5. The AK model

The AK model

5.1. Introduction

5.2. The simple AK model

Getting rid of diminishing returns

A Graphical Illustration

What happens when the saving rate increase?

The Harrod-Domar equation

No convergence

5.3. The AK model with endogenous savings

Adding the optimal consumption rule to the AK model

Transpiration responds to inspiration!

It depends!

Proximate versus fundamental causes of economic growth

5.4. Incarnations of the AK model

The Harrod-Domar model

A one sector model with Physical and Human Capital

A two-sector model of endogenous growth

A Neoclassical model with Endogenous growth

5.5. Empirical controversies

The ghost of financing gap

Box 5.2. The external aid controversy

Levels or changes?

Box 5.3. Cross-country growth regressions

5.6. Discussion

Key ideas of Chapter 5

Appendix 5.1 Unbalanced growth in the HD model

Problems and Exercises

Key concepts

Essay questions

Exercises
The AK model

“…a level effect can appear as a growth effect for long periods of time, since adjustments in real economies may take place over decades”. [Sachs and Warner].

Learning Goals:

• Understand why getting rid of diminishing returns one can obtain unceasing growth through factor accumulation.

• Review different models were simple factor accumulation can generate endogenous growth.

• Understand why in AK models an endogenous investment rate makes a difference.

• Understand the similarity between investment and R&D in terms of forego consumption.

• Acknowledge the empirical challenges raised by the abandonment of diminishing returns.

5.1. Introduction

Along the previous chapters, we learned that, if there are diminishing returns to the reproducible factors, factor accumulation cannot, by itself, explain the tendency for per capita income to grow over time. For this reason, steady state growth in the neoclassical model is achieved only by postulating an exogenous rate of technological progress.

In this chapter it is shown that, by getting rid of diminishing returns, one can obtain continuous growth of per capita income without the need to postulate an exogenous rate of technological progress. The basic model introduced in this chapter is the AK model. The AK model differs critically from the Solow model in that it relies on a
production function that is linear in the stock of capital. With this change, the model can deliver sustained per capita income growth without any tendency to approach a steady state. In this model, a rise in the saving rate has a proportional effect on the growth rate of per capita income, on a permanent basis. This contrasts with the Solow model, where a rise in the saving rate only delivers a “level effect”.

The pitfall of the AK model is that the assumption of diminishing returns plays a very central role in economic thinking. Hence, it shall not be abandoned without a well motivated story. Some of the sections below and in the following chapter describe alternative theories and models that have been proposed to motivate the abandonment of diminishing returns in endogenous growth models. This chapter also explains how endogenous growth can be obtained in the context of a neoclassical model with diminishing returns to capital. Although none the models described in this chapter shall be seen as the true model, they all offer alternative avenues to think economic development.

Sections 5.2 describes the AK model in its simpler formulation. Section 5.3 extends the AK model to the case of endogenous savings. Section 5.4 reviews alternative models that emulate the AK model. Section 5.5 addresses the empirical evidence on the relationship between savings on economic growth. Section 5.6 concludes.

5.2. The simple AK model

Getting rid of diminishing returns

Consider a closed economy where the population growth rate, the savings rate and the rate of depreciation the capital stock are all constant. In contrast to the Solow model, however, production is a linear function of the capital stock (K):

\[ Y_t = AK_t, \ A > 0 \quad (5.1) \]
In (5.1), the parameter \( A \) stands for the level of technology (or aggregate efficiency), and is assumed constant.

In light of (5.1), production depends only on capital and there are no diminishing returns. The reader may get suspicious about this formulation: after all, how can we model production without labour? In fact, we don’t need to rule out a role for labour in production: in the following sections, we will review alternative models that, while accounting for the role of labour, emulate versions of production function (5.1). For the moment, however just stick with this simple formulation.

Dividing (5.1) by \( N \), one obtains a linear relationship between per capita income and capital per worker:

\[
y_i = Ak_i
\]

The remaining equations of the model are the same as in the basic Solow model:

\[
Y_t = C_t + S_t
\]

\[
sY_t = I_t
\]

\[
\dot{K}_t = I_t - \delta K_t
\]

\[
n = \dot{N}_t / N_t
\]

From (2.7), (2.8) and (2.9), we obtain the dynamics of the capital labour ratio:

\[
\dot{k}_i = sAk_i - (n + \delta)k_i
\]

This equation is similar to (2.14), with the only difference that now \( \beta = 1 \). This small difference has an important implication: since both the production function and the break-even investment line are linear in \( k \), only by an exceptional coincidence of parameters would the two curves be the same. Hence, the general case in the AK model is one without steady state.

Dividing (5.3) by \( k \), one obtains the equation that describes the growth rate of capital per worker in this economy:
\[
\frac{\dot{k}}{k} = sA - (n + \delta)
\]

Since output is linear in K, the growth rate of capital per worker is also the growth rate of per capita income. That is:

\[
y = sA - (n + \delta) \quad (5.4)
\]

This equation states that the growth rate of per capita income rises with total factor productivity (A) and the saving rate (s) and declines with the depreciation of the capital-labour ratio (n and \(\delta\)). As long as \(sA > (n + \delta)\), per capita income will expand forever, at a constant growth rate. Note that this conformity with the real-world facts is achieved without the need to postulate any exogenous technological progress.

Because the growth rate of per capita income in (5.4) is influenced by the other parameters, instead as being given, the model is categorized as of endogenous growth.

**A Graphical Illustration**

Figure 5.1 describes the dynamics of the AK model. The horizontal axis measures the capital labour ratio (k). The vertical axis measures output per capita (y). The top line shows the production function in the intensity form, (5.2); the middle line corresponds to gross savings per capita (the first term in the right hand side of 5.3); the lower of the three lines is the break-even investment line (the second term in the right hand side of 5.3).

Since the production function is now linear in k, the locus representing gross savings never crosses the break-even investment line (compare with Figure 2.3). This means that there is no steady state: as long as \(sA > n + \delta\), per capita output will grow forever.
When the production function is linear, the curve describing per capita savings never crosses the break-even investment line. Hence, the capital-labour ratio and per capita output will expand without limits.

What happens when the saving rate increase?

The AK model differs drastically from the Solow model, in that changes in the exogenous parameters alter the long run growth rate of per capita income, rather than the level of per capita income.

Figure 5.2 compares the paths of per capita income in the AK model and in the Solow model following a once-and-for-all increase in the saving rate at time $t_0$ (the case with the Solow model was already discussed in detail in Figure 3.3). The top part of the diagram shows levels and the bottom part shows growth rates.
Figure 5.2: The AK model and the Solow model compared for a rise in the saving rate

The figure compares the response of per capita income to an exogenous increase in the saving rate in light of the AK model versus the Solow model. While in the Solow model this gives rise to a level effect, in the AK model, there is a growth effect. The figure also suggests that, in the short term, the Solow model and the AK model produce similar predictions.

In the Solow model, the rate of growth of per capita income jumps initially to a higher level, but then it declines slowly over time, until returning to the previous level (given by the exogenous rate of technological progress) (after $t_1$). Because of diminishing returns, the long run growth rate of per capita income is independent of the saving rate. Remember that the model without exogenous growth (Chapter 2) is just a case, with $\gamma=0$. 
In the AK model, the rise in the saving rate has a permanent effect on growth: there is no tendency for the growth rate of per capita income to decline as time goes by. The growth rate of per capita output is proportional to the saving rate.

**The Harrod-Domar equation**

A useful way to compare the AK model with the Solow model is looking at the long run relationship between the average product of capital and the growth rate of per capita income. For convenience, let’s use (5.1) in (5.4), to obtain:

\[ \gamma = s \frac{Y}{K} - (n + \delta) \]  

(5.5)

This equation is known as the Harrod-Domar equation. The difference between (5.5) and (3.11) (that holds in the Solow model in the long run) refers to the variables that are exogenous and endogenous in this equation. In both models, \( s, n \) and \( \delta \) are exogenous. But the two models differ in respect to the exogeneity of \( \delta \) and \( Y/K \): In the AK model, \( Y/K \) is exogenous and \( \gamma \) is endogenous. By contrast, in the Solow model, \( \gamma \) is exogenous and \( Y/K \) is endogenous.

Hence: In the Solow model, a rise in the saving rate leads to a lower average productivity of capital in the steady state. That is, \( Y/K \) declines from one steady state to the other (Figure 3.2). In the AK model, \( Y/K \) is constant (equal to \( A \)). Hence, a rise in the saving rate can only be accommodated in the model by an increase in the growth rate of per capita income, \( \gamma \).

Because the AK model predicts that changes in \( A \) or in the saving rate produce growth effects, it goes far beyond the neoclassical model in stressing the relationship between economic policies and economic growth: government policies, such as taxes and subsidies, that affect economic efficiency and consumption-saving decisions may alter the long run’ rate of economic growth, rather than simply the level of per capita income.
No convergence

The AK model does not predict convergence of per capita incomes, even among similar economies. According to (5.4), two economies having the same technology and savings rates will enjoy the same growth rate of per capita income, regardless of their starting position. This means that their per capita incomes will evolve in parallel and there will be no tendency for the poorer economy to “catch up” with the richest economy. This contrasts to the Solow model, where countries with similar parameters should approach the same per capita income level in the steady state.

Moreover, since changes in technology (A) and in the saving rate (s) affect growth rates permanently, countries with different parameters should exhibit different growth rates of per capita income. In a world where policies differ substantially across countries, the rule should be that of divergence of per capita incomes, rather than of convergence.

5.3. The AK model with endogenous savings

Thus far, the saving rate in the AK model has been assumed exogenous. In this section we show that, when the model is extended to allow individuals to optimally trade consumption today for consumption in the future, a second channel linking efficiency to growth is opened up.

Adding the optimal consumption rule to the AK model

In what follows, let’s recall the simplest possible optimal consumption rule, introduced in Section 2.6:

\[ \gamma = r - \rho \]  

(5.7)

This equation states that, as long as the interest rate is higher than the rate of time preference, there will be incentive for households to increase consumption over time.
This, in turn, is achieved through a higher saving rate. Note that in this model (because there is no transitional dynamics), consumption and income evolve in parallel each moment it time. Hence, (5.7) can be seen as describing simultaneously the growth rate of per capita consumption and the growth rate of per capita income.

To find out how the growth rate of per capita income relates with the remaining parameters of the AK model, one needs an interest rate. As before, we assume that firms are perfectly competitive and maximize profits. In this case, capital will be paid its marginal product, $A$. That is:

$$ r + \delta = A \quad (5.8) $$

Substituting (5.8) in (5.7) and rearranging, one obtains:

$$ \gamma = A - \delta - \rho \quad (5.9) $$

This equation describes the growth rate of per capita income in a version of the AK model where consumers are allowed to trade consumption today for consumption in the future at interest rate $r$. Comparing to (5.4), you see that now it is the rate of time preference, instead of the saving rate, that determines the rate of economic growth.

Transpiration responds to inspiration!

From the qualitative point of view, equation (5.9) brings no novelty relative to the case with exogenous savings, (5.5): a lower rate of time preference (that is, a change in consumption preferences in favour of more consumption in the future and less consumption today), by raising the saving rate, leads to a higher rate of capital accumulation and a higher growth rate of per capita income.

However, comparing (5.9) to (5.5), we observe that the impact of $A$ on growth is now much larger than in the case with exogenous savings. For instance, with a saving rate equal to 20%, the impact of a unitary change in $A$ on growth in light of (5.5) is 0.2. In light of (5.9), however, the impact of A on growth is one to one. That is: five times more.
What makes the assumption of endogenous savings so powerful that it can alter dramatically the relationship between the efficiency parameter and growth? The point is that, when $A$ rises, there are two effects: On one hand, when $A$ rises for a given $s$, the growth rate of per capita income rises, just like in (5.5); on the other hand, when $A$ rises, there is an additional impact through the interest rate, $r$: a higher marginal productivity of capital translates into a higher return on capital and this, in turn, will induce a higher saving rate, for each rate of time preference. Then, with a higher saving rate, the economy will grow faster.

Formally, one may substitute (5.9) in (5.5) and solve for $s$, to obtain the (endogenous) saving rate\(^1\): $s = 1 - \left( \rho - n \right) / A$. Taking the partial derivative in respect to $A$, we verify that the impact of a change in $A$ on the saving rate is $\partial s / \partial A = \left( \rho - n \right) / A^2$. The total impact of a change in $A$ is the sum of the direct impact of $A$ on $\gamma$ with the indirect impact, through $s$: $d\gamma / dA = \partial \gamma / \partial A + \left( \partial \gamma / \partial s \right) \left( \partial s / \partial A \right) = s + \left( \rho - n \right) / A = 1$.

This finding is of the utmost importance to understand the mechanics of many endogenous growth models. A typical assessment based on equation (5.5) is that a country may either grow through “inspiration” ($A$) or through “transpiration” ($s$). But, we just saw that “transpiration” responds to “inspiration”: that is, a more efficient resource allocation, leading to a higher marginal product of capital, implies a higher return on investment. Thus, agents will be willing to forego a higher proportion of their consumption to save more.

With this finding, one may rewrite equation (5.5) in the following form:

$$\gamma = s \left( \rho, A \right) - \left( n + \delta \right)$$

(5.10)

\(^{1}\) The restriction $\rho > n$ is necessary for the problem to be well-behaved. We skip, however this discussion.
It is important to distinguish the circumstances in which, when assessing the growth prospects of a given country, one shall refer to equation (5.10) or instead to (5.5). The difference is that the rule (5.9) presumes that the financial system is well developed, so that households are able to transfer units of the consumption good across time. Thus, in countries where the financial system is underdeveloped and households face borrowing constraints, equation (5.5) is thought to be more appropriated.

The implication of having two distinct equations, (5.10) and (5.5) depending on the quality of the financial system brings a new insight to our analysis: bad economic policies (as reflected in lower parameter $A$) are likely to impact more severely on growth in countries with developed financial systems than in countries with underdeveloped financial systems. Putting in other way, in countries with inefficient financial systems, people are more likely to tolerate bad government policies. This, in turn, may help perpetuate the bad policies!

This discussion adds to the general point that questions like “what happens to per capita income (or to economic growth) when some parameter increases” do not have a unique answer: it depends on specific circumstances.

**Proximate versus fundamental causes of economic growth**

According to equation (5.4), a low rate of economic growth can be explained either because a country does not invest enough ($s$) or because it has low productivity of capital ($A$). Dealing with the development question at a deeper level, however, one may ask why some countries save and invest more than others and why some countries reach higher levels of efficiency than others. In other words, one would like to take as endogenous the parameters that the model takes as exogenous.
To some extent, equation (5.10) is a step in that direction: according to this equation, individuals will save more in countries where the productivity of capital is higher$^2$. This gives parameter A a key role in the growth question.

The following chapters will be devoted to a better understanding of what is behind parameter $A$. In this quest, we will relate the level of A to the quality of economic policies and institutions. We will argue that countries with sound economic policies are expected to achieve higher efficiency levels than countries with poor economic policies.

But another question will immediately arise: why do some countries implement better policies than others? To answer this question, we need to address the incentives of policymakers. These, in turn depend on the quality of political institutions. These, in turn, are grounded in even deeper factors underlying human societies, such as social norms, culture and geography.

In a word, as one deepens the analysis, we move from the *proximate* causes of economic growth (the exogenous parameters in equation 5.4), to the *fundamental causes* of economic growth, which ultimately determine why in a given country the parameter values are what they are. These fundamental causes are essential to understand why some societies make choices that lead them to benefit from better policymaking and to adopt more modern technologies.

This is not to say that simple models like (5.4) are useless. On the contrary, they are essential to understand the *mechanics of economic growth*. In particular, the role of investment and technology as *mediators* between country characteristics and economic performances. But dealing with the growth question at a deeper level, one may want to understand what is behind the parameters that the model takes as exogenous.

---

$^2$ Equation (5.10) stresses the causality from “inspiration” to “transpiration”. However, the reversal may also be true. In Chapter 6 we’ll address precisely some theories according to which the level of A is enhanced by capital accumulation. The possibility of mutual causation implies that savings and efficiency may reinforce each other, both positively and negatively, raising the possibility of multiple equilibria and poverty traps.
5.4. Incarnations of the AK model

The Harrod-Domar model

The true predecessor of the AK model was developed independently by two economists, Roy Harrod, and Evsey Domar. The Harrod Domar model preceded that of Solow by several years and obviously it was not motivated by any explicit intention to improve on the Solow model. The HD model was developed in the aftermath of the Great Depression, as a dynamic extension of Keynes’ general theory, with the aim to discuss the business cycle in the U.S. economy. Since at that time, unemployment was very high, the focus of the model was on the relationship between investment in physical capital and output growth.

The main assumption of the Harrod-Domar model is that capital and labour are pure complements, meaning that they cannot substitute for each other in production. The underlying production function is Leontief:

\[ Y_t = \min \{ AK_t, BN_t \}, \]  
(5.11)

where A and B are positive constants.

The Leontief production function contrasts with the Cob-Douglas production function in that inputs cannot substitute for each other. As an illustration, suppose that your output (Y) was a meal consisting in a “steak with two eggs”. To produce any amount of this output you would need steaks (K) and eggs (N) in a proportion of two eggs per one steak. Figure 5.3 illustrates this, by displaying the isoquants corresponding to \( A=1 \) and \( B=0.5 \). Thus, to produce one meal (\( Y=1 \)), you need at least one steak and two eggs (point R). If you employed one steak and 8 eggs, your maximum production would still be equal to one meal (point S).

Now think that this production function applied to the economy as a whole and that K and N referred to capital and labour. If the economy’ endowments were \( K=1 \) and \( N=8 \), the economy would be producing \( Y=1 \) only, wasting 6 unit of labour (point S). From that point, expanding the quantity of labour would not deliver higher output, because labour cannot substitute for capital. The only way to expand production will be increasing the stock of physical capital. If one managed to increase the stock of capital to \( K=2 \), the output level would jump to \( Y=2 \) (point T), and unemployment would be reduced to 4 units of labour. Raising production by incrementing the stock of capital (\( K \)) in an economy with surplus labour (\( N \)) is basically how the Harrod-Domar model works.

Mathematically, a situation of employment surplus occurs when \( K/N < B/A \). In that case the relevant branch of the production function (5.11) is the first, implying a linear relationship between output and K, \( Y = AK \). This is basically the AK model. Then, given the exogenous saving rate and the population growth rate, from (2.5)-(2.9), you’ll obtain the growth rate of per capita income as described by (5.4).

**Figure 5.3: The Leontief production function**

The figure describes two isoquants where inputs to production are complementar. The straight line \( A/B \) corresponds to the efficient combinations. If the economy’ endowment point lies on the right-hand
side (left hand side) of this line, there will be unemployment of labour (capital).

The main limitation of the Harrod-Domar is that factor prices play no role in driving the economy towards full employment. This contrasts with the Solow model, where price flexibility ensures full employment each moment in time. Thus, a case may arise where, even if output is expanding over time at the rate $\dot{Y}/Y = sA - \delta$, this reveals less than population growth, implying a declining per capita output over time ($\gamma = \dot{y}/y = sA - \delta - n < 0$). If, in alternative, the combination of parameters is favourable, ($\gamma = sA - \delta - n > 0$) then eventually excess labour will be eliminated in the long run (see Appendix 5.1 for details)\(^4\).

\[ A \text{ one sector model with Physical and Human Capital} \]

Another way of accounting for the role of labour in production and obtain an AK type of model is by considering two types of capital, physical and human capital, and assuming that constant returns apply to the broad concept of capital\(^5\).

To see this formally, consider the following production function:

\[ Y = AK^\beta H^{1-\beta} \quad \text{(5.12)} \]

\(^4\) In the context of development economics, an influential cousin of the Harrod-Domar model was developed by Arthur Lewis (Lewis, W., 1954. “Economic Development with Unlimited Supplies of Labour”. The Manchester School of Economic and Social Studies 22, 139-191). In the Lewis formulation, surplus labour is living close to subsistence in a low productivity agriculture sector. The engine of economic growth is the accumulation of capital in the manufactures sector. As capital accumulates in manufactures, there is a higher demand for labour, allowing workers to migrate from agriculture. Along this process, the excess labour in agriculture will be eventually absorbed, allowing the economy to engage in modern economic growth.

In this production function, there are diminishing returns to physical capital and to human capital in isolation, but there are constant returns to scale in reproducible factors. This contrasts to the Solow and the MRW models, where returns to reproducible factors are decreasing due to the presence of a non-reproducible factor, labour.

Also note that this production function does not necessarily exclude raw labour from production. Indeed, one may think human capital, $H$, as measuring quality adjusted labour, that is, the number of workers, $N$, multiplied by the human capital of the typical worker ($h$):

$$H = hN. \quad (5.13)$$

The implied assumption in (5.13) is that the quantity of workers, $N$, and the quality of workers, $h$, are substitutes. With such a specification, raw labour needs no longer to be a source of diminishing returns: multiplying $h$ and $K$ by a given constant implies that production $Y$ will expand proportionally, even if $N$ remains constant. The CRS property ensures the linearity between production and reproducible factors, and this is what we need to generate economic growth.

To see this formally, let’s return to the MRW assumptions that people save a constant fraction of their incomes in the accumulation of human capital, just like they do for physical capital:

$$\dot{K}_s = sY - \delta K, \quad (5.14)$$

$$\dot{H}_s = s_h Y - \delta H. \quad (5.15)$$

Because of diminishing returns to each type of capital, it doesn’t make sense for people to accumulate one type of capital faster than other. Hence, the two types of capital will be expanding at the same rate $\dot{K}/K = \dot{H}/H$. Using (5.14) and (5.15) this implies

---

6 In alternative, you can solve the model assuming endogenous savings.
Substituting (5.16) in (5.11), we obtain a variant of the AK production function:

\[ Y = A \left( \frac{s_H}{s} \right)^{1-\beta} K \]  

(5.17)

Comparing to (5.1) we see that now the average product of capital embodies the propensity to invest in human relative to physical capital (that is, you can look at A in equation 5.1 as including this effect).7

Using this in (5.5), the growth rate of per capita income in this variant of the AK model is:

\[ \gamma = s^\beta \left( \frac{s_H}{s} \right)^{1-\beta} A - n - \rho, \]  

(5.18)

This equation shows that it is perfectly possible to have diminishing returns to physical capital alone and yet having sustained growth of per capita income. What we need is to have constant returns to all types of capital (or reproducible inputs) when considered together. Note that the MRW model differs from this one in that the non-reproducible factor (N) cannot be replaced by human capital: in the MRW returns to broad capital (K plus H) are decreasing.

**A two-sector model of endogenous growth**

An alternative version of the AK model that does not rely on constant returns to capital in the production function was proposed by Usawa, as early as in 19658. The author extended the Solow model, considering two sectors of production, one for...

---

7 The implication is that, if two countries differ in terms of these saving rates, the one investing more in human capital will exhibit a higher A.

consumption goods and the other for “technology”. In this model, the linearity that is needed to generate endogenous growth arises from the fact that the production function for technology does not exhibit diminishing returns.

The economy has two sectors, the production sector and the R&D sector. The production sector employs labour and capital, and produces goods and services, which are used for consumption and for investment in physical capital. The R&D sector employs labour only.

In the model, it is assumed that workers devote a fraction $1 - \mu$ of their working time to production of goods and the remaining $\mu$ to the development of new technologies. The production function for final goods is given by:

$$Y = AK^\beta [(1 - \mu)\lambda N]^{1 - \beta}$$  \hspace{1cm} (5.21)

The production function in the R&D sector is as follows:

$$\dot{\lambda} = b\mu\lambda$$  \hspace{1cm} (5.22)

The parameter $b$ shall be interpreted as the productivity in the research sector.

The production function (5.22) has a key property: a constant fraction of time devoted to R&D produces a constant growth rate of technology that is independent on the existing level of technology (in other words, there are no diminishing returns to technology on technology creation)\(^9\). With such an assumption, a policy change that successfully increases the proportion of time devoted to R&D ($\mu$) or that improves the productivity in the research sector ($b$) will impact positively and permanently on the growth rate of per capita income.

\(^9\) In terms of our previous terminology, the “standing on shoulders effect” delivers a proportional impact on the level of technology. If instead the productivity of R&D was decreasing (reflecting the dominance of a “fishing out effect”, leading to $\dot{\lambda} = b\mu\lambda^\theta$ with $\theta < 1$, then the growth rate of technology would tend to zero, no matter how much effort was devoted to accumulation of human capital, and sustained growth could not be achieved.
As for physical capital accumulation, the rule (5.14) is retained. This model can be solved in the same manner as the Solow model. For mathematical convenience, let’s rewrite the production function (5.21) as follows:

\[ \bar{y} = A(1-\mu)^{1-\beta} \bar{k}^{\beta} \quad (5.23) \]

Where \( \bar{y} = \frac{Y}{L} \), \( \bar{k} = \frac{K}{L} \), and \( L = \lambda N \). Proceeding as usual, the fundamental dynamic equation becomes:

\[ \bar{k} = sA(1-\mu)^{1-\beta} \bar{k}^{\beta} - [n + \delta + b\mu] \bar{k} \quad (5.24) \]

Comparing with (3.8) you can verify how similar this model is with the Solow model. The main difference is that the parameter determining the effectiveness of labour, rather than growing exogenously, is now dependent of other parameters in the model.

Figure 5.4: the steady state in the two-sector model

This model is hybrid, in the sense that it shares characteristics with the AK model and with the Solow model. It shares with the neoclassical model the feature that it has a stable steady state. To find the steady state, we just need to solve for \( \bar{k} = 0 \). Figure 5.4
illustrates the steady state of the model. Like in the Solow model, changes in $s$ produce “level effects”, causing the steady state level of output per unit of efficiency to increase. Because both $\bar{y}$ and $\bar{k}$ are constant in the steady state, the output-capital ratio is constant and so will do the interest rate. In the steady state: $\dot{\bar{y}} = \dot{\bar{k}} = \gamma + n$ and $\dot{\bar{y}} = \dot{\bar{Y}} - n = b\mu$.

Contrasting to the Solow model, the long run growth rate of per capita income ($b\mu$) is explained inside the model. It depends on the proportion of working time that people allocate to R&D and on the effectiveness of the research activity, $b$. Thus, for instance, a policy that is successful in inducing an increase in the proportion of time devoted to R&D raises the growth rate of the economy on a permanent basis (growth effect). The model is capable of generating sustained growth of per capita income without the need to assume exogenous shifts in the production function.

Technically, sustained growth is obtained in this model because the production function for R&D is free of diminishing returns. In other words, the model overcomes
diminishing returns to physical capital by postulating a linear production function for technology. Physical capital can then be accumulated without seeing its productivity declining, because technology is expanding. Rewriting the production function (5.21), we see that:

$$Y = A \left( \frac{1 - \mu}{k} \right)^{1-\beta} K$$  \hspace{1cm} (5.25)

Since in the steady state $\tilde{k}$ is constant, the long run version of the model is no more than another incarnation of the AK model. In the short run, however, $\tilde{k}$ is not in general constant, so the model also displays a transition dynamic.

Figure 5.5 describes the path of per capita income in this economy following an increase in the time devoted to R&D: at the impact, there is a negative effect on per capita income, because less time is devoted to production. As the times go by, however, the growth rate of per capita output accelerates, due to the faster rate of technological expansion. Note that in Figure 5.4 the production function and the break-even investment line shift in opposite directions. Hence, $\tilde{k}$ starts declining, implying that during the transition period the growth rate of per capita output will fall short the corresponding level in the steady state. As the economy approaches the new steady state, the decline in $\tilde{k}$ decelerates, implying a convergence of per capita income growth to the new steady state growth rate, $b\mu$.

A question that naturally arises is how people decide the optimal level of $\mu$. Intuitively, the optimal investment in R&D shall depend on a variety of factors, such as the productivity of research ($b$), the cost of devoting time to research, the level of impatience of people (in their consumption-saving decisions), and so on. However, with the present formulation, we cannot explore the microeconomic incentives for R&D.

*A Neoclassical model with Endogenous growth*
The models above generate endogenous growth by abandoning the assumption of diminishing returns. This is not, however, a necessary condition: one may generate endogenous growth even without departing from the assumption of diminishing returns to capital\(^ \text{(10)} \).

To see this, let’s consider again the optimal consumption rule (5.7): as already explained in Section 2.6 (Figure 2.12), the Solow model cannot deliver long-run growth because the marginal product of capital falls down to zero as the capital labour ratio increases: at the time the interest rate equals the discount rate, the desired consumption becomes constant over time and the process of capital accumulation stops.

These considerations suggest an avenue to generate endogenous growth: what we need is simply to prevent the interest rate from falling below the rate of time preference. In the AK model, this is achieved because the marginal product of capital is a constant. Thus, as long as \( A > \delta + \rho \), per capita income will grow forever.

The same can be achieved in the context of the neoclassical model. Note that the assumption of diminishing returns only requires the marginal product of capital to be a decreasing function of the capital stock. The Solow model goes a bit further, by postulating an aggregate production function (as exemplified by the Cobb-Douglas) with marginal returns falling asymptotically to zero. If however the marginal product of capital never approached zero, the model could display endogenous growth.

Thus, the only modification we need in the neo-classical model to generate sustained growth of per capita income is to postulate that the marginal product of capital is bounded below by a positive constant. As an example, consider the production function \( Y = AK + BK^\beta N^{1-\beta} \). This production function exhibits diminishing marginal returns: as the amount of capital per worker increases, the marginal product of capital decreases. It converges however asymptotically to A, without falling below this constant.

Thus, as long as $A > \delta + \rho$, the model will display unceasing growth. the economy will expand without bound.  

A neoclassical growth model, suitably modified along these lines is capable of generating at the same time endogenous growth (as the AK model) and transition dynamics. In such a model, two economies differing only in terms of their initial per capital incomes will exhibit a tendency to approach each other, with the one with less capital per worker growing faster. At the same time any government policy that was successful in raising the saving rate would have a permanent effect in the growth rate of per capita income. This class of models is labelled neoclassical models of endogenous growth.

5.5. Empirical controversies

The ghost of financing gap

One of the reasons why the Harrod Domar equation (5.5) became so popular is that it offers a simple and appealing formula to forecast economic growth. This formula was also extensively used by international organizations, such as the World Bank, to calculate a country’ financial needs.

If equation (5.5) was true, one could easily forecast a country’ economic growth, using the saving rate, the depreciation rate and an estimate for the average product of capital, $A$. Since the later is not readily available in national accounts, a possible proxy would be the ratio of net investment to the change in real GDP over two consecutive years:

---

11 Actually, this is what we did by introducing exogenous technological progress (Chapter 3): the effect of technological progress is to raises the productivity of capital, offsetting the diminishing returns.
\[ \text{ICOR} = \frac{\Delta K}{\Delta Y} = \frac{\text{net investment}}{\text{change in GDP}} \]

This is the known as the "Incremental Capital-Output Ratio", ICOR.

As an example, consider a poor economy where the ICOR = 3 and the observed investment ratio (s) is 15%. Assuming a depreciation rate equal to 4%, equation (5.4) implies that output will grow at \(0,15/3 - 0,04 = 0,01\). Now suppose you were a consultant for that economy, advising on poverty alleviation. You could well conclude that the saving rate in this economy was too low. If, for instance, population was growing at 2%, that would imply a fall in per capita income...

You could, then, use the HD equation the other way around: how much should the investment rate in this country, for per capita income to increase at some desired rate? Suppose you wanted income per capita to expand at 2% per year. With the population growing at 2% and a measured ICOR equal to 3, following (5.5), you would need a net investment amounting to 24% of GDP. Since domestic savings were only 15%, you could request the international donors to fill the "financing gap", equal to 9% of GDP.

Economists in international institutions, such as the World Bank, the IMF, the Inter-American Development Bank, the European Bank for Reconstruction and Development used models based on the HD equation to estimate the amount of savings (and/or aid) necessary for poor countries to achieve a minimum rate of economic growth. This philosophy was supported by the understanding that people living near the subsistence level cannot save the same as rich people. In theory, foreign aid could fix this: if the external aid succeeded in raising per capita output above a critical level, it...
could be that the domestic saving responded, allowing the country to engage in a self-sustained growth path. In this case, foreign aid would need only to be temporary\(^\text{12}\).

This interpretation of the HD equation was seriously criticised by William Easterly in an insightful paper called “The ghost of financing gap”\(^\text{13}\). The author noted that, over the past four decades, large amounts of international financial assistance to the developing world did not translate into faster economic growth. Using a sample of 146 countries along the period from 1950 to 1992, the author failed to find a robust positive linear relationship between aid and economic growth.

Does this mean that the HD equation is wrong? Not necessarily. But probably one should not trust too much historical values of the average product of capital (the ICOR) to guess the marginal impact of new investments: when, for instance, part of the external aid is diverted into unproductive uses (frivolous expenses, corruption fees), then much of the higher saving rate will be offset by a lower A. Arguably, the impact of external aid on growth shall depend on the quality of policies and institutions of the recipient country, that may be more of less favourable to an efficient allocation of resources. The discussion in Box 5.2 reviews some empirical evidence along this question.

**Box 5.2. The external aid controversy**

---

\(^\text{12}\) You are invited to demonstrate that replacing the assumption of a constant saving rate by a saving rate that depends positively on per capita income in the context of the AK model raises the possibility of a bifurcated growth pattern, whereby per capita income rises or decreases forever, depending on the initial level of capital per worker. Recent proponents of idea include Sachs (2005) and the United Nations Millennium Development Goals Project. Sachs (2005): “(…) if the foreign assistance is substantial enough, and lasts long enough, the capital stock rises sufficiently to lift households above subsistence…growth becomes self-sustained through households savings and public investments supported by taxation of households” (p. 246). United Nations (2005, p. 19): “The key to escape the trap is to raise the economy’s capital stock to the point where the downwards spiral ends and self-sustained economic growth takes over”. (p. 19). Sachs, J., 2005. The end of poverty: economic possibilities for our times. New York: Penguin Press. United Nations, 2005. Millenium Development, Project Report, United Nations, New York.

The question as to whether external aid helps or not a country achieve faster economic growth is obviously very important from the policy point of view. With no surprise, this question has been subject to empirical scrutiny.

A branch in the literature has investigated the possibility of the impact of aid being conditional on the recipient country’s characteristics. A particularly influential study was a background paper to the 1998 World Bank *Assessing Aid* report, by Burnside and Dollar\(^\text{14}\). The authors run some regressions trying to explain the growth rates of per capita income along the period from 1970 to 1993, using a sample of 56 developing countries. The original results of Burnside and Dollar are reproduced in columns (1) and (2) of Table 5.1. In equation (1), the growth rate of per capita GDP is correlated with: the logarithm of initial per capita GDP (capturing conditional convergence); the degree of ethnic fractionalisation, the rate of political assassinations and the product of these two variables (to capture political instability); an index of institutional quality; the ratio of money to GDP (to capture financial development); two regional dummies, for sub-Saharan Africa and East Asia; a “policy index” (compounding the government budget surplus, inflation and openness to international trade, to capture the quality of domestic policies); and external aid as a percentage of GDP.

In column (1), we see that the t-ratio on AID/GDP is too low (0.28 in column 1). The authors then concluded that aid, by itself, does not explain growth. Column (2) differs from column (1) by adding an interaction term, given by the product of the variables AID/GDP and the Policy Index. Because this last variable was found to be significant while AID/GDP alone was not, the authors concluded that aid only leads to more growth in a sound policy framework\(^\text{15}\).

---


\(^{15}\) The authors also tested the possibility of aid to be detrimental to policies. However, no significant relationship was found between the amount of aid received and the quality of the domestic policies.
These results caused a significant reaction in the economic profession, as it implied that foreign aid is useless in countries pursuing bad policies. Not surprisingly, they were subject to an intense scrutiny by other researchers. In general, this further investigation revealed sensitivity of the Burnside and Dollar results in respect to alternative specifications of the regression model or of the sample period\textsuperscript{16}. Burnside and Dollar then shifted their focus from the quality of policies to the quality of institutions. Using a cross section of 124 countries over the 1990s, they found that, while aid alone is not significant related to growth, aid interacted with the degree of institutional quality is significant\textsuperscript{17}.


Table 5.1. Growth regressions explaining the growth rate of per capita GDP in 56 developing countries

<table>
<thead>
<tr>
<th>Estimation method</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial GDP</td>
<td>-0.61</td>
<td>-0.60</td>
<td>-0.54</td>
</tr>
<tr>
<td></td>
<td>(1.09)</td>
<td>(1.05)</td>
<td>(0.96)</td>
</tr>
<tr>
<td>Ethnic fractionalization</td>
<td>-0.54</td>
<td>-0.42</td>
<td>0.12</td>
</tr>
<tr>
<td></td>
<td>(0.75)</td>
<td>(0.58)</td>
<td>(0.16)</td>
</tr>
<tr>
<td>Assassinations</td>
<td>-0.44*</td>
<td>-0.45*</td>
<td>-0.38</td>
</tr>
<tr>
<td></td>
<td>(1.69)</td>
<td>(1.73)</td>
<td>(1.55)</td>
</tr>
<tr>
<td>Ethnic fractionalization * Assassinations</td>
<td>0.82*</td>
<td>0.79*</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>(1.86)</td>
<td>(1.80)</td>
<td>(1.63)</td>
</tr>
<tr>
<td>Institutional quality</td>
<td>0.64**</td>
<td>0.69**</td>
<td>0.69**</td>
</tr>
<tr>
<td></td>
<td>(3.76)</td>
<td>(4.06)</td>
<td>(4.02)</td>
</tr>
<tr>
<td>M2/GDP (lagged)</td>
<td>0.014</td>
<td>0.012</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(1.06)</td>
<td>(0.86)</td>
<td>(1.54)</td>
</tr>
<tr>
<td>Sub-Saharan Africa</td>
<td>-1.60**</td>
<td>-1.87**</td>
<td>-1.58**</td>
</tr>
<tr>
<td></td>
<td>(2.19)</td>
<td>(2.49)</td>
<td>(2.04)</td>
</tr>
<tr>
<td>East Asia</td>
<td>0.91*</td>
<td>1.31**</td>
<td>1.57**</td>
</tr>
<tr>
<td></td>
<td>(1.65)</td>
<td>(2.26)</td>
<td>(2.63)</td>
</tr>
<tr>
<td>Burnside-Dollar policy index</td>
<td>1.00**</td>
<td>0.71**</td>
<td>0.78**</td>
</tr>
<tr>
<td></td>
<td>(7.14)</td>
<td>(3.74)</td>
<td>(4.05)</td>
</tr>
<tr>
<td>Aid/GDP</td>
<td>0.034</td>
<td>-0.021</td>
<td>1.49**</td>
</tr>
<tr>
<td></td>
<td>(0.28)</td>
<td>(0.13)</td>
<td>(3.92)</td>
</tr>
<tr>
<td>(Aid/GDP) * policy index</td>
<td>0.19**</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.71)</td>
<td>(1.34)</td>
<td></td>
</tr>
<tr>
<td>Fraction of land in tropics</td>
<td></td>
<td></td>
<td>-0.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1.32)</td>
</tr>
<tr>
<td>(Aid/GDP) * fract. of land in tropics</td>
<td></td>
<td></td>
<td>-1.52**</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(4.02)</td>
</tr>
</tbody>
</table>

Observations 275 270 270
Countries 56 56 56
R^2 0.36 0.36

Sources: Burnside and Dollar (2002) for columns (1) and (2) (regressions (3) and (4) in the original paper; Dalgaard et al., (2004) for column (3) (column 5 in the original).

Notes: The dependent variable is real per capita GDP growth. All regressions include time dummies. Robust t-statistics in parentheses.* significant at 10%; ** significant at 5%.
This was not however a final word. In column 3 of table 5.1 we reproduce the results of another study, by Daalgard et al. (2004)\(^{18}\). The authors stressed the role of geography in explaining growth and accordingly they included the fraction of a country’s land located in the tropics as explanatory variable. In column (3), we see that the policy-aid interaction becomes insignificant, while aid and aid interacted with the climate became significant. These results suggest that aid has a positive impact on growth, but the impact decreases for countries located in the tropics. This last result is, of course, disappointing because it points to a critical role of geography - which cannot be changed by human actions - rather than of policy, which people can change.

In the last few years, many other studies have investigating the extent to which the impact of aid on growth is conditional on third variables. The main conclusion is that there is no definitive answer regarding the variable that better interacts with aid: some studies suggests it is policy, others point to the critical role of institutions, and some others to geography. This disparity of results suggests that the inter-play between aid, local circumstances and growth is eventually too complex to be captured by a simple estimation. It also points to the difficulty in establishing empirically the right determinants of economic growth, when the candidate variables are so many, as compared to the limited number of observations.

*Levels or changes?*

The AK model differs dramatically from the exogenous growth model, in terms of the relationship it establishes between the investment rate and economic growth. This prediction suggests an avenue to find out which model better complies with the real-world facts: investigating whether shifts in the investment rate have *permanent* or *temporary* effects on economic growth.

In practice, a major difficulty in disentangling whether the true model is the Solow model or the AK model is that the two models are observationally equivalent for long periods of time: remember that the Solow model predicts a positive relationship between investment and growth while the economy moves from one steady state to the other. Since this transition period can be quite long (in the MRW formulation, for instance, half of the transition dynamics takes as long as 35 years) and because most reliable datasets with comparable data start after 1950, it is not easy to assess whether changes in the investment rate have long run level effects or long run growth effects.

The most common approach to assess whether the true model is the AK model or the neoclassical growth model consists in testing for conditional convergence. The conditional convergence hypothesis states that countries converge to the respective steady states, and that the speed of convergence varies in direct proportion to the distance to the steady state. This property of the neoclassical model contrasts to the AK model, according to which parameter shifts alter the growth rate of per capita income permanently, without any tendency for per capita income to return to a previous path.

Tests on conditional convergence are typically implemented using cross-country growth regressions (see Box 5.3). Basically, the method consists in estimating the growth rate of per capita income as a function of a range of explanatory variables that control for different levels of $A$, and of the initial level of per capita GDP. Conditional convergence is assessed by the significance of the coefficient on the initial level of per capita income. If this coefficient is not significant, this means that changes in the variables proxying $A$ produce growth effects and eventually the AK model is true.

In general, the evidence with cross-country growth regressions using large samples of countries has been favourable to the conditional convergence hypothesis (an illustration in table 5.1). That is, the coefficient on the initial level of per capita income has been found to be, in general, negative and significant in cross-country growth regressions. This fact inspired Robert Barro to state: “It is surely an irony that one of the lasting contributions of endogenous growth theory is that it stimulated empirical work
that demonstrated the explanatory power of the neoclassical growth model” (Barro (1997), p. x).

Box 5.3. Cross-country growth regressions

Cross-country growth regressions are the workhorse of empirical research on economic growth\(^{19}\). Basically, the approach consists in estimating equations relating the growth rate of per capita GDP to a range of possible determinants. The later may include variables capturing factor accumulation (such as the investment rate and the population growth rate), and variables that are more likely to exert its influence through the productivity parameter, A. Examples of cross-country growth regressions are presented in Tables 4.2, and 5.1.

Growth regressions have a natural interpretation in terms of equation (5.5): in that model, growth appears as a function of the investment rate and of the productivity term, A. The advantage of cross-country growth regressions relative to simple growth accounting is that, rather than estimating A as a residual, they try to identify the policies and other factors that underlie the cross-country differences in A. Variables that have been found significant in empirical work include proxies for the quality of policies and institutions (such as trade openness, the rule law, political risk, inflation, financial depth) and of geographical conditions. The evidence in Table 5.1 illustrates this.

Cross-country growth regressions also have an interpretation in light of the neoclassical growth model. In that case, the regression equation is obtained substituting the steady state level of per capita income 4.18 in 3.14 (see Box 4.4). This procedure allows testing for conditional convergence, controlling for differences in the steady


09/11/2019
https://mlebredefreitas.wordpress.com/teaching-materials/
states, as implied for instance by investment rates in human and physical capital, population growth rates, or variables influencing the parameter A.

Putting the pieces together, the equation to be estimated in an extended neoclassical framework is:

\[ \ln y_t - \ln y_0 = a - b \ln y_0 + \psi X + \chi Z + u_t, \quad (5.25) \]

where X is a vector of variables capturing factor accumulation that are present in the MRW model (propensities to invest in physical and human capital, and the population growth rates) and Z is a vector of other variables determining the level of A.

Conditional convergence is assessed investigating the significance of \( b \): if its found to be zero, then changes in the other explanatory variables impact on the growth rate permanently, supporting the endogenous growth model (5.5); if, instead, \( b \) is found to be significant and negative, this suggests that growth rates are proportional to the distance to steady states, which accords to the idea of conditional convergence.

The Empirical analysis using cross country growth regressions face a number of limitations²⁰.

First, because the theory does not provide an unambiguous guide to the choice of elements of Z, there is a lot of uncertainty regarding the right model specification. In practice researchers have proposed more and more variables to complement the baseline specification, each one stressing a causal relationship between a particular variable and growth. This, in turn, brings a familiar econometric problem: because explanatory variables tend to be correlated to each other (countries performing badly in a given indicator also tend to perform badly in other indicators), there is a large scope for multicollinearity: the significance of each variable in the equation is influenced by the particular combination of variables included in the regression. This problem makes very

difficult to assess empirically which variable is more correlated to growth and how much (e.g., if inflation rates, exchange rate volatility and political instability go wrong together, how one can disentangle the various contributions to growth?).

Second, there is a problem of endogeneity: although it may appear natural that the parameter estimates ($\psi$ and $\chi$ in equation 5.25) contain information of causal effects on economic growth, this is not necessarily true. Some right-hand-side variables may be econometrically endogenous in the sense that they are jointly determined with the rate of economic growth: for instance, the same factors that make a country invest more in physical capital may also have a direct effect on a country growth rate. In that case, the estimated parameter will be biased and will provide little information regarding the direction of causality.

Third, even if all variables on the right-hand side were exogenous, many of them could be “symptoms”, rather than “syndromes”. For instance, consider the measurement of human capital. Shall we choose the secondary school enrolment or the primary school enrolment? Since these tend to be correlated to each other, they render one another insignificant when both are included in the regression equation. So which one should we choose? Moreover, a given symptom may be interpreted as capturing different syndromes. For example, a negative correlation between inflation and growth means bad macroeconomic management or a large tax evasion that forces the government to rely on revenues from money creation?

Fourth, there is a problem of parameter heterogeneity: parameter values estimated with cross section exercises that pool together very different countries may fail to accurately capture any of each. As once stated by Arnold Harberger: “What do Thailand, the Dominican Republic, Zimbabwe, Greece and Bolivia have in common that merits their being put in the same regression analysis?.

Fifth, the lack of a structural model stating how much the parameter $A$ depends on each policy variables makes it difficult to go beyond general statements on observed correlations and to provide a convincing interpretation of the results.
Other problems of cross-country-growth regressions include: the presence of outliers, measurement errors, and model linearity. Despite the extensive econometric improvements that have been adopted to overcome these limitations, the results of cross-country growth regressions have still to be taken with caution.

5.6. Discussion

The AK model stresses the relationship between policies and economic growth. This contrasts with the Solow model, whereby changes in the key parameters only produce level effects. The empirical evidence has not been, however, very favourable to the simpler version of the AK model. In general, country characteristics, such as the saving rate and aggregate efficiency are found to influence the levels of per capita income, rather than growth rates. This view is supported by an extensive empirical literature favourable to the conditional convergence hypothesis.

Does this mean that we shall abandon the AK model? The answer is no.

First, remember that the important link between efficiency and growth is also present in the neoclassical model: the difference is that in the later the growth effect will be transitory. That is, you may interpret the AK model as a short-run version of the neoclassical growth model. With half of the transition period between steady states in the neoclassical model taking as long as 35 years, whatever the true model is, we are doomed.

to accept that policy actions may influence economic growth for a considerable period of time.\(^\text{(22)}\)

Second, the AK model is much easier to solve than the Solow model. Because of this, from the expositional point of view, it is often more convenient to study the impact of particular policies in the context of the AK model than in the context of the Solow model, especially when the math becomes too complex. Of course, in doing so, one shall take into account that any conclusion regarding the impact of the policy on growth would be spelled out in terms of level effects, if adapted to the context of the neo-classical model. In some of the upcoming chapters, we will follow this approach.

Last, but not at all the least, the basic AK model illustrates how linearity avoids the basic problem of diminishing returns, generating long-term growth. Linearity is a basic feature of most endogenous growth models, including those focusing on technological change, like the Usawa model explored in this chapter. The AK model can therefore be interpreted as a toy version of more complex endogenous growth models, whereby knowledge expands through investments in R&D. Knowledge shares with capital the characteristic that it can be built over time by sacrificing some of today’s consumptions. So, interpreting investment as foregone consumption in a broad sense (that is, including physical assets, human capital and R&D), one can interpret the AK model as a general framework to think the mechanics of economic growth through technological change.

\(^{22}\) Easterly (2005) calibrated a simple neoclassical growth model with a share of total capital equal to 2/3 (which accords to MRW) and with other reasonable values for the remaining parameters. He found that a tax decrease from 30% to zero raises per capita income by a factor of 2.25 times. The author also showed that immediately after the change in policy, the growth rate of the economy jumps up by almost 8 percentage points relative to its steady state. Only in the very long run (more than 5 decades after), the growth effect wears off and the growth rate returns to its long run level. The author concluded that policies have significant effects in the neoclassical model, too (pp. 1024-1026).
Key ideas of Chapter 5

- The AK model reveals in a simple manner that getting rid of diminishing returns, factor accumulating alone can generate continuous growth of per capita income. In the context of the AK model, changes in the saving rate produce “growth effects” rather than “level effects”.

- Extending the AK model to the case with endogenous savings, the direct effect of aggregate efficiency on growth is reinforced by an indirect effect via a higher return on savings. The implication is that, wherever financial markets are more developed, the impact of policy changes on growth is more dramatic.

- The model with endogenous savings appeals to the distinction between proximate causes of growth, like the savings rate and aggregate efficiency, and fundamental causes of growth, that determine why some countries have higher investment rates and better efficiency than others.

- Extending the model in a variety of ways, we found that one can interpret K in a broad sense, including other reproducible inputs to production and technology.

- In its simpler formulation, the AK models displays no transitional dynamics. There are some hybrid models, however, like the Usawa model and an extended version of the neo-classical model that at the same time have transition dynamics and display unceasing growth.

- The Harrod Domar equation inspired the idea that complementing low domestic savings in poor countries by foreign aid would be a key to generate economic growth. In practice, however, the impact of external aid on the growth varied significantly across countries, depending on the quality of domestic policies, institutions, and geography.
• The empirical evidence of conditional convergence has been more favorable to the neo-classical model than to the simpler versions of the AK model whereby higher saving rates generate faster economic growth.

• The AK model can be seen as a toy version of more complex models of endogenous growth based on technological change.

**Appendix 5.1 Unbalanced growth in the HD model**

Because prices play no role in the Harrod-Domar model, no force will act to eliminate unemployment of labour or of capital. To see the long-run implications of this, let’s consider first the knife-hedge case in which the parameters of the model are such that output and population grow exactly at the same rate (that is, \( sA - \delta = n \)). In that case, the capital-labour ratio and per capita income remain constant over time. In terms of figure 5.1, this exceptional case occurs when the break-even investment line coincides with the saving line. Thus, any starting point will be a steady state.

Note however that even in this exceptional case, the economy will not in general grow with full employment. To see this, consider Figure A5.1: if, for instance, the economy started out with labour surplus (point S), then it would move along a path with a constant capital-labour ratio but with increasing unemployment (\( k_s \) in the figure). The only case in which the economy expands along the full employment locus is when \( sA - \delta = n \) and simultaneously the economy starts out without surplus labour (point R).

A less fortunate case occurs when \( sA - \delta < n \). In that case, the economy does not save the enough to keep the capital labour ratio unchanged. In terms of (5.5), we see that in this case, the growth rate of per capita income is negative. In terms of figure A5.1, if the economy starts out in point S, surplus labour will be increasing over time along the path \( k_U \), implying chronic underproduction.
The best scenario in the HD model occurs when the parameters in the economy are such that the capital stock grows faster than population (that is, when $sA - \delta > n$). In this case, per capita income increases until the surplus labour is completely eliminated. Still, the mechanics of the model is such that per capita income cannot grow indefinitely. The reason is that at the time the full employment line is crossed (point R in Figure A5.1), the binding constraint in production becomes the availability of labour (that is, the relevant segment of the production function in (5.11) shifts to $Y = BN$). Hence, beyond this point output will be bound to expand at the same rate as population, implying a constant level of per capita income thereafter.

To see how the steady state in that case looks like, let’s divide both terms of $BN$ by $K$, and substitute this for $Y/K$ in the Harrod-Domar equation (5.5). The growth rate of the economy in this segment becomes: $\gamma = \frac{sB}{n + \delta}$. Since this expression depends negatively on $k$ (that is, as $k$ rises, its growth rate declines), this segment of the model has a stable equilibrium. Solving for $\gamma = 0$, one obtains the steady state level of capital per worker: $k^* = \frac{sB}{n + \delta} > B/A$. The implication is that, after

Figure 5.4. Unbalanced growth in the Harrod Domar Model

![Figure 5.4. Unbalanced growth in the Harrod Domar Model](image-url)
crossing the full employment locus, the economy will evolve along the path \( k = k^* \), with unemployment of capital (path \( k^* \) in Figure A5.1).

**Problems and Exercises**

**Key concepts**

- Aggregate efficiency
- Harrod Domar equation
- Level effect versus growth effect
- Proximate versus fundamental causes of economic growth
- Returns on a broad concept of capital
- Research sector
- Cross-country growth regressions

**Essay questions:**

a) Referring to the Harrod-Domar equation, compare the AK model and the Solow model in respect to the variables that are exogenous and endogenous. In particular, examine the impact of an increase in the saving rate in light of the two models.

b) Comment: “Poor countries, with underdeveloped financial markets, are more likely to tolerate bad policies than rich countries with developed capital markets”.

c) Explain why the Usawa model is hybrid. In the context of this model, which policies could influence the rate of economic growth?

**Exercises**

5.1.
Consider an economy where the production function is given by \( Y = AK \). In this economy, the saving rate is \( s \), the population grows at rate \( n \) and the capital depreciation rate is \( \delta \).

a) Does this production function satisfy the usual neoclassical properties? Why?

b) Describe analytically and graphically the dynamics of per capita income in this economy. Is there any stable equilibrium?

c) Does this model predict convergence of per capita incomes across economies?

d) Describe, comparing with the Solow model, the impact of: (i) a fall in the population growth rate; (ii) An increase in \( A \).

5.2.

Consider an economy, where the production function is given by \( Y=0.2K \), the population grows at 2% per year, the capital depreciates at 3% and the saving rate is 25%.

a) Find out the growth rate of per capita income in this economy.

b) What will be the effect of \( A \) increasing to 0.25?

c) Now assume that the saving rate was endogenous, as implied by the following optimal consumption rule: \( \gamma_t = r_t - 0.17 \). Analyse in this case the implications of an increase in efficiency from 0.2 to 0.25.

d) Comparing the two models, find out the expression that relates the saving rate to efficiency \( A \). Explain why a change in the efficiency parameter \( A \) impacts more on growth when savings are endogenous.

5.3.

In Micronésia, the aggregate production function is given by \( Y = K^{0.5}H^{0.5} \), where \( H=hN \), \( N \) is the number of workers, and \( h \) measures the amount of human capital per worker. In this economy, the saving rate is given by \( s=25\% \), the population is constant and the rate of depreciation of physical capital is equal to 5%.

a) Assume for the moment that \( h=l \). Find out the equilibrium values of \( k=K/N \) e \( y=Y/N \). Explain the dynamics of the model with the help of a graph.
Assume now that $\dot{h} = s_h y - \delta h = 0.25 y - 0.05 h$.

b) Explain this specification.

c) In this case, the properties of the neoclassical model are satisfied? Why?

d) Find out the growth rate of per capita income in this economy.

e) Compare, in the light of both models: (i) the short run and the long run effects of a rise in the saving rate (ii) The convergence hypothesis

5.3.

Consider the following production function and law of motion of per capita consumption:

$$Y_t = A_t K_t^\beta N_t^\alpha H_t^{1-\beta-\alpha}, \text{ with } \alpha, \beta \leq 1$$

$$\gamma = r - \rho.$$

Assume that the depreciation rate is identical for the two capital types and that population does not grow over time.

a) Suppose that $\alpha + \beta = 1$, $\alpha, \beta \neq 0$.

i. Explain if it is possible to obtain sustained growth of per capita income in the long-run through factor accumulation.

ii. Describe the impact of an increase in $\rho$ in the interest rate and in per capita income.

b) Suppose that $\alpha + \beta < 1$, $\alpha, \beta \neq 0$. Discuss the advantages of this parameterization comparing them to the results obtain in (a).

c) Finally, suppose that $\alpha = 0$, $\beta < 1$.

iii. Explain if it is possible to obtain sustained growth in the long-run through factor accumulation.

iv. Describe the impact of an increase of $\rho$ in the interest rate and in per capita income.

5.4.

Consider a closed economy without government, where population is equal to one thousand inhabitants, and constant over time. In this economy, the relevant production
function is given by \( Y = 0.5K \), capital deteriorates at the rate of \( \delta = 0.04 \) per year, and consumption is a linear function of income, according to \( C = 8000 + 0.84Y \).

a) Suppose the initial capital stock in this economy was exactly \( K = 200,000 \). In this case, per capital income: (i) will be growing over time; (ii) will be decreasing over time; (iii) will be stagnant.

b) Suppose that, due to some external support, the capital stock in this economy jumped temporarily to \( K = 250,000 \). In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04.

c) Suppose this economy started out with a capital stock equal to \( K = 250,000 \), but the production function was actually given by \( Y = \min\{0.5K, 150N\} \). In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04.

d) In the conditions set out in (c), the long run value of the capital-labour ratio will be: (i) decreasing over time; (ii) \( k=300 \); (iii) \( k=400 \); (iv) increasing over time.

5.5.

Consider an economy where the production function is given by \( Y_t = (1-\mu)^{\lambda/2} K_t^{\lambda/2} (N\lambda)^{\mu/2} \), where \( \mu = 0.75 \) is the fraction of time devoted to production. In this economy, the saving rate is 15%, the population is constant and capital does not depreciate. The productivity of labour accumulates at the rate \( \gamma = \dot{\lambda}/\lambda = b\mu \), where \( b=0.02 \).

e) Explain the equation describing technological progress.

f) Using the equation describing the change in \( k = K/L \) (the fundamental dynamic equation), find out the steady state values of \( \dot{y} \) and \( \dot{k} \).

g) Examine the implications of an increase in the saving rate from \( s=15\% \) to \( s=18\% \). In particular, compute the new equilibrium values of \( \dot{y} \) and \( \dot{k} \). Describe the change in a graph and explain what will happen to the interest rate.

h) Returning to the initial figures, examine the implication of an increase in \( b \) to \( b=0.04 \). In particular, compute the new equilibrium values of \( \dot{y} \) and \( \dot{k} \). Describe the change in a graph and explain what will happen to the interest rate.

i) Compare the effects on the path of per capita income, \( y=Y/N \), of the changes described in c) and in (d).
5.6.

Consider an economy where the aggregate production function is given by $Y_t = A_t K_t^{1/2} N_t^{1/2}$. In this economy, the saving rate is 20%, capital depreciates at 5% per year, and population is constant and equal to $N=100$.

a) Assume that $A_t = 1$. (Find out the steady state values for: (a1) per capita income; (a2) interest rate; (a3) capital and labor income shares; (a4) wage rate. (a5) To what extent does this model comply with the Kaldor stylized facts? (a6) Represent the equilibrium in a graph and discuss its stability.

b) Departing from (a), assume instead that $A_t = h_t^{0.5}$. (b1) Will the implied production function have the neo-classical properties? Why? Assume that human capital accumulated according to $\dot{h} = 0.018y - 0.05h$. (b2) Explain this equation. (b3) Assuming again a saving rate on physical capital equal to 20% and a depreciation rate for physical capital equal to 5%, what would be the growth rate of this economy in the long run? (b4) Describe the dynamics of the model with the help of a graph.

5.7.

Consider and econony where $Y_t = K_t^{1/2} [(1 - \mu) (N \lambda)]^{1/2}$, with $\mu = 0.36$ being the time devote to work. In this economy, the saving rate is 20%, population is constant and the depreciation of physical capital is 2.2% per year. Human capita per worker evolves according to $\gamma = \lambda / \lambda = b \mu$, with $b=0.05$.

a) Explain the equation describing the accumulation of knowledge.

b) From the equation that describe the dynamics of $\dot{k} = K / L$, find out the equilibrium levels of $\bar{p}$. Represent in a graph.

c) In the steady state, per capita income and the interest rate evolve according to the Kaldor stylized facts?

d) Analyse the effects if a change in $\mu$ to 0.64. (f1) Which phenomenon this change is intended to capture? Quantify the new steady state and show the change in a graph.

e) Comparing to (b), what happens to per capita income and to the interest rate? Discuss.