soft interphase. The components (inclusions) are supposed to be non-overlapping disks forming a doubly periodic structure. The case of steady-state conduction is assumed, i.e., the Laplace equation governs this process. This may be seen as a natural continuation of the works of Castro et al. [5] and Drygas, Mityushev [7] in the sense that some important improvements are introduced. Namely, in [5],[7] it is considered a special temperature distribution directed along the Ox-axis. In contrast to this problem now we introduce more natural conditions of periodicity of the flux on the boundary of the minimal representative cell (see [11]) which make the problem more complicated for a solution analysis in comparison with the problem formulated in [5] or [7]. We treat the problem by using the methods developed in [5], [7], reducing the corresponding boundary value problem to a system of functional equations with respect to certain doubly periodic analytical functions. The algorithm for the numerical calculations presented in [5] is mostly oriented to find the effective conductivity. Here, we modify the algorithm in order to increase the accuracy of the numerical computations an any point in the distance from the centers of the inclusions and to find the temperature field (with accuracy to an arbitrary constant). The proposed modification allows us to find the flux distribution in an explicit form containing all parameters of the considered model such as conductivities of the matrix and inclusions, radii and centers of inclusions, an intensity and an angle of the flux.

The paper is organized as follows. In Section 2 we describe the geometry of the considered composites and formulate the mathematical problem. In Section 3 we solve the problem, describe a new algorithm in detail and show that both components of the solution, the flux and the temperature, can be computed. Finally, numerical calculations are performed and discussed in Section 4.

2 Statement of the Problem

Let a lattice L be defined by the two fundamental translation vectors "1" and "$\imath$" (where $\imath^2 = -1$) in the complex plane $\mathbb{C} \cong \mathbb{R}^2$ (with elements identified in the form $z = x + \imath y$). Here, the representative cell is the square

$$Q_{(0,0)} := \left\{ z = t_1 + \imath t_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2},\, p = 1,2 \right\}. \tag{2.1}$$

Let $\mathcal{E} := \bigcup\limits_{m_1,m_2} \{m_1 + \imath m_2\}$ be the set of the lattice points, where $m_1, m_2 \in \mathbb{Z}$. The cells corresponding to the points of the lattice $\mathcal{E}$ will be denoted by

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \left\{ z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)} \right\}. \tag{2.2}$$

It is considered the situation when mutually disjoint disks (inclusions) of different radii $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ with boundaries $\partial D_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$ (for $k = 1, 2, \ldots, N$) are located inside the cell $Q_{(0,0)}$ and periodically repeated in all cells $Q_{(m_1,m_2)}$. Let us denote by

$$D_0 := Q_{(0,0)} \setminus \left(\bigcup_{k=1}^{N} D_k \cup \partial D_k \right) \tag{2.3}$$

the connected domain obtained by removing the inclusions from the cell $Q_{(0,0)}$.

Let us consider the problem of determination of the temperature and flux of an unbounded double periodic composite material with matrix

$$D_{matrix} = \bigcup_{m_1,m_2} \left((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2 \right) \tag{2.4}$$

and inclusions

$$D_{inc} = \bigcup_{m_1,m_2} \bigcup_{k=1}^{N} (D_k + m_1 + \imath m_2) \tag{2.5}$$

occupied by materials of conductivities $\lambda_m > 0$ and $\lambda_k > 0$, respectively. This problem is equivalent to the determination of the potential of the corresponding fields, i.e., a function T satisfying the Laplace equation in each component of the composite material

$$\Delta T(z) = 0, \qquad z \in D_{matrix} \cup D_{inc}, \tag{2.6}$$

which have to satisfy the following boundary conditions on all ∂D_k, $k = 1, 2, \ldots, N$:

$$\lambda_m \frac{\partial T}{\partial n}(t) = \lambda_k \frac{\partial T_k}{\partial n}(t), \tag{2.7}$$

$$\lambda_k \frac{\partial T_k}{\partial n}(t) + \gamma_k (T_k(t) - T(t)) = 0, \qquad t \in \bigcup_{m_1, m_2} \partial D_k, \tag{2.8}$$

where $\gamma_k > 0$, the vector $n = (n_1, n_2)$ is the outward unit normal vector to ∂D_k, and

$$\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}$$

is the outward normal derivative,

$$T(t) := \lim_{D_0 \ni z \to t} T(z), \qquad T_k(t) := \lim_{D_k \ni z \to t} T(z). \tag{2.9}$$

The conditions (2.7)–(2.8) form the so-called *imperfect contact conditions.* In addition, we assume that the heat flux is periodic on y. Thus,

$$\lambda_m T_y \Big(x, \frac{1}{2}\Big) = \lambda_m T_y \Big(x, -\frac{1}{2}\Big) = -A \sin \theta + q_1(x), \tag{2.10}$$

where A is the intensity of an external flux. The heat flux is periodic on x, and, consequently,

$$\lambda_m T_x \Big(-\frac{1}{2}, y\Big) = \lambda_m T_x \Big(\frac{1}{2}, y\Big) = -A \cos \theta + q_2(y). \tag{2.11}$$

To complement the average flux conditions at infinity, the latter immediately proves that the equalities

$$\int_{-1/2}^{1/2} q_j(\xi) d\xi = 0 \tag{2.12}$$

are valid for the unknown functions q_j $(j = 1, 2)$. As a result of (2.10) and (2.11), the heat flux has a zero mean value along the boundary of the cell

$$\int_{\partial Q_{(m_1, m_2)}} \frac{\partial T(s)}{\partial n} ds = 0, \qquad \int_{\partial D_k + m_1 + \imath m_2} \frac{\partial T(s)}{\partial n} ds = 0. \tag{2.13}$$

The condition (2.13) is the consequence of the fact that no source (sink) exists in the cells.

We introduce complex potentials $\varphi(z)$ and $\varphi_k(z)$ which are analytic in D_0 and D_k, and continuously differentiable in the closures of D_0 and D_k, respectively, by using the following relations

$$T(z) = \begin{cases} \operatorname{Re}\, (\varphi(z) + Bz), \;\; z \in D_{matrix}, \\ \\ \frac{2\lambda_m}{\lambda_m + \lambda_k} \operatorname{Re} \varphi_k(z), \;\; z \in D_{inc}, \end{cases} \tag{2.14}$$

where B is an unknown constant belonging to $\mathbb{C}$. Besides, we assume that the real part of φ is doubly periodic in D_0, i.e.,

$$\operatorname{Re} \varphi(z + 1) - \operatorname{Re} \varphi(z) = 0, \qquad \operatorname{Re} \varphi(z + \imath) - \operatorname{Re} \varphi(z) = 0.$$

Note that in general the imaginary part of φ is not doubly periodic in D_0.

It is shown in [11] that φ is a single-valued function in D_{matrix}. The harmonic conjugate to T is a function v which has the following form:

$$v(z) = \begin{cases} \operatorname{Im}(\varphi(z) + Bz), & z \in D_{matrix}, \\ \frac{2\lambda_m}{\lambda_m+\lambda_k} \operatorname{Im}\varphi_k(z), & z \in D_{inc}, \end{cases} \tag{2.15}$$

with the same unknown constant B.

Following the procedure exhibited in [5], we rewrite the conditions (2.7)–(2.8) in terms of the complex potentials $\varphi(z)$ and $\varphi_k(z)$:

$$\varphi(t) = \varphi_k(t) - \rho_k\overline{\varphi_k(t)} + \mu_k(t-a_k)(\varphi_k)'(t) + \mu_k\frac{r_k^2}{t-a_k}\overline{(\varphi_k)'(t)} - Bt, \qquad |t-a_k| = r_k, \tag{2.16}$$

where $\rho_k = \frac{\lambda_k-\lambda_m}{\lambda_k+\lambda_m}$ and $\mu_k = \frac{\lambda_m(\rho_k+1)}{2r_k\gamma_k}$.

For the determination of the flux $\nabla u(x,y)$, we introduce the derivatives of the complex potentials:

$$\begin{aligned} \psi(z) &:= \frac{\partial\varphi}{\partial z} = \frac{\partial T}{\partial x} - \imath\frac{\partial T}{\partial y} - B, && z \in D_0, \\ \psi_k(z) &:= \frac{\partial\varphi_k}{\partial z} = \frac{\lambda_m+\lambda_k}{2\lambda_m}\left(\frac{\partial T_k}{\partial x} - \imath\frac{\partial T_k}{\partial y}\right), && z \in D_k. \end{aligned} \tag{2.17}$$

Representing the function φ in the form $\varphi(z) = \sum\limits_{l=0}^{\infty} \alpha_k(z-a_k)^l$, $|z-a_k| \leq r_k$, and by using the relation $t = \frac{r_k^2}{\overline{t-a_k}} + a_k$ on the boundary $|t-a_k| = r_k$, one can get

$$[\overline{\varphi(t)}]' = -\left(\frac{r_k}{t-a_k}\right)^2 \overline{\varphi'(t)}, \qquad [\overline{\varphi'(t)}]' = -\left(\frac{r_k}{t-a_k}\right)^2 \overline{\psi'(t)}, \qquad |t-a_k| = r_k. \tag{2.18}$$

Thus, after differentiating (2.16), we arrive at the following $\mathbb{R}$-*linear boundary value problem* on each contour $|t-a_k| = r_k$, $k = 1, 2, \dots, N$,

$$\psi(t) = (1+\mu_k)\psi_k(t) + (\rho_k-\mu_k)\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)} + \mu_k(t-a_k)\psi_k'(t) - \mu_k\frac{r_k^4}{(t-a_k)^3}\overline{\psi_k'(t)} - B \tag{2.19}$$

with the unknown constant B.

We will seek a solution $\psi(z)$, $\psi_k(z)$ of the problem (2.19) as a sum $\psi(z) = \psi^{(1)}(z) + \psi^{(2)}(z)$, $\psi_k(z) = \psi_k^{(1)}(z) + \psi_k^{(2)}(z)$ of solutions of the following two boundary value problems:

$$\psi^{(1)}(t) = (1+\mu_k)\psi_k^{(1)}(t) + (\rho_k - \mu_k)\left(\frac{r_k}{t-a_k}\right)^2 \overline{\psi_k^{(1)}(t)}$$
$$+\mu_k(t-a_k)(\psi_k^{(1)}(t))' - \mu_k \frac{r_k^4}{(t-a_k)^3}\overline{(\psi_k^{(1)}(t))'} - B_1 \tag{2.20}$$

$$\psi^{(2)}(t) = (1+\mu_k)\psi_k^{(2)}(t) + (\rho_k - \mu_k)\left(\frac{r_k}{t-a_k}\right)^2 \overline{\psi_k^{(2)}(t)}$$
$$+\mu_k(t-a_k)(\psi_k^{(2)}(t))' - \mu_k \frac{r_k^4}{(t-a_k)^3}\overline{(\psi_k^{(2)}(t))'} - \imath B_2, \tag{2.21}$$

where $\psi_k^{(1)}$ and $\psi_k^{(2)}$ are analytical doubly periodic functions, $B = B_1 + \imath B_2$.

3 Solution of the Problem

To look for the solution of the problem (2.20), we use the auxiliary problem discussed in [5]. Here, for the reader convenience, we shortly overview this problem.

Let $\widetilde{T}$ be a solution of the boundary value problem (2.6)-(2.8) with a constant jump corresponding to the external field applied in the x-direction

$$\widetilde{T}(z+1) = \widetilde{T}(z) + 1, \quad \widetilde{T}(z+\imath) = \widetilde{T}(z). \tag{3.1}$$

The complex potentials $\widetilde{\varphi}^{(1)}(z)$ and $\widetilde{\varphi}_k^{(1)}(z)$ are introduced as follows

$$\widetilde{T}(z) = \begin{cases} \mathrm{Re}\,(\widetilde{\varphi}^{(1)}(z) + z), \ z \in D_{matrix}, \\ \\ \frac{2\lambda_m}{\lambda_m+\lambda_k}\,\mathrm{Re}\,\widetilde{\varphi}_k^{(1)}(z), \ z \in D_{inc}. \end{cases} \tag{3.2}$$

Note that $\widetilde{\varphi}^{(1)}(z)$ and $\widetilde{\varphi}_k^{(1)}(z)$ are analytic in D_0 and D_k, and continuously differentiable in the closures of D_0 and D_k, respectively. Besides, the real part of $\widetilde{\varphi}^{(1)}$ is doubly periodic in D_0, i.e.,

$$\mathrm{Re}\,\widetilde{\varphi}^{(1)}(z+1) - \mathrm{Re}\,\widetilde{\varphi}^{(1)}(z) = 0, \qquad \mathrm{Re}\,\widetilde{\varphi}^{(1)}(z+\imath) - \mathrm{Re}\,\widetilde{\varphi}^{(1)}(z) = 0. \tag{3.3}$$

In general, the imaginary part of $\widetilde{\varphi}^{(1)}$ is not doubly periodic in D_0. The problem (2.6)-(2.8), (3.1) can be reduced to the following $\mathbb{R}$-linear conjugation boundary value problem (cf. [5]):

$$\widetilde{\varphi}^{(1)}(t) = \widetilde{\varphi}_k^{(1)}(t) - \rho_k \overline{\widetilde{\varphi}_k^{(1)}(t)} + \mu_k(t-a_k)(\widetilde{\varphi}_k^{(1)})'(t) + \mu_k \frac{r_k^2}{t-a_k}\overline{(\widetilde{\varphi}_k^{(1)})'(t)} - t, \ |t-a_k| = r_k. \tag{3.4}$$

Differentiating the last equality, we obtain the following $\mathbb{R}$-linear conjugation boundary value problem for analytical doubly periodic functions $\widetilde{\psi}^{(1)}, \widetilde{\psi}_1^{(1)}, \ldots, \widetilde{\psi}_N^{(1)}$ (cf. [5]):

$$\begin{aligned} \widetilde{\psi}^{(1)}(t) &= (1+\mu_k)\widetilde{\psi}_k^{(1)}(t) + (\rho_k - \mu_k)\left(\frac{r_k}{t-a_k}\right)^2 \overline{\widetilde{\psi}_k^{(1)}(t)} \\ &\quad + \mu_k(t-a_k)(\widetilde{\psi}_k^{(1)}(t))' - \mu_k \frac{r_k^4}{(t-a_k)^3}\overline{(\widetilde{\psi}_k^{(1)}(t))'} - 1 \end{aligned} \tag{3.5}$$

with

$$\frac{\partial \widetilde{T}}{\partial x} - \imath \frac{\partial \widetilde{T}}{\partial y} = \begin{cases} \widetilde{\psi}^{(1)}(z) + 1, \ z \in D_{matrix}, \\ \frac{2\lambda_m}{\lambda_m+\lambda_k}\, \widetilde{\psi}_k^{(1)}(z), \ z \in D_{inc}, \end{cases} \tag{3.6}$$

and

$$\begin{aligned} \widetilde{\psi}^{(1)}(z) &:= \tfrac{\partial \widetilde{\varphi}^{(1)}}{\partial z} = \tfrac{\partial \widetilde{T}}{\partial x} - \imath \tfrac{\partial \widetilde{T}}{\partial y} - 1, && z \in D_0, \\ \widetilde{\psi}_k^{(1)}(z) &:= \tfrac{\partial \widetilde{\varphi}_k^{(1)}}{\partial z} = \tfrac{\lambda_m+\lambda_k}{2\lambda_m}\left(\tfrac{\partial \widetilde{T}_k}{\partial x} - \imath \tfrac{\partial \widetilde{T}_k}{\partial y}\right), && z \in D_k. \end{aligned} \tag{3.7}$$

Notice that when the temperature has a constant jump corresponding to the external field applied in the y-direction

$$\widetilde{T}(z+1) = \widetilde{T}(z), \quad \widetilde{T}(z+\imath) = \widetilde{T}(z) - 1,$$

the temperature is defined as

$$\widetilde{T}(z) = \begin{cases} \mathrm{Re}\,(\widetilde{\varphi}^{(2)}(z) + \imath z), \ z \in D_{matrix}, \\ \frac{2\lambda_m}{\lambda_m+\lambda_k}\, \mathrm{Re}\, \widetilde{\varphi}_k^{(2)}(z), \ z \in D_{inc}, \end{cases} \tag{3.8}$$

with corresponding functions $\widetilde{\varphi}^{(2)}$ and $\widetilde{\varphi}_k^{(2)}$ possessing the same properties as the functions $\widetilde{\varphi}^{(1)}$ and $\widetilde{\varphi}_k^{(1)}$, and

$$\widetilde{\varphi}^{(2)}(t) = \widetilde{\varphi}_k^{(2)}(t) - \rho_k \overline{\widetilde{\varphi}_k^{(2)}(t)} + \mu_k(t-a_k)(\widetilde{\varphi}_k^{(2)})'(t) + \mu_k \frac{r_k^2}{t-a_k}\overline{(\widetilde{\varphi}_k^{(2)})'(t)} - \imath t, \ |t-a_k| = r_k.$$

The corresponding $\mathbb{R}$-linear conjugation boundary value problem has the form

$$\begin{aligned} \widetilde{\psi}^{(2)}(t) &= (1+\mu_k)\widetilde{\psi}_k^{(2)}(t) + (\rho_k - \mu_k)\left(\frac{r_k}{t-a_k}\right)^2 \overline{\widetilde{\psi}_k^{(2)}(t)} \\ &\quad + \mu_k(t-a_k)(\widetilde{\psi}_k^{(2)}(t))' - \mu_k \frac{r_k^4}{(t-a_k)^3}\overline{(\widetilde{\psi}_k^{(2)}(t))'} - \imath. \end{aligned} \tag{3.9}$$

The problems (2.20) and (2.21) can be equivalently reduced to the problems (3.5) and (3.9), by using the following replacements:

$$\psi^{(1)}(z) = B_1 \widetilde{\psi}^{(1)}(z), \quad \psi_k^{(1)}(z) = B_1 \widetilde{\psi}_k^{(1)}(z), \tag{3.10}$$

$$\psi^{(2)}(z) = B_2 \widetilde{\psi}^{(2)}(z), \quad \psi_k^{(2)}(z) = B_2 \widetilde{\psi}_k^{(2)}(z). \tag{3.11}$$

Remark 3.1 *It is easy to verify that the functions* $\widetilde{\psi}^{\perp}(z) := \imath\widetilde{\psi}^{(2)}(\imath z)$ *and* $\widetilde{\psi}_k^{\perp}(z) := \imath\widetilde{\psi}_k^{(2)}(\imath z)$ *satisfy the following* $\mathbb{R}$*-linear conjugation boundary value problem*

$$\begin{aligned}\widetilde{\psi}^{\perp}(t) &= (1+\mu_k)\widetilde{\psi}_k^{\perp}(t) + (\rho_k - \mu_k)\left(\frac{r_k}{t-b_k}\right)^2 \overline{\widetilde{\psi}_k^{\perp}(t)} \\ &\quad + \imath\mu_k(t-b_k)(\widetilde{\psi}_k^{\perp}(t))' + \imath\mu_k\frac{r_k^4}{(t-b_k)^3}\overline{(\widetilde{\psi}_k^{\perp}(t))'} + 1, \end{aligned} \tag{3.12}$$

where $|t-b_k| = r_k$, $b_k = -\imath a_k$.

Note that $\widetilde{\psi}^{(2)}(z) = -\imath\widetilde{\psi}^{\perp}(-\imath z)$ and $\widetilde{\psi}_k^{(2)}(z) = -\imath\widetilde{\psi}_k^{\perp}(-\imath z)$.

Thus, to find a solution of the problem (2.19), it is sufficient to find solutions $\widetilde{\psi}^{(1)}(z), \widetilde{\psi}_k^{(1)}(z)$ and $\widetilde{\psi}^{\perp}(z), \widetilde{\psi}_k^{\perp}(z)$ of the problems (3.5) and (3.12), respectively.

For obtaining of the real constants B_1 and B_2, we use the same approach presented in [11] and get the following relations:

$$B_1 = \frac{-A\cos\theta}{\lambda_m(I+1)}, \qquad B_2 = \frac{-A\sin\theta}{\lambda_m(I^{\perp}-1)}, \tag{3.13}$$

where

$$I := \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{Re}\,\widetilde{\psi}^{(1)}\Big(\frac{1}{2} + \imath y\Big)\,dy, \quad I^{\perp} := \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{Re}\,\widetilde{\psi}^{\perp}\Big(\frac{1}{2} + \imath y\Big)\,dy.$$

Now we can formulate the statement about how to find the temperature flux.

Theorem 3.2 *Let* $T = T(x,y)$ *and* $T_k = T_k(x,y)$ *be the solution of the problem (2.7)-(2.8), (2.10) and (2.11). The temperature flux is defined in the following form:*

$$\frac{\partial T(x,y)}{\partial x} - \imath\frac{\partial T(x,y)}{\partial y} = \begin{cases} \psi(z) + B, \ z = x + \imath y \in D_{matrix}, \\ \frac{2\lambda_m}{\lambda_m + \lambda_k}\,\psi_k(z), \ z = x + \imath y \in D_{inc}, \end{cases} \tag{3.14}$$

where

$$B = \frac{-A\cos\theta}{\lambda_m(I+1)} - \frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\imath,$$

and

$$\begin{aligned}\psi(z) &:= \frac{-A\cos\theta}{\lambda_m(I+1)}\widetilde{\psi}^{(1)}(z) + \imath\frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\widetilde{\psi}^{\perp}(-\imath z), \quad z \in D_{matrix}, \\ \psi_k(z) &:= \frac{-A\cos\theta}{\lambda_m(I+1)}\widetilde{\psi}_k^{(1)}(z) + \imath\frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\widetilde{\psi}_k^{\perp}(-\imath z), \quad z \in D_{inc}.\end{aligned}$$

To find the temperature it is sufficient to find the functions $\varphi, \varphi_1, \dots, \varphi_N$ (cf. (2.14)). These functions can be represented as sums $\varphi(z) = \varphi^{(1)}(z) + \varphi^{(2)}(z)$, $\varphi_k(z) = \varphi_k^{(1)}(z) + \varphi_k^{(2)}(z)$ of two functions $\varphi^{(1)}$ and $\varphi^{(2)}$ which have to satisfy the following boundary value problems:

$$\varphi^{(1)}(t) = \varphi_k^{(1)}(t) - \rho_k \overline{\varphi_k^{(1)}(t)} + \mu_k(t-a_k)(\varphi_k^{(1)}(t))' + \mu_k \frac{r_k^2}{t-a_k}\overline{(\varphi_k^{(1)}(t))'} - B_1 t, \quad (3.15)$$

$$\varphi^{(2)}(t) = \varphi_k^{(2)}(t) - \rho_k \overline{\varphi_k^{(2)}(t)} + \mu_k(t-a_k)(\varphi_k^{(2)}(t))' + \mu_k \frac{r_k^2}{t-a_k}\overline{(\varphi_k^{(2)}(t))'} - \imath B_2 t. \quad (3.16)$$

Analogously to (3.10)–(3.11), we have $\varphi^{(1)}(z) = B_1\widetilde{\varphi}^{(1)}(z)$, $\varphi_k^{(1)}(z) = B_1\widetilde{\varphi}_k^{(1)}(z)$ and $\varphi^{(2)}(z) = B_2\widetilde{\varphi}^{(2)}(z)$, $\varphi_k^{(2)}(z) = B_2\widetilde{\varphi}_k^{(2)}(z)$. A straightforward computation shows that

$$\widetilde{\varphi}^{(2)}(z) = \widetilde{\varphi}^{\perp}(-\imath z),\ \widetilde{\varphi}_k^{(2)}(z) = \widetilde{\varphi}_k^{\perp}(-\imath z)$$

with $\widetilde{\varphi}^{\perp}$ and $\widetilde{\varphi}_k^{\perp}$ satisfy the following $\mathbb{R}$-linear conjugation boundary value problem:

$$\widetilde{\varphi}^{\perp}(t) = \widetilde{\varphi}_k^{\perp}(t) - \rho_k \overline{\widetilde{\varphi}_k^{\perp}(t)} + \mu_k(t-b_k)(\widetilde{\varphi}_k^{\perp}(t))' + \mu_k \frac{r_k^2}{t-b_k}\overline{(\widetilde{\varphi}_k^{\perp}(t))'} + t,$$

where $|t-b_k| = r_k$, $b_k = -\imath a_k$. The functions $\widetilde{\varphi}^{(1)}$ and $\widetilde{\varphi}_k^{(1)}$ can be found up to an arbitrary constant as indefinite integrals of the functions $\widetilde{\psi}^{(1)}$ and $\widetilde{\psi}_k^{(1)}$, respectively (cf. (3.7)). Thus, we arrive at the following result.

Theorem 3.3 *Let $T = T(x,y)$ and $T_k = T_k(x,y)$ be the solution of the problem (2.7)-(2.8), (2.10) and (2.11). The temperature distribution can be found up to an arbitrary constant and is defined in the form (2.14), where*

$$\begin{aligned}
B &= \frac{-A\cos\theta}{\lambda_m(I+1)} - \frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\imath,\\
\varphi(z) &= \frac{-A\cos\theta}{\lambda_m(I+1)}\widetilde{\varphi}^{(1)}(z) - \frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\widetilde{\varphi}^{\perp}(-\imath z),\\
\varphi_k(z) &= \frac{-A\cos\theta}{\lambda_m(I+1)}\widetilde{\varphi}_k^{(1)}(z) - \frac{A\sin\theta}{\lambda_m(I^{\perp}-1)}\widetilde{\varphi}_k^{\perp}(-\imath z).
\end{aligned}$$

The solvability of the problem (3.5) is described in [5] in details, where we used the method of functional equations (cf. [4, 14]). Here, we describe a new algorithm for obtaining the solution of the problem (3.5) which gives more accurate numerical results. The problem (3.12) can be solved analogously. For convenience, we omit the upper index in $\widetilde{\psi}^{(1)}$ and will write $\widetilde{\psi}$ below. Here, we use some facts and notation of the paper [5].

Notice that we have N contours ∂D_k and N complex conjugation conditions on each contour ∂D_k but we need to find $N+1$ functions $\widetilde{\psi}, \widetilde{\psi}_1, \dots, \widetilde{\psi}_N$. This means that we need one additional condition to close up the system. For this reason we introduce a new doubly periodic function Φ which is sectionally analytic in $Q_{(0,0)}$ and in $\bigcup_{k=1}^{N} D_k$ and has zero jumps along each ∂D_k, $k = 1, 2, \dots, N$. Such consideration will give an additional condition on $\widetilde{\psi}, \widetilde{\psi}_1, \dots, \widetilde{\psi}_N$. In fact, we will show that $\Phi \equiv 0$.

Let us introduce the sectionally analytic doubly periodic function Φ by the following formula:

$$\Phi(z) = \begin{cases} \Phi_{(k)}(z),\ |z-a_k| \le r_k, \\ \\ \Phi_{(0)}(z),\ z \in D_0, \end{cases} \tag{3.17}$$

where

$$\Phi_{(k)}(z) = (1+\mu_k)\widetilde{\psi}_k(z) + \mu_k(z-a_k)\widetilde{\psi}_k'(z) - \sum_{m=1}^{N}\sum_{m_1,m_2}{}^{*} (\rho_m-\mu_m)W_{m_1,m_2,m}\widetilde{\psi}_m(z)$$
$$+\sum_{m=1}^{N}\sum_{m_1,m_2}{}^{*}\mu_m W'_{m_1,m_2,m}\widetilde{\psi}_m'(z) - 1, \tag{3.18}$$

$$\Phi_{(0)}(z) = \widetilde{\psi}(z) - \sum_{m=1}^{N}\sum_{m_1,m_2} (\rho_m-\mu_m)W_{m_1,m_2,m}\widetilde{\psi}_m(z) + \sum_{m=1}^{N}\sum_{m_1,m_2} \mu_m W'_{m_1,m_2,m}\widetilde{\psi}_m'(z). \tag{3.19}$$

Here,

$$W_{m_1,m_2,k}\widetilde{\psi}_k(z) = \left(\frac{r_k}{z-a_k-m_1-\imath m_2}\right)^2 \overline{\widetilde{\psi}_k\left(\frac{r_k^2}{\overline{z-a_k-m_1-\imath m_2}}+a_k\right)}, \tag{3.20}$$

$$W'_{m_1,m_2,k}\widetilde{\psi}_k'(z) = \frac{r_k^4}{(z-a_k-m_1-\imath m_2)^3}\overline{\widetilde{\psi}_k'\left(\frac{r_k^2}{\overline{z-a_k-m_1-\imath m_2}}+a_k\right)}, \tag{3.21}$$

$$\sum_{m=1}^{N}\sum_{m_1,m_2}{}^{*}(\rho_m-\mu_m)W_{m_1,m_2,m} := \sum_{m\neq k}\sum_{m_1,m_2}(\rho_m-\mu_m)W_{m_1,m_2,m} + \sum_{m_1,m_2}{}'(\rho_k-\mu_k)W_{m_1,m_2,k}. \tag{3.22}$$

The “prime” notation in $\sum_{m_1,m_2}{}'$ means that the summation occurs in all m_1 and m_2 except at $(m_1,m_2)=(0,0)$.

Applying the *Analytic Continuation Principle* and *Liouville's theorem* for doubly periodic functions, we have that $\Phi = c$.

Let $\widetilde{\psi}$ and $\widetilde{\psi}_k$ be solutions of the system $\Phi(z)=c$. Then, in D_0, we have

$$\widetilde{\psi}(z) = \widetilde{\psi}'(z) + c \tag{3.23}$$

with some doubly periodic function $\widetilde{\psi}'$. Inserting the last equality in (3.7) and then in (3.2), we obtain

$$T(z) = \mathrm{Re}\,(\widetilde{\varphi}'(z)+cz+z), \quad z \in D_0, \tag{3.24}$$

with some function $\widetilde{\varphi}'$ which yields $c=0$. Thus, we have $\Phi(z)\equiv 0$. Writing $\Phi(z)\equiv 0$, we obtain the following system of linear functional equations

$$\widetilde{\psi}_k(z) = -\frac{\mu_k}{1+\mu_k}(z-a_k)\widetilde{\psi}_k'(z) + \frac{1}{1+\mu_k}\sum_{m=1}^{N}\sum_{j}{}^{*}(\rho_m-\mu_m)W_{j,m}\widetilde{\psi}_m(z)$$
$$-\frac{1}{1+\mu_k}\sum_{m=1}^{N}\sum_{j}{}^{*}\mu_m W'_{j,m}\widetilde{\psi}_m'(z) + \frac{1}{1+\mu_k}, \qquad k=1,2,\ldots,N, \tag{3.25}$$

which is uniquely solvable with respect to $\widetilde{\psi}_k$ in the space of analytical functions (for more details cf. [5]).

The function $\widetilde{\psi}$ has the form

$$\widetilde{\psi}(z) = \sum_{m=1}^{N} \sum_{j} (\rho_m - \mu_m) W_{j,m} \widetilde{\psi}_m(z) - \sum_{m=1}^{N} \sum_{j} \mu_m W'_{j,m} \widetilde{\psi}'_m(z). \tag{3.26}$$

Let us expand $\widetilde{\psi}_k(z)$ into Taylor series,

$$\widetilde{\psi}_k(z) = \sum_{l=0}^{\infty} \widetilde{\psi}_{lk}(z-a_k)^l, \qquad \widetilde{\psi}'_k(z) = \sum_{l=1}^{\infty} \widetilde{\psi}_{lk} l (z-a_k)^{l-1}, \tag{3.27}$$

in order to sum up $W_{m_1,m_2,k}\widetilde{\psi}_k(z)$ and $W'_{m_1,m_2,k}\widetilde{\psi}'_k(z)$ over all translations $m_1 + \imath m_2$.

The series $\sum\limits_{j} W_{j,k}\widetilde{\psi}_k(z)$, where $j = (m_1, m_2)$ and k is a fixed number, can be represented via the elliptic Eisenstein functions $E_l(z)$ of order l (see [16, 5]):

$$\sum_{j} W_{j,k}\widetilde{\psi}_k(z) = \sum_{l=0}^{\infty} \overline{\widetilde{\psi}_{lk}} r_k^{2(l+1)} E_{l+2}(z-a_k), \tag{3.28}$$

$$\sum_{j} W'_{j,k}\widetilde{\psi}'_k(z) = \sum_{l=1}^{\infty} \overline{\widetilde{\psi}_{lk}}\, l\, r_k^{2(l+1)} E_{l+2}(z-a_k). \tag{3.29}$$

The series

$$\sum_{j}{}' W_{j,k}\widetilde{\psi}_k(z) := \sum_{j} W_{j,k}\widetilde{\psi}_k(z) - \left(\frac{r_k}{z-a_k}\right)^2 \overline{\widetilde{\psi}_k\left(\frac{r_k^2}{\overline{z-a_k}} + a_k\right)}$$

and

$$\sum_{j}{}' W'_{j,k}\widetilde{\psi}'_k(z) := \sum_{j} W'_{j,k}\widetilde{\psi}'_k(z) - \frac{r_k^4}{(z-a_k)^3} \overline{\widetilde{\psi}'_k\left(\frac{r_k^2}{\overline{z-a_k}} + a_k\right)}$$

can be written in the form

$$\sum_{j}{}' W_{j,k}\widetilde{\psi}_k(z) = \sum_{l=0}^{\infty} \overline{\widetilde{\psi}_{lk}} r_k^{2(l+1)} \sigma_{l+2}(z-a_k), \tag{3.30}$$

$$\sum_{j}{}' W'_{j,k}\widetilde{\psi}'_k(z) = \sum_{l=1}^{\infty} \overline{\widetilde{\psi}_{lk}}\, l\, r_k^{2(l+1)} \sigma_{l+2}(z-a_k), \tag{3.31}$$

where σ_l is the modified Eisenstein function defined by the formula $\sigma_l(z) := E_l(z) - z^{-l}$. The Eisenstein functions E_l converges absolutely and uniformly for $l = 3, 4, \ldots$ and conditionally for $l = 2$ (cf. [16]).

Thus, we can rewrite the equations (3.25) and (3.26), for $\widetilde{\psi}_k$ and $\widetilde{\psi}$, as follows:

$$\begin{aligned}
\widetilde{\psi}_k(z) \;=\; & -\frac{\mu_k}{1+\mu_k}(z-a_k)\widetilde{\psi}'_k(z) + \frac{1}{1+\mu_k}\sum_{m\neq k}^{N}\sum_{l=0}^{\infty}(\rho_m-\mu_m)\overline{\widetilde{\psi}_{lm}}\, r_m^{2(l+1)}\, E_{l+2}(z-a_m) \\
& + \frac{\rho_k-\mu_k}{1+\mu_k}\sum_{l=0}^{\infty}\overline{\widetilde{\psi}_{lk}}\, r_k^{2(l+1)}\,\sigma_{l+2}(z-a_k) \\
& - \frac{1}{1+\mu_k}\sum_{m\neq k}^{N}\sum_{l=1}^{\infty}\mu_m\overline{\widetilde{\psi}_{lm}}\, l\, r_m^{2(l+1)}\, E_{l+2}(z-a_m) \\
& - \frac{\mu_k}{1+\mu_k}\sum_{l=1}^{\infty}\overline{\widetilde{\psi}_{lk}}\, l\, r_k^{2(l+1)}\,\sigma_{l+2}(z-a_k) + \frac{1}{1+\mu_k},
\end{aligned} \tag{3.32}$$

$$\widetilde{\psi}(z) = \sum_{m=1}^{N}\sum_{l=0}^{\infty}(\rho_m-\mu_m)\overline{\widetilde{\psi}_{lm}}\, r_m^{2(l+1)}\, E_{l+2}(z-a_m) - \sum_{m=1}^{N}\sum_{l=1}^{\infty}\mu_m\overline{\widetilde{\psi}_{lm}}\, l\, r_m^{2(l+1)}\, E_{l+2}(z-a_m). \tag{3.33}$$

Using (3.27), we have

$$\widetilde{\psi}_k(z) + \frac{\mu_k}{1+\mu_k}(z-a_k)\widetilde{\psi}'_k(z) = \sum_{l=0}^{\infty}\widetilde{\psi}_{lk}(z-a_k)^l\left(1+\frac{l\mu_k}{1+\mu_k}\right).$$

Now we need to find the numerical coefficients $\widetilde{\psi}_{lm}$ of the system (3.32). Note that the equation (3.33) for $\widetilde{\psi}$ has the same coefficients $\widetilde{\psi}_{lm}$. Taking a partial sum of the Taylor series with M first items

$$\widetilde{\psi}_k(z) = \widetilde{\psi}_{0k} + \widetilde{\psi}_{1k}(z-a_k) + \widetilde{\psi}_{2k}(z-a_k)^2 + \cdots + \widetilde{\psi}_{Mk}(z-a_k)^M$$

and collecting the coefficients of the consequent powers of $z-a_k$, we obtain the formula for definition of $\widetilde{\psi}_{jk}$:

$$\widetilde{\psi}_{jk} = \frac{1+\mu_k}{j!(1+(1+j)\mu_k)}\widetilde{\psi}_k^{(j)}\Big|_{z=a_k}, \tag{3.34}$$

where $\widetilde{\psi}_k^{(j)}$ is the derivative of order j of the function $\widetilde{\psi}_k$. Then, we get

$$\begin{aligned}
\widetilde{\psi}_{jk} \;=\; & \frac{1}{j!(1+(1+j)\mu_k)}\sum_{m\neq k}^{N}\sum_{l=0}^{M}(\rho_m-\mu_m)\overline{\widetilde{\psi}_{lm}} r_m^{2(l+1)}(-1)^j\frac{(l+j+1)!}{(l+1)!}E_{l+j+2}(a_k-a_m) \\
& + \frac{\rho_k-\mu_k}{j!(1+(1+j)\mu_k)}\sum_{l=0}^{M}\overline{\widetilde{\psi}_{lk}} r_k^{2(l+1)}(-1)^j\frac{(l+j+1)!}{(l+1)!}\sigma_{l+j+2}(0) \\
& - \frac{1}{j!(1+(1+j)\mu_k)}\sum_{m\neq k}^{N}\sum_{l=1}^{M}\mu_m\overline{\widetilde{\psi}_{lm}} r_m^{2(l+1)}(-1)^j\frac{l(l+j+1)!}{(l+1)!}E_{l+j+2}(a_k-a_m) \\
& - \frac{\mu_k}{j!(1+(1+j)\mu_k)}\sum_{l=1}^{M}\overline{\widetilde{\psi}_{lk}} r_k^{2(l+1)}(-1)^j\frac{l(l+j+1)!}{(l+1)!}\sigma_{l+j+2}(0) + I_j,
\end{aligned} \tag{3.35}$$

where $I_j = \begin{cases} \frac{1}{1+\mu_k}, \ j = 0, \\ 0, \ j = 1, \ldots, M. \end{cases}$

Thus, we arrive at a system with $N(M+1)$ unknown constants $\widetilde{\psi}_{jk}$ and $N(M+1)$ equations which can be solved numerically. This system is obtained for an arbitrary number N of inclusions of different radii and parameters γ_k.

Remark 3.4 *Note that for finding the flux distribution (namely, the functions $\widetilde{\psi}$, $\widetilde{\psi}_k, \ldots,$ $\widetilde{\psi}_N$) in explicit form, we change an algorithm for the solution of the equations (3.32) and (3.33) in comparison with the algorithm proposed in [5]. The new algorithm allows us to get more accurate numerical values of the flux in each point of considered composite material.*

4 Numerical Results

This section is devoted to the presentation of the algorithm mentioned above. The algorithm is realized in Maple 14 software.

We consider the case when four inclusions are situated within one cell (i.e. $N = 4$). We suppose that a heat flux of fixed intensity $A = -1$ flows in different directions with respect to the main axis. Here the minus sign shows that the flux is directed from the right to the left (or from the top to the bottom) depending on the angle θ. The conductivity of the matrix is $\lambda_m = 1$, and the conductivity of inclusions λ_k take different values.

We take a non-symmetrical configuration of non-overlapping inclusions with the centers

$$a_1 = -0.18 + 0.2\imath, a_2 = 0.33 - 0.34\imath, a_3 = 0.33 + 0.35\imath, a_4 = -0.18 - 0.2\imath \qquad (4.1)$$

and the same radius $r_k = R$ of value 0.145. In this case some of inclusions are situated very close to inclusions of the neighboring cells. For symmetrical configurations of the inclusions or small radius, the accuracy is higher. Therefore, we choose this configuration to check the accuracy of the calculations in the worse situation.

We calculate the flux components in the center a_k of the k-inclusion

$$Q_x^{(k)}(a_k) \equiv \lambda_k \frac{\partial T_k(a_k)}{\partial x} = \frac{2\lambda_k \lambda_m}{\lambda_m + \lambda_k} \cdot \operatorname{Re} \psi_k(a_k),$$

$$Q_y^{(k)}(a_k) \equiv \lambda_k \frac{\partial T_k(a_k)}{\partial y} = -\frac{2\lambda_k \lambda_m}{\lambda_m + \lambda_k} \cdot \operatorname{Im} \psi_k(a_k),$$

and at the matrix point $z = 0$, when the flux components in any point of the matrix can be found as

$$Q_x^{(m)}(z) = \lambda_m \cdot \operatorname{Re}(\psi(z) + B), \quad Q_y^{(m)}(z) = -\lambda_m \cdot \operatorname{Im}(\psi(z) + B)$$

in accordance with the formula (3.14). Computations in the Table 1 are given for the first eight consecutive values of the number M $(M = 0, 1, ..., 7)$ showing how many terms are selected for computations in the Taylor series (3.35).

Table 1: The flux components for different numbers of M, while other problem parameters are: $\theta = 0$, $\gamma_k = 1/10$, $R = 0.145$, $\lambda_m = 1$, $\lambda_k = 100$ and the configuration of the inclusions being defined by (4.1).

M	$Q_x^{(1)}(a_1)$	$Q_y^{(1)}(a_1)$	$Q_x^{(m)}(0)$	$Q_y^{(m)}(0)$
0	0.041704	0.000305	1.505039	0.007381
1	0.042568	0.000358	1.494873	0.007442
2	0.042683	0.000354	1.481236	0.007406
3	0.042687	0.000357	1.482434	0.007526
4	0.042705	0.000359	1.483022	0.007527
5	0.042712	0.000360	1.483523	0.007539
6	0.042713	0.000360	1.483721	0.007543
7	0.042714	0.000360	1.483787	0.007545

Computations show that taking $M = 7$ the accuracy is between five or six valid units depending on where the flux is computed. Note that for the same configuration but $\gamma_k = 0$ (ideal-contact conditions) we get the same accuracy for $M = 4$ (see [11]).

Table 2: Temperature in two points of the model for different numbers of M, while other problem parameters are: $\theta = 0$, $\gamma_k = 1/10$, $R = 0.145$, $\lambda_m = 1$, $\lambda_k = 100$ and the configuration of the inclusions being defined by (4.1).

M	$T(a_1)$	$T(0)$
0	-0.319549	0.071912
1	-0.324873	0.085084
2	-0.326249	0.085099
3	-0.326794	0.084942
4	-0.327119	0.084877
5	-0.327239	0.084842
6	-0.327280	0.084829
7	-0.327294	0.084826

Taking the same parameters as for the flux, we calculate the temperature in two points a_1 and 0 for different M and get the accuracy five valid units for $M = 6$. It is worth to mention that an accuracy depends on a position (a_i), size (R) of inclusions and other parameters of a model. However, our calculations show that change of the parameter λ_k with fixed other parameters a_i, R, N, λ_m, γ_k, A, θ does not influence on the accuracy.

As an example, we find the temperature distribution $T(x, y)$ presented on Figures 1-2 for the following parameters: $\lambda_m = 1$, $R = 0.145$, $\theta = 0; \pi/4$, $\lambda_k = 100$ and $\lambda_k = 0.01$.

We also show the flux distribution inside the cell $Q_{(0,0)}$ for different γ_k, angles and conductivities of inclusions on Figures 3-5. Note that in the case of ideal-contact conditions when the conductivity of the matrix $\lambda_m = 1$ and the conductivity of inclusions

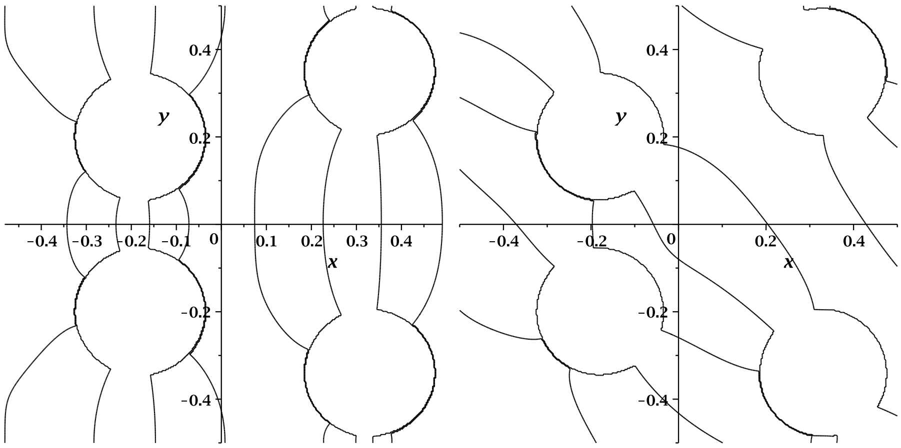

Figure 1: The temperature distribution inside $Q_{(0,0)}$ for $\lambda_k = 100$, $\theta = 0; \pi/4$.

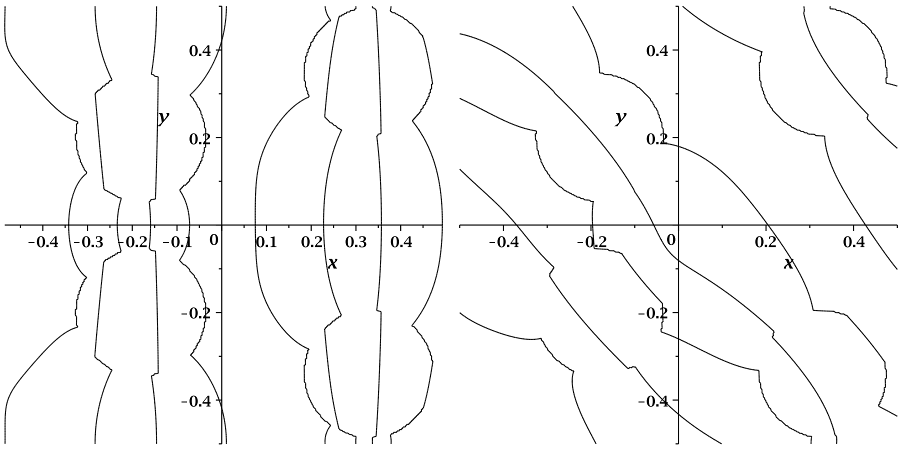

Figure 2: The temperature distribution inside $Q_{(0,0)}$ for $\lambda_k = 0.01$, $\theta = 0; \frac{\pi}{4}$.

$\lambda_k > 1$, the flux in the inclusions is more intensive in comparison with the flux in the matrix. For $\lambda_k < 1$, we have the opposite result (see [11]). In the case of imperfect contact conditions, the situation depends not only on λ_k with $\lambda_m = 1$ but also on the parameter γ_k as it can be seen by analyzing formula (2.8). To show this effect, we give several examples with different parameters λ_k and γ_k fixing $\lambda_m = 1$ (cf. Figures 3-5).

In the general case of composites with different random non-overlapping inclusions the tensor of effective conductivity Λ_e has a form

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{yx} & \lambda_e^y \end{pmatrix} \tag{4.2}$$

with components which can be found from the well-known equation

$$\langle \mathbf{q} \rangle = -\Lambda_e \cdot \langle \nabla T \rangle, \tag{4.3}$$

where $\langle \mathbf{q} \rangle = (\mathfrak{q}_1, \mathfrak{q}_2)$ is the average flux, and $\langle \nabla T \rangle = (T_1, T_2)$ is the average temperature

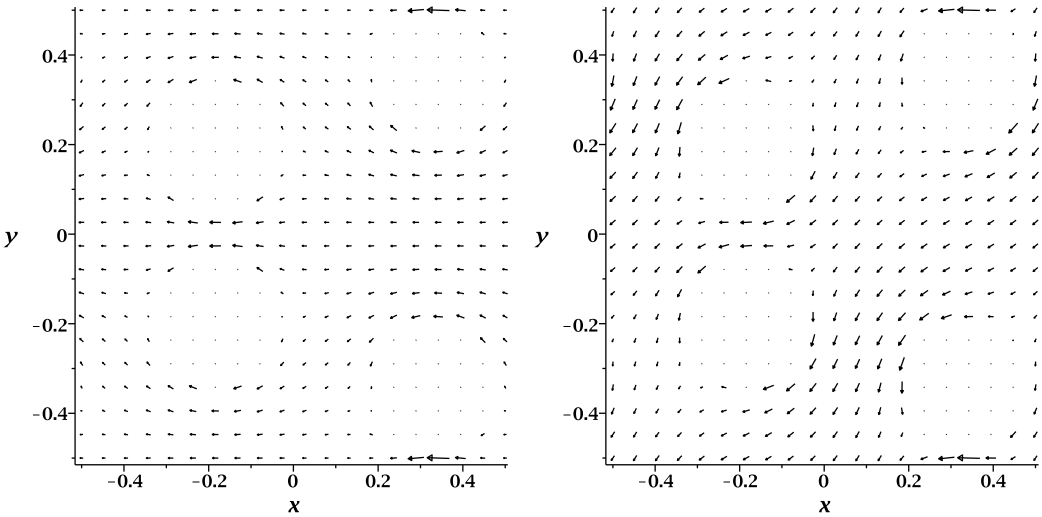

Figure 3: The flux distribution inside $Q_{(0,0)}$ for $\gamma_k = 1/10$, $\lambda_k = 100$, $\theta = 0; \pi/4$.

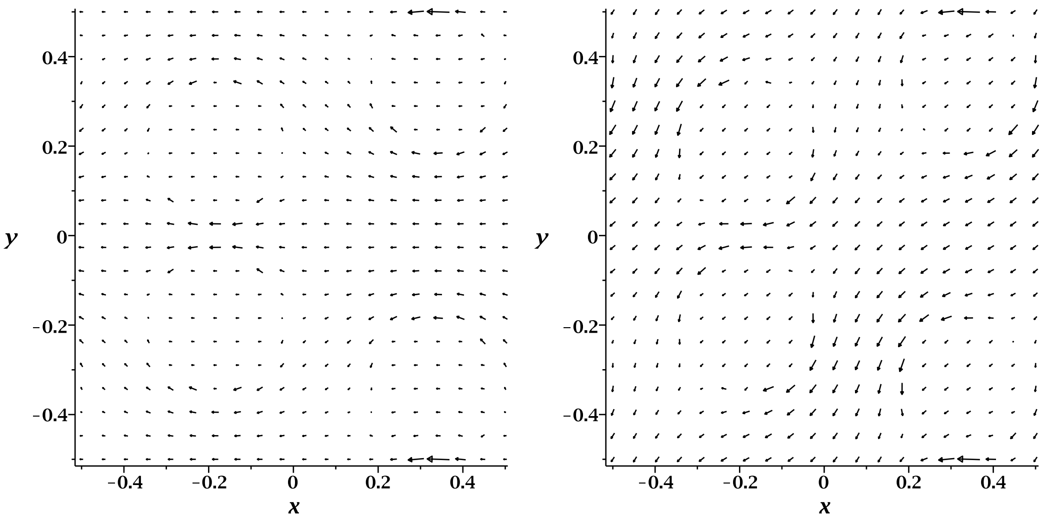

Figure 4: The flux distribution inside $Q_{(0,0)}$ for $\gamma_k = 1/10$,$\lambda_k = 0.01$, $\theta = 0; \frac{\pi}{4}$.

gradient with

$$\mathfrak{q}_j = \lambda_m \iint\limits_{D_0} \frac{\partial T}{\partial x_j}\, dx_1 dx_2 + \sum_{k=1}^{N} \lambda_k \iint\limits_{D_k} \frac{\partial T_k}{\partial x_j}\, dx_1 dx_2, \tag{4.4}$$

and

$$T_j = \iint\limits_{D_0} \frac{\partial T}{\partial x_j}\, dx_1 dx_2 + \sum_{k=1}^{N} \iint\limits_{D_k} \frac{\partial T_k}{\partial x_j}\, dx_1 dx_2, \tag{4.5}$$

where $j = 1, 2$ and $x_1 = x$ and $x_2 = y$. These components were found in terms of the obtained solution in [11] and contain all parameters of the model. As since all formulas are valid in the case of imperfect conditions on the boundaries of components,

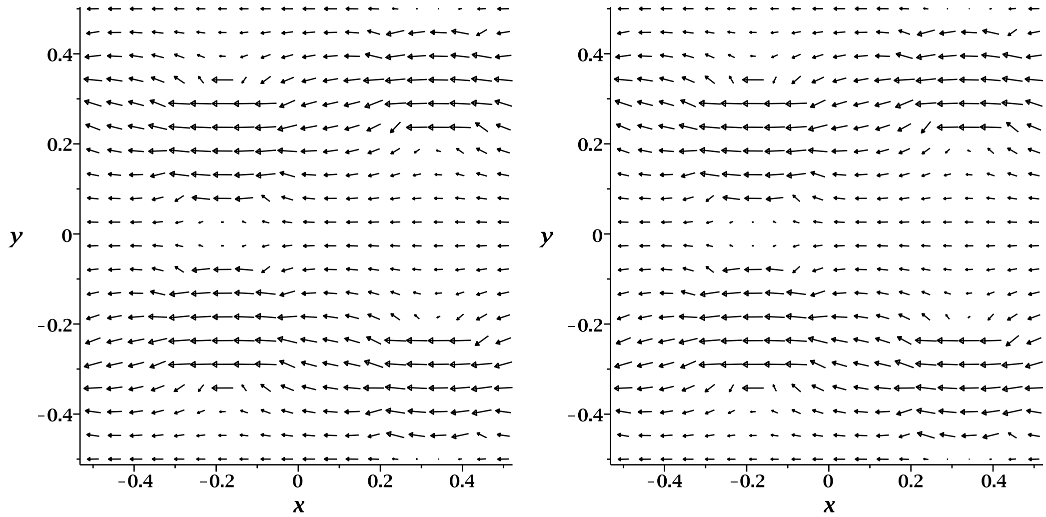

Figure 5: The flux distribution inside $Q_{(0,0)}$ for $\gamma_k = 100; 1000, \lambda_k = 100$, $\theta = 0$.

we represent here only final formulas:

$$\mathfrak{q}_1 = -A\cos\theta, \qquad \mathfrak{q}_2 = -A\sin\theta,$$

$$\begin{aligned} T_1 - \imath T_2 &= \frac{-Ae^{-\imath\theta}}{\lambda_m} + 2\sum_{k=1}^{N} \frac{\lambda_m - \lambda_k}{\lambda_m + \lambda_k} \iint\limits_{D_k} \psi_k(z)\, dxdy \\ &= \frac{-Ae^{-\imath\theta}}{\lambda_m} - 2\pi \sum_{k=1}^{N} \rho_k r_k^2\, \psi_k(a_k). \end{aligned}$$

We present the computations for the material conductivities $\lambda_m = 1$ and $\lambda_k = 100; 0.01$. Values of all components of the tensor Λ_e as a function on the radius R are presented in Tables 3-5.

Table 3: The components of the effective conductivity tensor Λ_e for the configuration of the inclusions given in (4.1) for the material constants $\lambda_m = 1, \lambda_k = 100$, $\gamma_k = 1/10$, $M = 7$.

R	λ_e^x	λ_e^{yx}	λ_e^y	λ_e^{xy}
0	1	0	1	0
0.05	1.000323	$-1.3 \cdot 10^{-9}$	1.000316	$-1.2 \cdot 10^{-9}$
0.11	1.004139	$-2.9 \cdot 10^{-7}$	1.003657	$-9.0 \cdot 10^{-8}$
0.135	1.008968	$-3.7 \cdot 10^{-7}$	1.007228	$-2.5 \cdot 10^{-6}$
0.145	1.012247	$-6.8 \cdot 10^{-7}$	1.009277	$-6.1 \cdot 10^{-6}$

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Table 4: The components of the effective conductivity tensor Λ_e for the configuration of the inclusions given in (4.1) for the material constants $\lambda_m = 1$,$\lambda_k = 0.01$, $\gamma_k = 1/10$, $M = 7$.

R	λ_e^x	λ_e^{yx}	λ_e^y	λ_e^{xy}
0	1	0	1	0
0.05	0.978881	$8.2 \cdot 10^{-8}$	0.979340	$8.0 \cdot 10^{-8}$
0.11	0.834698	$3.1 \cdot 10^{-6}$	0.851255	$5.8 \cdot 10^{-6}$
0.135	0.722485	$8.8 \cdot 10^{-6}$	0.764132	$2.6 \cdot 10^{-5}$
0.145	0.664977	$1.4 \cdot 10^{-5}$	0.724652	$5.0 \cdot 10^{-5}$

Table 5: The components of the effective conductivity tensor Λ_e for the configuration of the inclusions given in (4.1) for the material constants $\lambda_m = 1$,$\lambda_k = 100$, $\gamma_k = 100$, $M = 7$.

R	λ_e^x	λ_e^{yx}	λ_e^y	λ_e^{xy}
0	1	0	1	0
0.05	1.052660	$1.4 \cdot 10^{-7}$	1.053442	$4.5 \cdot 10^{-7}$
0.11	1.306726	$6.0 \cdot 10^{-6}$	1.340042	$2.3 \cdot 10^{-4}$
0.135	1.501538	$2.0 \cdot 10^{-5}$	1.602899	0.001508
0.145	1.601547	$3.3 \cdot 10^{-5}$	1.761323	0.003240

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References

[1] Andrianov, I. V., Bolshakov, V. I., Danishevskyy, V. V., Weichert, D., Asymptotic study of imperfect interfaces in conduction through a granular composite material, *Proc. R. Soc. A*, 466 (2010), 2707-2725.

[2] Benveniste, Y., Miloh, T., The effective conductivity of composites with imperfect thermal contact at constituent interfaces, *International Journal of Engineering Science*, 24 (9) (1986), 1537-1552.

[3] Benveniste, Y., Miloh, T., Imperfect soft and stiff interfaces in two-dimensional elasticity, *Mechanics of Materials*, 33 (2001), 309–323.

[4] Berlyand, L., Mityushev, V., Generalized Clausius–Mossotti formula for random composite with circular fibers, *J. Stat. Phys.* **102** (2001), No. 1-2, 115–145.

[5] Castro, L.P., Pesetskaya, E., Rogosin, S., Effective conductivity of a composite material with non-ideal contact conditions, *Complex Var. Elliptic Equ.* **54** (2009), No. 12, 1085–1100.

[6] Cheng, H., Torquato, S., Effective Conductivity of Periodic Arrays of Spheres with Interfacial Resistance, *Proceedings of the Royal Society of London A*, 45 (1997), 145–161.

[7] Drygas P., Mityushev V., Effective conductivity of arrays of unidirectional cylinders with interfacial resistance, *The Quarterly Journal of Mechanics and Applied Mathematics* 62 (2009), 235–262.

[8] Goncalves, L. C., Kołodziej, J. A., Determination of effective thermal conductivity in fibrous composites with imperfect contact between constituents, *Inter. Communications in Heat and Mass Transfer*, 20 (1993), 111–121.

[9] Graham, S., McDowell, D. L., Numerical analysis of the transverse thermal conductivity of composites with imperfect interfaces, *Journal of Heat Transfer*, 125 (2003), 389-393.

[10] Hasselman, D.P.H., Johnson, L., Effective thermal conductivity of composites with interfacial thermal barrier resistance, *Journal of Composite Materials* 21 (1987), 508-515.

[11] Kapanadze, D., Mishuris, G., Pesetskaya, E., Improved algorithm for analytical solution of the heat conduction problem in composites, *Complex Var. Elliptic Equ.*, to appear.

[12] Kushch, V.I., Sevostianov, Igor, V.S. Chernobai, V.S., Effective conductivity of composite with imperfect contact between elliptic fibers and matrix: Maxwells homogenization scheme *International Journal of Engineering Science*, in press.

[13] Lipton, R., Vernescu, B., Composites with imperfect interface, *Proceedings of the Royal Society, London A* 452 (1996), 329-358.

[14] Mityushev, V., Rogosin, S., *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions. Theory and Applications.* Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[15] Riva, M., Musolino, P., A singularly perturbed nonideal transmission problem and application to the effective conductivity of a periodic composite, *SIAM J. Appl. Math.*, 73(1) (2013), 2446.

[16] Weil, A., *Elliptic Functions According to Eisenstein and Kronecker* (reprint of the 1976 edition), Classics in Mathematics, Springer-Verlag, Berlin, 1999.

[17] Wu, L., Bounds on the effective thermal conductivity of composites with imperfect interface, *International Journal of Engineering Science*, 48 (2010), 783-794.