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The AK model

“…a level effect can appear as a growth effect for long periods of time, since adjustments in real economies may take place over decades”. [Sachs and Warner].

Learning Goals:

• Understand why getting rid of diminishing returns one can obtain unceasing growth through factor accumulation.
• Review different models were simple factor accumulation can generate endogenous growth.
• Understand why in AK models an endogenous investment rate makes a difference.
• Understand the similarity between investment and R&D in terms of forego consumption.
• Acknowledge the empirical controversies underlying the abandonment of the diminishing returns assumption.

5.1. Introduction

Along the previous chapters, we learned that, if there are diminishing returns to the reproducible factors, factor accumulation cannot, by itself, explain the long-term growth of per capita income. For this reason, growth in the neoclassical model is achieved only by postulating an exogenous rate of technological progress.

In this chapter it is shown that, by getting rid of diminishing returns, one can obtain continuous growth of per capita income without the need to postulate an exogenous rate of technological progress. The basic model introduced in this chapter is the AK model. The AK model differs critically from the Solow model in that it relies on a
production function that is linear in the stock of capital. With this change, the model implies a sustained growth of per capita income without any tendency for the economy to approach a steady state. In this model, a rise in the saving rate has a proportional effect on the growth rate of per capita income, on a permanent basis. This contrasts with the Solow model, where a rise in the saving rate only delivers a “level effect”.

The pitfall of the AK model is that the assumption of diminishing returns plays a very central role in economic thinking. Hence, it shall not be abandoned without a well motivated story. Some of the sections below and in the following chapter describe alternative theories and models that have been proposed to motivate the abandonment of diminishing returns in endogenous growth models. This chapter also explains how endogenous growth can be obtained in the context of a neoclassical model with diminishing returns to capital. Although none the models described in this chapter shall be seen as the true model, they all offer avenues to think about the various ingredients that are behind economic development.

Sections 5.2 describes the AK model in its simpler formulation. Section 5.3 extends the AK model to the case of endogenous savings. Section 5.4 reviews alternative models that emulate the AK model. Section 5.5 reviews some empirical controversies regarding the impact of policies and savings on economic growth. Section 5.6 concludes.

5.2. The basic AK model

Getting rid of diminishing returns

Consider a closed economy where the population growth rate, the savings rate and the rate of depreciation the capital stock are all constant. In contrast to the Solow model, however, production is a linear function of the capital stock (K):

\[ Y_t = AK_t, \quad A > 0 \] (5.1)
In (5.1), the parameter $A$ stands for the level of technology (or *aggregate efficiency*), and is assumed constant.

Comparing (5.1) with (2.1), we see that now production depends only on the reproducible factor and that there are no diminishing returns to this factor. The reader may get suspicious about this formulation. Doesn’t labour have any role in production? Actually, we don’t need to rule out a role for labour input: in the following sections, we will review different models that, while accounting for labour in production, emulate this simple AK model. For the moment, just stick with this simple formulation.

Dividing (5.1) by $N$, one obtains a linear relationship between per capita income and capital per worker:

$$ y_t = Ak_t $$

(5.2)

The remaining equations of the model are the same as in the basic Solow model:

$$ Y_t = C_t + S_t $$

(2.5)

$$ sY_t = I_t $$

(2.7)

$$ \dot{K}_t = I_t - \delta K_t $$

(2.8)

$$ n = \frac{N_t}{N_t} $$

(2.9)

From (2.7), (2.8) and (2.9), we have the dynamics of the capital labour ratio:

$$ \dot{k}_t = sAk_t - (n + \delta)k_t $$

(5.3)

This equation is similar to (2.14), with the only difference that now $\beta=1$. This small difference has an important implication: since both the production function and the break even investment line are linear in $k$, only by an exceptional coincidence of parameters would the two curves cross each other. Hence, the general case in the AK model is one without steady state.
In particular, if \( sA > (n + \delta) \), the capital labour ratio will expand forever, at a constant growth rate. Note that this conformity with real world experience is now achieved without the need to postulate any exogenous technological progress.

Dividing (5.3) by \( k \), one obtains the equation that describes the growth rate of capital per worker in this economy:

\[
\frac{\dot{k}}{k} = sA - (n + \delta)
\]

Since output is linear in \( K \), the growth rate of capital per worker is also the growth rate of per capita income. That is:

\[
\gamma = sA - (n + \delta) \quad (5.4)
\]

This equation states that the growth rate of per capita income rises with total factor productivity \( A \) and the saving rate \( s \) and declines with the depreciation of the capital-labour ratio \( (n \) and \( \delta) \). Because the growth rate of per capita income in this model is influenced by the other parameters, instead as being given, the model is labelled as of endogenous growth.

**A Graphical Illustration**

Figure 5.1 describes the dynamics of the AK model. The horizontal axis measures the capital labour ratio \( (k) \). The vertical axis measures output per capita \( (y) \). The top line shows the production function in the intensity form, (5.2); the middle line corresponds to gross savings per capita (the first term in the right hand side of 5.3); the lower of the three lines is the break-even investment line (the second term in the right hand side of 5.3).

Since the production function is now linear in \( K \), the locus representing gross savings never crosses the break-even investment line (compare with Figure 2.3). This means that there is no steady state: as long as \( sA > n + \delta \), per capita output will grow forever.
When the production function is linear, the curve describing per capita savings never crosses the break even investment line. Hence, the capital-labour ratio will increase forever and so will do per capita output.

What happens when the saving rate increase?

The AK model differs drastically from the Solow model, in that change in the exogenous parameters alter the long run growth rate of per capita income, rather than the level of per capita income.

Figure 5.2 compares how the paths of per capita income would differ in first the Solow model with exogenous growth and second in the AK model, following a once-and-for-all rise in the saving rate at time $t_0$ (the case with the Solow model was already discussed in detail in Figure 3.3). The top part of the diagram shows levels and the bottom part shows growth rates.
The figure compares the responds of the AK model versus the Solow model to an exogenous increase in the saving rate. While in the Solow model this gives rise to a level effect, in the AK model, there is a growth effect. The figure also suggests that, in the short term, the Solow model and the AK model produce similar predictions.

Comparing the two models, one concludes that:

- In the Solow model, the rate of growth of per capita income jumps initially to a higher level, but then it declines slowly over time, until returning to the previous level (given by the exogenous rate of technological progress) (after \( t_1 \)). Because of diminishing returns, the long run growth rate of per capita income is independent of the saving rate. Remember that the model without exogenous growth (Chapter 2) is just a particular case, with \( \gamma = 0 \).
- In the AK model, the rise in the saving rate has a permanent effect on growth: there is no tendency for the growth rate of per capita income to decline as time goes by. The growth rate of per capita output is proportional to the saving rate.

**The Harrod-Domar equation**

A useful way to compare the AK model with the Solow model is looking at the long run relationship between the average product of capital and the growth rate of per capita income, in light of the two models (equations 3.11 and 5.4). For convenience, let’s use (5.1) in (5.4), to obtain a formula that applies to both models in the long run:

$$\gamma = s \frac{Y}{K} - (n + \delta)$$  \hspace{1cm} (5.5)

This equation is known as the Harrod-Domar equation. The difference between the two models refers to the variables that are exogenous and endogenous in this equation. In both models, $s$, $n$ and $\delta$ are exogenous. But the two models differ in respect to the exogeneity of $\delta$ and $Y/K$: In the AK model, $Y/K$ is exogenous and $\gamma$ is endogenous. By contrast, in the Solow model, $\gamma$ is exogenous and $Y/K$ is endogenous.

Hence:

- In the Solow model, a rise in the saving rate leads to a lower average productivity of capital in the steady state. That is, $Y/K$ declines from one steady state to the other (Figure 3.2).

- In the AK model, $Y/K$ is constant (equal to $A$). Hence, a rise in the saving rate can only be accommodated in the model by an increase in the growth rate of per capita income, $\gamma$.

Because the AK model predicts changes in $A$ or in the saving rate to produce growth effects, it goes far beyond the neoclassical model in stressing the relationship between economic policies and economic growth: government policies, such as taxes and
subsidies, that affect economic efficiency and consumption-saving decisions may alter the long run’ rate of economic growth.

No convergence

Another important implications of the AK model is that it does not predict convergence of per capita incomes, even among similar economies. According to (5.4), two economies having the same technology and savings rates will enjoy the same growth rate of per capita income, regardless of their starting position. This means that their per capita incomes will evolve in parallel and there will be no tendency for the poorer economy to “catch up” with the richest economy.

Moreover, since changes in technology (A) and in the saving rate (s) affect growth rates permanently, the most likely case is that countries with different parameters will grow apart. In a world where policies differ substantially across countries, the rule should be that of divergence of per capita incomes, rather than of convergence.

5.3. The AK model with endogenous savings

Thus far, the saving rate in the AK model has been assumed exogenous. In this section we show that, when the model is modified so as to allow individuals to optimally trade consumption today for consumption in the future, a second channel linking efficiency to growth is opened up.

Adding an optimal consumption rule to the AK model

In what follows, let’s recall the simplest possible optimal consumption rule, introduced in Section 2.6:

$$\gamma = r - \rho$$  \hspace{1cm} (5.7)
This equation states that, as long as the interest rate is higher than the rate of time preference, there will be incentive for economic agents to increase consumption over time. This, in turn, is achieved through a higher saving rate.

Since the AK model has no transitional dynamics, consumption and income evolve in parallel each moment it time. Hence, (5.7) can be seen as describing simultaneously the growth rate of per capita consumption and the growth rate of per capita income.

To find out how the growth rate of per capita income relates with the remaining parameters of the AK model, one needs to solve for the interest rate. As before, we assume that firms are perfectly competitive and maximize profits. In this case, capital will be paid its marginal product, \( A \). That is:

\[
r + \delta = A \quad (5.8)
\]

Substituting (5.8) in (5.7) and rearranging, one obtains:

\[
\gamma = A - \delta - \rho \quad (5.9)
\]

This equation describes the growth rate of per capita income in an AK model where consumers are allowed to trade consumption today for consumption in the future at a given interest rate, \( r \). Comparing to (5.4), you see that now it is the rate of time preference, instead of the saving rate, that determines the rate of economic growth.

Transpiration responds to inspiration!

From the qualitative point of view, equation (5.9) brings no novelty relative to the case with exogenous savings, (5.5): a lower rate of time preference (that is, a change in consumption preferences in favour of more consumption in the future and less consumption today), by raising the saving rate, leads to a higher rate of capital accumulation and a higher growth rate of per capita income.

However, comparing (5.9) to (5.5), we observe that the impact of \( A \) on growth is now much larger than in the case with exogenous savings. For instance, with a saving
rate equal to 20%, the impact of a unitary change in $A$ on growth according to equation (5.5) is 0.2. In equation (5.9), the impact of a unitary change in $A$ on growth is one. That is: five times more.

What makes the assumption of consumption smoothing so powerful that it can alter dramatically the relationship between the efficiency parameter and growth? The point is that, when $A$ rises, there are two effects:

- On one hand, when $A$ rises for a given $s$, the growth rate of per capita income rises, just like in (5.5);

- On the other hand, when $A$ rises, there is now an additional effect through the interest rate, $r$: a higher marginal productivity of capital translates into a higher return on capital and this, in turn, will induce a higher saving rate, for each rate of time preference. Then, with a higher saving rate, the economy will grow faster.

Formally, one may substitute (5.9) in (5.5) and solve for $s$, to obtain the (endogenous) saving rate: $s = 1 - \left( \rho - n \right) / A$ (the restriction $\rho > n$ is necessary for the problem to be well-behaved). Taking the partial derivative in respect to $A$, we verify that the impact of a change in $A$ on the saving rate is $\partial s / \partial A = (\rho - n) / A^2$. The total impact of a change in $A$ is the sum of the direct impact of $A$ on $\gamma$ with the indirect impact, through $s$: $d\gamma / dA = \partial \gamma / \partial A + \left( \partial \gamma / \partial s \right) \left( \partial s / \partial A \right) = s + (\rho - n) / A = 1$.

This finding is of the upmost importance to understand the mechanics of many endogenous growth models. A typical assessment based on equation (5.5) is that a country may either grow through “inspiration” ($A$) or through “transpiration” ($s$). But, we just saw that “transpiration” responds to “inspiration”: that is, a more efficient resource allocation, leading to a higher marginal product of capital, implies a higher return on investment. Thus, agents will be willing to forego a higher proportion of their consumption to save more.

With this finding, one may rewrite equation (5.5) in the following form:
\[ \gamma = s \left( \rho, A \right) A - \left( n + \delta \right) \]  

(5.10)

*It depends!*

It is important to distinguish the circumstances in which one shall refer to equation (5.10) from the case in which (5.5) is more appropriated.

Remember that the rule (5.9) presumes that the financial system is well developed, so that households are able to smooth their consumption over time. When instead the financial system is underdeveloped and households face borrowing constraints, equation (5.5) is with no question more appropriated.

The implication is that bad economic policies (as reflected in low levels of A) are likely to impact more severely in countries with developed financial systems than in countries with underdeveloped financial systems. Putting in other way, in countries with inefficient financial systems, people are more likely to tolerate bad government policies. This, in turn, may help perpetuate the bad policies!

This discussion adds to the general point that questions like “what happens to per capita income (or to economic growth) when some parameter increases” do not have a unique answer: it depends on a country economic conditions.

*Proximate versus fundamental causes of economic growth*

According to equation (5.4), a low rate of economic growth can be explained either because a country does not invest enough (s) or because it does not achieve a minimum productivity of capital (A). Dealing with the development question at a deeper level, however, one may like to understand why some countries save and invest more than others and why some countries reach higher returns on their investment than others. In other words, one would like to take as endogenous the parameters that the models so far see as exogenous.
To some extent, equation (5.10) is a step in that direction: according to this equation, individuals will save more wherever the productivity of capital is higher\(^1\). The following chapters will be devoted to a better understanding of what is behind parameter \(A\), which we have by now loosely relate to “efficiency” and “technology”.

In this quest, we will relate the level of \(A\) to the quality of economic policies and institutions. We will argue that countries with sound economic policies are expected to achieve higher efficiency levels and to employ better technologies than countries with poor economic policies.

But another question will immediately arise: why do some countries implement better policies than others? To answer this question, we need to address the incentives of policymakers. These, in turn depend on the quality of political institutions. These, in turn, are grounded in even deeper factors underlying human societies, such as social norms, culture and geography.

In a word, as one deepens the analysis, we move from the *proximate* causes of economic growth, which are captured by the parameter values in equation (5.4), to the *fundamental causes* of economic growth, which ultimately determine why in a given country the parameter values are what they are. These fundamental causes are essential to understand why some societies make choices that lead them to benefit from better policymaking and to adopt more modern technologies.

This is not to say that simple models like (5.4) are useless. On the contrary, they are essential to understand *the mechanics of economic growth*. In particular, the role of investment and technology as *mediators* between a country fundamental characteristics and its economic performance. But dealing with the development question at a deeper

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\(^1\) Equation (5.10) stresses the causality from “inspiration” to “transpiration”. However, the reversal may also be true. In Chapter 6 we’ll address precisely some theories according to which the level of \(A\) is enhanced by capital accumulation. The possibility of mutual causation implies that savings and efficiency may reinforce each other, both positively and negatively, raising the possibility of multiple equilibria and poverty traps.
level, one may want to understand what is behind the parameters that the simple models we are using take as exogenous.

5.4. Incarnations of the AK model

The Harrod-Domar model

The true predecessor of the AK model is a model that was developed independently by two economists, Roy Harrod, and Evsey Domar\(^2\). The Harrod Domar model preceded that of Solow by several years and obviously it was not motivated by any explicit intention to improve on the Solow model. Actually, the HD model was developed in the aftermath of the Great Depression, as a dynamic extension of Keynes’ general theory, with the aim to discuss the business cycle in the U.S. economy. Since at that time, unemployment was very high, the focus of the model was on the relationship between investment in physical capital and output growth.

The main assumption of the Harrod-Domar model is that capital and labour are pure complements, meaning that they cannot substitute for each other in production. The underlying production function is of the Leontief type:

\[
Y_t = \min\{A K_t, B N_t\},
\]

where A and B are positive constants.

The Leontief contrasts with the Cob-Douglas production function, where inputs are also essential to production, but they can substitute for each other. As an illustration, suppose that you were producing meals (Y) each one consisting in one steak (K) and two eggs (N). That is, to produce any amount of this output you would need steaks and eggs

in a proportion of two eggs per one steak and you could not get around this rule: if you did, you would not be producing meals labelled as “steaks with two eggs”.

Figure 5.3: The Leontief production function

![Leontief production function diagram]

The figure describes two isoquants where inputs to production are complementar. In this case, increasing one input only does not deliver a higher level of production. The straight line $A/B$ corresponds to the efficient combinations of the two inputs. If the economy endowments are at the right hand side (left hand side) of this line, there will be unemployment of labour (capital).

Figure 5.3 illustrates this, by displaying the isoquants corresponding to $A=1$ and $B=0.5$. Thus, to produce one meal ($Y=1$), you need at least one steak and two eggs (Point R). If you had one steak and 8 eggs, your maximum production would still be equal to one meal (point S). Now think that this production function applies to the economy as a whole and that K and N refer to capital and labour. If the economy’ endowments were K=1 and N=8, the economy would be producing $Y=1$ only, and there would be unemployment of labour (in point S, for instance, 6 unit of labour are wasted). From that point, expanding the endowments of amount of will not deliver higher output. If however we managed to increase the stock of capital to $K=2$, the output level would jump to $Y=2$ (point T), and unemployment would be reduced to 4 units of labour. Raising production
by incrementing the stock of capital \((K)\) in an economy with surplus labour \((N)\) is basically how the Harrod-Domar model works.

The main limitation of the Harrod-Domar is that factor prices play no role in driving the economy towards full employment. This contrasts with the Solow model, where price flexibility ensures full employment each moment in time. Hence, there will be unemployment of labour or unemployment of capital, depending on whether the proportion of labour relative to capital in the economy as a whole starts out with more than \(A/B\) or with less than \(A/B\).

The case of interest is when the economy is endowed with too much labour. Since labour cannot substitute for capital, the existence of excess supply of labour implies that the only way to expand production is by increasing the stock of physical capital. In that case, the relevant branch of the production function (5.11) is the first one, implying a linear relationship between output and \(K\), just like in (5.1). Then, given the exogenous saving rate and the population growth rate, from (2.5)-(2.9), you’ll obtain the growth rate of per capita income as described by (5.4).

Depending on the model parameters, however, it may be that (5.4) implies a positive growth rate of per capita output, in which case capital expands faster than labour. Hence, by a favourable combination of parameters, the excess labour will be eventually eliminated, and the economy will move to the other branch of the production function, with unemployment of capital and the growth rate of output limited by the availability of labour. In that case, the growth rate of per capita output becomes zero, just like in the basic Solow model (see Appendix 5.1 for details).

* A one sector model with Physical and Human Capital

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21/02/2016

http://sweet.ua.pt/afreitas/growthbook/capa.htm
Another way of accounting for the role of labour in production and obtain the AK model is by considering constant returns apply to a broad concept of capital, which includes both physical and human capital$^3$.

To see this formally, consider the following production function:

$$Y = AK^\beta H^{1-\beta}$$  \hspace{1cm} (5.12)

In this production function, there are diminishing returns to physical capital and to human capital in isolation, but there are constant returns to scale in reproducible factors. This contrasts to the Solow and the MRW models, where returns to reproducible factors are decreasing.

Also note that this production function does not necessarily exclude raw labour. Indeed, one may think human capital, $H$, as measuring quality adjusted labour, that is, the number of workers, $N$, multiplied by the human capital of the typical worker ($h$):

$$H = hN.$$  \hspace{1cm} (5.13)

The implied assumption in (5.13) is that the quantity of workers, $N$, and the quality of workers, $h$, are substitutes. With such a specification, raw labour needs no longer to be a source of diminishing returns: multiplying the use of $h$ and $K$ by two causes the production level $Y$ to double, even if $N$ is held constant. Hence, provided the two capital inputs grow at the same rate, the CRS property will assure the linearity between production and reproducible factors that characterizes the AK class of models.

To see this formally, let’s return to the MRW assumptions that people save a constant fraction of their incomes in the accumulation of human capital, just like they do for physical capital$^4$:

$$\dot{K}_t = sY_t - \delta K_t$$  \hspace{1cm} (5.14)

---


$^4$ In alternative, you can solve the model assuming endogenous savings.
Because of diminishing returns to each type of capital, it doesn’t make sense for people to accumulate one type of capital faster than other. Hence, the two types of capital will be expanding at the same rate \( \dot{K}/K = \dot{H}/H \). Using (5.14) and (5.15) this implies

\[
\frac{\dot{H}}{K} = s \frac{\dot{H}}{s} \quad (5.16)
\]

Substituting (5.16) in (5.11), the following variant of the AK production function obtains:

\[
Y = A \left( s \frac{\dot{H}}{s} \right)^{1-\beta} K \quad (5.17)
\]

Comparing to (5.1) we see that now the average product of capital embodies the propensity to invest in human relative to physical capital (that is, you can look at \( A \) in equation 5.1 as including this effect)\(^5\).

Using this in (5.5), the growth rate of per capita income in this variant of the AK model is:

\[
\gamma = s^{\beta} s h^{1-\beta} A - n - \rho, \quad (5.18)
\]

This equation shows that it is perfectly possible to have diminishing returns to physical capital alone and yet having sustained growth of per capita income. What we need is to have constant returns to all types of capital (or reproducible inputs) when considered together. Note that the MRW model differs from this one in that the non-reproducible factor (N) cannot be replaced by human capital: in the MRW the returns on K and H when considered together are decreasing.

---

\(^5\) The implication is that, if two countries differ in terms of these weights, then the one investing more in human capital will show up with a higher productivity parameter.
The models above generate endogenous growth by abandoning the assumption of diminishing returns. This is not, however, a necessary condition: one may generate endogenous growth even without departing from the assumption of diminishing returns to capital\(^6\).

To see this, let’s consider again the optimal consumption rule (5.7): as already explained in Section 2.6 (Figure 2.12), the Solow model cannot deliver long-run growth because the marginal product of capital falls down to zero as the capital labour ratio increases: at the time the interest rate equals the discount rate, the desired consumption becomes constant over time and the process of capital accumulation stops.

These considerations suggest an avenue to generate endogenous growth: what we need is to prevent the interest rate from falling below the rate of time preference. In the AK model, this is possible because the marginal product of capital is a constant parameter. Thus, as long as \( A > \delta + \rho \), per capita income will grow forever.

The same can be achieved in the context of the neoclassical model. Note that the assumption of diminishing returns only requires the marginal product of capital to be a decreasing function of the capital stock. The Solow model goes a bit further, by postulating an aggregate production function (as exemplified by the Cobb-Douglas) with marginal returns falling asymptotically to zero. If however the marginal product of capital never approached zero, the model could display endogenous growth.

Thus, the only requirement to generate sustained growth of per capita income in the neoclassical framework is to postulate that the marginal product of capital is bounded below by a positive constant. In that case, as the amount of capital per worker increases,

the marginal product of capital approaches that constant. If it happens that this constant is higher than the rate of time preference, then the economy will expand without bound\(^7\).

As an example, consider the production function \(Y = AK + BK^{\beta}N^{1-\beta}\). This production function exhibits diminishing marginal returns. It converges however asymptotically to the AK form. In other words, the average product of capital is bounded below by the parameter \(A\). Thus, as long as \(A > \delta + \rho\), the model will display unceasing growth.

A neoclassical growth model, suitably modified along these lines is capable of generating at the same time endogenous growth (as the AK model) and transition dynamics. In such a model, two economies differing only in terms of their initial per capital incomes will exhibit a tendency to approach each other, with the one with less capital per worker growing faster. At the same time any government policy that was successful in raising the saving rate would have a permanent effect in the growth rate of per capita income. This is why this class of models is labelled \textit{neoclassical models of endogenous growth}.

\textbf{A two sector model of endogenous growth}

An alternative version of the AK model that does not rely on constant returns to capital was proposed by Usawa, as early as in 1965\(^8\). The author extended the Solow model, considering two sectors of production, one for consumption goods and the other for “technology”. In this model, the linearity that is needed to generate endogenous growth arises in that the production function for technology does not exhibit diminishing returns. This section reviews the main assumptions of that model.

\(^7\) Actually, this is what we did by introducing exogenous technological progress (Chapter 3): the effect of technological progress is to raises the productivity of capital, offsetting the diminishing returns. In the long run the interest rate is kept above the discount rate and the economy displays positive growth forever.

The economy has two sectors, the production sector and the R&D sector. The production sector employs both labour and capital, and produces goods and services, which are used for consumption and for investment in physical capital. The R&D sector employs labour only, and its output consists in the expansion of the level of technology.

In the model, it is assumed that workers devote a fraction $1 - \mu$ of their working time to production of goods and the remaining $\mu$ to the development of new technologies. The production function for final goods is given by:

$$Y = AK^\beta \left[(1 - \mu)\lambda N\right]^{-\beta} \quad (5.21)$$

The production function in the R&D sector is as follows:

$$\dot{\lambda} = b\mu\lambda \quad (5.22)$$

The parameter $b$ shall be interpreted as the productivity in the research sector.

The production function (5.22) has a critical property: a constant fraction of time devoted to R&D produces a constant growth rate of technology that is independent on the existing level of technology (in other words, there are no diminishing returns to technology on technology creation). With such an assumption, a policy change that successfully increases the proportion of time devoted to R&D ($\mu$) or that improves the productivity in the research sector ($b$) will impact positively and permanently on the growth rate of per capita income.

As far as the physical capital accumulation is concerned, the equation of motion (5.14) is retained. This model can be solved in the same manner as the Solow model. For mathematical convenience, let’s rewrite the production function (5.21) as follows:

$$\tilde{y} = A(1 - \mu)^{-\beta} \tilde{k}^\theta \quad (5.23)$$

Note that, if the productivity of R&D was decreasing (due to some fishing out effect, leading to $\dot{\lambda} = b\mu\lambda^\theta$ with $\theta < 1$, then the growth rate of technology would tend to zero, no matter how much effort was devoted to accumulation of human capital, and sustained growth could not be achieved.
Where $\bar{y} = Y / L$, $\bar{k} = K / L$, and $L = \lambda N$.

Proceeding as usual, the fundamental dynamic equation becomes:

$$\dot{\bar{k}} = sA(1-\mu)^{1-\beta}\bar{k}^\beta - [n + \delta + b\mu]\bar{k}$$  \hspace{1cm} (5.24)

Comparing with (3.8) you can verify how similar this model is with the Solow model. The main difference is that the parameter determining the effectiveness of labour, rather than growing exogenously, is now dependent of other parameters in the model.

Figure 5.4: the steady state in the two-sector model

This model is hybrid, in the sense that it shares characteristics with the AK model and with the Solow model.

It shares with the neoclassical model the feature that it has a transitional dynamics and a stable steady state (to find the steady state, just solve for $\dot{\bar{k}} = 0$). Figure 5.4 describes this. In this model, the saving rate, $s$, determines the steady state and changes in $s$ produce “level effects” (just like in the Solow model). Because $\bar{y}$ and $\bar{k}$ are both
constant in the steady state, the output-capital ratio is constant and so will do the interest rate. Hence, in the steady state, \( \dot{Y} = \dot{K} = \gamma + n \) and \( \dot{y} = \dot{Y} - n = b\mu \).

Figure 5.5 – The path of output per capita following an increase in \( \mu \)

Contrasting to the Solow model, the long run growth rate of per capita income (\( b\mu \)) is explained in the model: it depends on the proportion of working time people allocate to R&D; and on \( b \), the effectiveness of the research activity. Thus, for instance, a policy that is successful in inducing an increase in the proportion of time devoted to R&D raises the growth rate of the economy on a permanent basis (growth effect).

Figure 5.5 describes the path of per capita income in this economy following an increase in the time devoted to R&D: at the impact, there is a negative effect on per capita income, because less time is devoted to production. As the times go by, however, the growth rate of per capita output accelerates, due to the faster rate of technological expansion. Note however that, in terms of Figure 5.4, the production function and the break-even investment line shift in opposite directions. Hence, \( \ddot{k} \) starts declining, implying that during the transition period the growth rate of per capita output will be lower than in the steady state. As the economy approaches the new steady state, the fall in
\( \tilde{k} \) becomes smaller, so that the growth rate of per capita income approaches the new steady state growth rate, \( b\mu_t \).

Technically, sustained growth is obtained in this model because the production function for R&D is free of diminishing returns. In other words, the model overcomes diminishing returns to physical capital by postulating a linear production function for technology. Physical capital can then be accumulated without seeing its productivity declined, because it will be increasing at the same rate as technology. Rewriting the production function (5.21), we see that:

\[
Y = A\left(1 - \frac{\mu}{k}\right)^{1-\beta} K \quad (5.25)
\]

Since in the steady state levels of \( \mu \) and \( \tilde{k} \) are both constant, this means that in the long run we have no more than another incarnation of the AK model. The model is capable of generating sustained growth of per capita income without the need to assume exogenous shifts in the production function. This is why the model belongs to the general category of endogenous growth models.

In the short run, however, \( \tilde{k} \) is not in general constant, so the model also displays a transition dynamics. On the other hand, as long as the parameter \( \mu \) is held constant, this model behaves like the Solow model: a change in the saving rate will produce a level effect, only.

A question that naturally arises is how people decide the optimal level of \( \mu \). Intuitively, the optimal investment in R&D shall depend on a number of factors, such as on the productivity of the research effort (\( b \)), on the opportunity costs of the research, the level of impatience of people (in their consumption-saving decisions), and so on. However, with the present formulation, we cannot go far in that direction. Note that technology is still assumed to be freely available to all agents in the economy, so we cannot model the private incentives to innovation.
5.5. Empirical controversies

The ghost of financing gap

One of the reasons why the Harrod Domar equation (5.5) became so popular is that it offers a simple and appealing formula to forecast economic growth\(^{10}\). This formula was also extensively used by international organizations, such as the World Bank, to calculate a country’ financial needs.

Indeed, if equation (5.5) was true, one could easily forecast a country’ economic growth, using the saving rate, the depreciation rate and an estimate for the average product of capital, \(A\). Since the later is not readily available in national accounts, a possible proxy would be the ratio of net investment to the change in real GDP over two consecutive years:

\[
ICOR = \frac{\Delta K}{\Delta Y} = \frac{\text{net investment}}{\text{change in GDP}}
\]

This is the known as the "Incremental Capital-Output Ratio", ICOR.

As an example, consider a poor economy where the ICOR = 3 and the observed investment ratio (s) is 15%. Assuming a depreciation rate equal to 4%, equation (5.4) implies that output will grow at \(0,15/3 - 0,04 = 0,01\). Now suppose you were a World Bank economist for that economy, charged with advising on poverty alleviation. You could well regard the predicted growth rate as being too low. If, for instance, population was growing at 2%, that would imply a fall in per capita income…

\(^{10}\) In the context of development economics, a very influential cousin of the Harrod-Domar model was developed by Arthur Lewis (Lewis, W., 1954. “Economic Development with Unlimited Supplies of Labour”. The Manchester School of Economic and Social Studies 22, 139-191). In the Lewis model, the engine of economic growth is the accumulation of capital in manufactures; this does not only delivers higher production in manufactures, but also a higher demand for labour in manufactures, allowing workers to migrate from the low productivity agriculture, where they live close to subsistence. Along this process, the excess labour in agriculture will be eventually absorbed and the economy will enter in modern economic growth.
You could, then, use the HD equation the other way around: how much should the investment rate be for this country to achieve some desired rate of output growth? Suppose you wanted income per capita to grow at 2%. With the population expanding at 2% and a measured ICOR equal to 3, following (5.5), you would need a net investment amounting to 24% of GDP. Since domestic savings were only 15%, you could request the international donors to fill the "financing gap", amounting to 9% of GDP.

Economists in international institutions, such as the World Bank, the IMF, the Inter-American Development Bank, the European Bank for Reconstruction and Development used and still use models based on the HD equation to estimate the amount of savings (and/or aid) necessary for poor countries to achieve a minimum rate of economic growth. This philosophy is supported by the understanding that people living near the subsistence level cannot save the same as rich people. In theory, foreign aid could fix this: if the external aid succeeded in raising per capita output above a critical level, it could be that the domestic saving responded, allowing the country to engage in a self-sustained growth path. In this case, foreign aid would need only to be temporary.11

This interpretation of the HD equation was seriously criticised by William Easterly in an insightful paper called “The ghost of financing gap”12. The author noted that, over the past four decades, large amounts of international financial assistance to the developing world did not translate into faster economic growth. Using a sample of 146

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11 You are invited to demonstrate that replacing the assumption of a constant saving rate by a saving rate that depends positively on per capita income in the context of the AK model raises the possibility of a bifurcated growth pattern, whereby per capita income rises or decreases forever, depending on the initial level of capital per worker. Recent proponents of idea include Sachs (2005) and the United Nations Millennium Development Goals Project. Sachs (2005): “(…) if the foreign assistance is substantial enough, and lasts long enough, the capital stock rises sufficiently to lift households above subsistence…growth becomes self-sustained through households savings and public investments supported by taxation of households” (p. 246). United Nations (2005, p. 19): “The key to escape the trap is to raise the economy’s capital stock to the point where the downwards spiral ends and self-sustained economic growth takes over”. (p. 19). Sachs, J., 2005. The end of poverty: economic possibilities for our times. New York: Penguin Press. United Nations, 2005. Millennium Development, Project Report, United Nations, New York.

countries along the period from 1950 to 1992, the author failed to find a robust positive linear relationship between aid and economic growth.

Does this mean that the HD equation is wrong? Not necessarily. But probably one should not trust too much historical values of the average product of capital to guess the marginal impact of new investments: when, for example, part of the external aid is diverted into unproductive uses (frivolous expenses, corruption fees), then much of the higher saving rate will be offset by a lower $A$. Large amounts of financial aid to countries with poor institutions may have perverse effects, such as encouraging aid-dependence, by increasing the size of resources that organized groups compete for, and by helping perpetuate bad governments in power. On the other hand, it may be that the marginal product of capital is not simply independent of the level of capital per worker, as assumed in the AK model.

**Levels or changes?**

The AK model differs dramatically from the exogenous growth model, in terms of the relationship it establishes between the investment rate and economic growth. This prediction suggests an obvious avenue to find out which model conforms better to the real world facts: to investigate whether shifts in the investment rate have *permanent* or *temporary* effects on economic growth. Along this avenue, critics of the AK model have pointed out that, among OECD countries for instance, richer economies tend to exhibit higher investment rates in physical and in human capital than poorer economies, and yet they do not enjoy faster growth (actually, the data in Figure 3.6 points to the opposite case).

In practice, a major difficulty in disentangling whether the *true model* is the Solow model or the AK model is that the two models are observationally equivalent for long periods of time: remember that the Solow model predicts a positive relationship between investment and growth while the economy moves from one steady state to the other. Since this transition period can be quite long (in the MRW formulation, for instance, half of the transition dynamics takes as long as 35 years) and because most
reliable datasets with comparable data start after 1950, it is not easy to assess whether changes in the investment rate have long run level effects or long run growth effects.

The most common approach to assess whether the true model is the AK model or the neoclassical growth model is testing for conditional convergence. The conditional convergence hypothesis states that countries tend to approach their respective steady states, and that the speed of adjustment varies in direct proportion to the distance to the steady state. This property of the neoclassical model contrasts to the AK model, according to which parameter shifts alter the growth rate of per capita income permanently, without any tendency for per capita income to return to a previous path.

The most common approach to test for conditional convergence is to run cross-country growth regressions (see Box 5.3). Basically, the method consists in estimating the growth rate of per capita income as a function of a range of explanatory variables, including the initial level of per capita GDP. Then, conditional convergence is assessed by investigating the significance of the coefficient in the initial level of per capita income.

As an illustration, consider the regressions displayed in Box 5.2. In all regressions, the initial level of per capita GDP helps explain the subsequent growth rate of per capita GDP. That is, controlling for the other variables, an initial lower level of per capita GDP is positively related to subsequent growth. This evidence is favourable to the conditional convergence hypothesis, which holds in the Solow model but not in the basic formulation of the AK model.

In general, the evidence with cross-country growth regressions using large samples of countries has been favourable to the conditional convergence hypothesis. That is, the coefficient on the initial level of per capita income has been found to be, in general, negative and significant in cross-country growth regressions. This fact inspired a famous quote from Robert Barro: “It is surely an irony that one of the lasting contributions of endogenous growth theory is that it stimulated empirical work that demonstrated the explanatory power of the neoclassical growth model” (Barro (1997), p. x).
Box 5.2. The external aid controversy

The question as to whether external aid helps or not a country achieve faster economic growth is obviously very important from the policy point of view. With no surprise, this question has been subject to empirical investigation.

A branch in the literature has explored the possibility of the impact of aid being conditional on the recipient country characteristics. A particularly influential study was a background paper to the 1998 World Bank *Assessing Aid* report, by Burnside and Dollar\(^\text{13}\) (2000). Working with a sample of 56 developing countries along the period from 1970 to 1993, the authors distinguished two sub-groups of countries: those that pursued “sound policies” and those that pursued “poor policies”. Policy soundness was assessed by a compounded index of trade openness, fiscal discipline and low inflation. Focusing on the “sound policies” sub-sample, the authors found that those countries receiving large amounts of aid grew, on average 3.5%, while those receiving small amounts of aid grew, on average, 2%. In the “poor policies” sub-sample, the authors found virtually no growth, irrespectively of the amount of aid received. These findings suggest that external aid boosts growth, but only if domestic policies are sound.

To further investigate this hypothesis, Burnside and Dollar run some regressions trying to explain the growth rates of per capita income achieved by the countries in the sample. Their central results are reproduced in columns (1) to (3) of Table 5.1. The dependent variable is the average growth rate of per capita GDP along the period 1970-93. In equation (1), the growth rate of per capita GDP is correlated with: the logarithm of initial GDP per capita GDP (capturing conditional convergence); the degree of ethnic fractionalisation, the rate of political assassinations and the product of these two variables (to capture political instability); an index of institutional quality; the ratio of money to

GDP (to capture financial development); two regional dummies, for sub-Saharan Africa and East Asia; and the government budget surplus, inflation and an index of openness to international trade (to capture the quality of domestic policies).

Burnside and Dollar used the coefficients of the last three variables in regression (1) to compute a “policy index”. Then, they estimated equation (2), replacing the variables capturing the quality of domestic policies by this “policy index”. In this equation, they also added the external aid as a percentage of GDP. Since the t-ratio of the later was found to be very low (0.28 in column 2), the authors concluded that aid, by itself, does not explain growth. In column (3), a similar regression is performed, but including an “interaction term”, given by the product of the variables AID/GDP and the Policy Index. Because this last variable was found to be significant while AID/GDP alone is not, the authors concluded that aid only leads to more growth in a sound policy framework. The authors also tested the possibility of aid to be detrimental to policies. However, no significant relationship was found between the amount of aid received and the quality of the domestic policies.

These results caused a significant reaction in the economic profession, as it implied that foreign aid is useless in countries pursuing bad policies. Not surprisingly, they were subject to an intense scrutiny by other researchers. In general, this further investigation revealed that the conclusions of Burnside and Dollar were, in general, fragile to alternative specifications of the regression model or the sample period\textsuperscript{14}. Burnside and Dollar then shifted their focus from the quality of policies to the quality of institutions. Using a cross section of 124 countries over the 1990s, they found that, while aid alone is not significantly related to growth, the degree of institutional quality interacted with aid, is\textsuperscript{15}.


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http://sweet.ua.pt/afreitas/growthbook/capa.htm
Other authors turned to the impact of a variable that cannot be changed by policy: geography. The reasoning is that a bad location may help explain why in some countries external aid produced less effect than in other countries. Columns (4) in Table 5.1 displays the estimation results by Daalgard et al. (2004)\(^\text{16}\). The authors measured the fraction of a country’s land located in the tropics and interacted this variable with aid, to evaluate the aid-growth relationship, using the same data-set as Burnside and Dollar. As shown in the table, the policy-aid interaction variable becomes insignificant, while aid and aid interacted with the climate became significant. These results suggest that aid has a positive impact on growth, but the impact decreases for countries located in the tropics. This last result is, of course, disappointing because it points to a critical role of geography - which cannot be changed by human actions - rather than of policy, which people can change.

### Table 5.1. Growth regressions explaining the growth rate of per capita GDP in 56 developing countries

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation method</td>
<td>OLS</td>
<td>OLS</td>
<td>OLS</td>
<td>OLS</td>
</tr>
<tr>
<td>Initial GDP</td>
<td>-0.65</td>
<td>-0.61</td>
<td>-0.60</td>
<td>-0.54</td>
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<td>(1.18)</td>
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<td></td>
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<tr>
<td>Ethnic fractionalization</td>
<td>-0.58</td>
<td>-0.54</td>
<td>-0.42</td>
<td>0.12</td>
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<tr>
<td>(0.79)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Assassinations</td>
<td>-0.44*</td>
<td>-0.44*</td>
<td>-0.45*</td>
<td>-0.38</td>
</tr>
<tr>
<td>(1.63)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ethnic fractionalization * Assassinations</td>
<td>0.81*</td>
<td>0.82*</td>
<td>0.79*</td>
<td>0.70</td>
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<td>(1.80)</td>
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<tr>
<td>Institutional quality</td>
<td>0.64**</td>
<td>0.64**</td>
<td>0.69**</td>
<td>0.69**</td>
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<tr>
<td>(3.76)</td>
<td></td>
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</tr>
<tr>
<td>M2/GDP (lagged)</td>
<td>0.015</td>
<td>0.014</td>
<td>0.012</td>
<td>-0.02</td>
</tr>
<tr>
<td>(1.00)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub-Saharan Africa</td>
<td>-1.53**</td>
<td>-1.60**</td>
<td>-1.87**</td>
<td>-1.58**</td>
</tr>
<tr>
<td>(2.10)</td>
<td></td>
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<tr>
<td>East Asia</td>
<td>0.89</td>
<td>0.91*</td>
<td>1.31**</td>
<td>1.57**</td>
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<tr>
<td>(1.59)</td>
<td></td>
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<tr>
<td>Budget surplus</td>
<td>6.85**</td>
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<tr>
<td>(2.02)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Inflation</td>
<td>-1.40**</td>
<td></td>
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<tr>
<td>(3.41)</td>
<td></td>
<td></td>
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<tr>
<td>Openness</td>
<td>2.16**</td>
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<tr>
<td>(4.24)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Burnside-Dollar policy index</td>
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<td>1.00**</td>
<td>0.71**</td>
<td>0.78**</td>
</tr>
<tr>
<td>(Aid/GDP) * policy index</td>
<td></td>
<td>0.034</td>
<td>-0.021</td>
<td>1.49**</td>
</tr>
<tr>
<td>(0.28)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Fraction of land in tropics</td>
<td></td>
<td>0.19**</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td>(Aid/GDP) * fract. of land in tropics</td>
<td></td>
<td>(2.71)</td>
<td>(1.34)</td>
<td></td>
</tr>
<tr>
<td>(Aid/GDP) * fract. of land in tropics</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observations</td>
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<td>270</td>
<td>270</td>
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<tr>
<td>Countries</td>
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<td>56</td>
<td>56</td>
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<tr>
<td>R²</td>
<td>0.35</td>
<td>0.36</td>
<td>0.36</td>
<td></td>
</tr>
</tbody>
</table>

Sources: Burnside and Dollar (2002) for columns (1), (2) and (3) (regressions (1), (3) and (4) in the original paper; Dalgaard et al., (2004) for column (4) (column 5 in the original) Notes: The dependent variable is real per capita GDP growth. All regressions include time dummies. Robust t-statistics in parentheses. * significant at 10%; ** significant at 5%.

In the last years, a number of other papers came out, addressing the question as to whether the impact of aid on growth is conditional on third variables. The literature has
still no definitive answer regarding the variable that better interacts with aid: some papers say is policy, others say is institutions; some others say it is geography and others say there is no interaction at all. This evidence suggests that the inter-play between aid, local circumstances and growth is eventually more complex than that captured by the initial Burnside and Dollar estimation. It also point to the difficulty in establishing empirically the right determinants of economic growth, when the candidate variables so many, as compared to the limited number of observations.

Box 5.3. Cross-country growth regressions

Cross-country growth regressions are the workhorse of empirical research on economic growth. Basically, the approach consists in estimating equations relating the growth rate of per capita GDP to a range of possible determinants. The later may include variables capturing factor accumulation (such as the investment rate and the population growth rate), and variables that are more likely to exert its influence on growth through the productivity parameter, A. Examples of cross-country growth regressions are presented in Tables 4.2, and 5.1.

Growth regressions have a natural interpretation in terms of equation (5.5): in that model, growth appears as a function of the investment rate and of the productivity term, A. The advantage of cross-country growth regressions relative to simple growth accounting is that, rather than estimating A as a residual, they try to identify the policies and other factors that underlie the cross-country differences in A. In practice, it has been common to find indicators capturing country idiosyncrasies that are strongly correlated to growth. This includes either variables measuring the quality of policies and institutions (such as trade openness, the rule law, political risk, inflation, financial depth) and the geographical conditions. The evidence in Table 5.1 illustrates this.

Often, the researcher compares the coefficient of the variable of interest in two different regressions, one controlling for the investment rate and other not controlling for the investment rate. The idea is that any particular variable may have a direct influence on growth, or an influence that is indirect, via its influence in the rate at which individuals accumulate capital (remember equation 5.10). Thus, for instance, if the variable of interest has a smaller coefficient or looses significance when the investment rate is included among the regressors, this suggests that its influence on growth occurs mainly through its impact on the investment rate\(^{18}\).

Cross-country growth regressions also have an interpretation in light of the neoclassical growth model. In its basic formulation, the *conditional convergence* test (Box 4.4) consists in estimating the growth rate of per capita income as a function of investment rates in human and physical capital, population growth rates, and initial income. This test intends to control for differences in the steady states (remember that the regression equation is obtained substituting the steady state level of per capita income 4.18 in 3.14). The MRW formulation fails, however, to control for policies impacting on the level of \(A\).

Putting all the pieces together, new research on conditional convergence turned to extended versions of the MRW test, by adding other variables to the regression equation. Formally, the equation to be estimated in an extended neoclassical framework is:

\[
\ln y_t - \ln y_0 = a - b \ln y_0 + \psi X + \chi Z + u_t, \quad (5.25)
\]

where \(X\) is a vector of variables capturing factor accumulation that are present in the MRW model (propensities to invest in physical and human capital, and the population growth rates) and \(Z\) is a vector of other variables determining the level of \(A\).

Conditional convergence is assessed investigating the significance of $b$: if its found to be zero, then changes in the other explanatory variables impact on the growth rate permanently, supporting the endogenous growth model (5.5); if, instead, $b$ is found to be significant and negative, this suggests that growth rates are proportional to the distance to steady states, which accords to the idea of conditional convergence.

The Empirical analysis using cross country growth regressions face a number of limitations.\footnote{For a detailed discussion, see Durlauf, S., Johnson, P., 1995. Multiple regimes and cross-country growth behaviour. Journal of Applied Econometrics, 10, 365-384.}

First, because the theory does not provide an unambiguous guide to the choice of elements of $Z$, there is a lot of uncertainty regarding the right model specification. In practice researchers have proposed more and more variables to complement the baseline specification, each one stressing a causal relationship between a particular variable and growth. This, in turn, brings a familiar econometric problem: because explanatory variables tend to be correlated to each other (countries performing badly in a given indicator also tend to perform badly in other indicators), there is a large scope for multicollinearity: the significance of each variable in the equation is influenced by the particular combination of variables included in the regression.

In practice, although many variables have been found to be correlated with growth, most of them loose significance when other variables are included in the same equation. This problem makes very difficult to assess empirically which variable is more correlated to growth and how much (e.g, if inflation rates, exchange rate volatility and political instability go wrong together, how one can disentangle the various contributions to growth?)\footnote{Attempts to resolve this problem in a systematic way include: Levine, R. and D. Renelt (1992), “A sensitivity analysis of cross-country growth regressions”, American Economic Review 82(4), 942-63; Sala-i-Martin, X., “I just ran two million regressions”, American economic review 97 (2), 178-83.}.
Second, there is a problem of endogeneity: although it may appear natural that the parameter estimates ($\varphi$ and $\chi$ in equation 5.25) contain information of causal effects on economic growth, this is not necessarily true. Some right-hand-side variables may be econometrically endogenous in the sense that they are jointly determined with the rate of economic growth: for instance, the same factors that make a country invest little in physical capital may also have a direct effect on its growth rate. In that case, the estimated parameter will be biased and will provide little information regarding the direction of causality.

Third, even if all variables on the right hand side were actually exogenous, many of them could be “symptoms”, rather than “syndromes”. For instance, consider the measurement of human capital. Shall we choose the secondary school enrolment or the primary school enrolment? Since these tend to be correlated to each other, they render one another insignificant when both are included in the regression equation. So which one should we choose? Moreover, a given symptom may be interpreted as capturing different syndromes. For example, a negative correlation between inflation and growth means bad macroeconomic management or a large tax evasion that forces the government to rely on revenues from money creation?

Fourth, there is a problem of parameter heterogeneity: parameter values estimated with cross section exercises that pool together very different countries may fail to accurately capture any of each. As once stated by Harberger: “What do Thailand, the Dominican Republic, Zimbabwe, Greece and Bolivia have in common that merits their being put in the same regression analysis?.

Fifth, the lack of a structural model stating how much the parameter $A$ depends on each policy variables makes it difficult to go beyond general statements on observed correlations and to provide a convincing interpretation of the results.

Other problems of cross-country-growth regressions include: the presence of outliers, measurement errors, and model linearity. Despite the extensive econometric improvements that have been adopted to overcome these limitations, the results of cross-country growth regressions have still to be taken with caution.
5.6. Discussion

The AK model stresses the relationship between policies and economic growth. This contrasts with the Solow model, whereby changes in the key parameters only produce level effects. The empirical findings are not, however, very favourable to the simpler version of the AK model. In general, country characteristics, such as the saving rate and aggregate efficiency are found to influence the levels of per capita income, rather than growth rates. This view is supported by an extensive empirical literature favourable to the conditional convergence hypothesis.

Does this mean that we shall abandon the AK model? The answer is no.

First, remember that the important link between efficiency and growth is also present in the neoclassical model: the difference is that in the later the growth effect will be transitory. That is, you may interpret the AK model as a short-run version of the neoclassical growth model. With half of the transition period between steady states in the neoclassical model taking as long as 35 years, whatever the true model is, we are doomed to accept that policy actions may influence economic growth for a considerable period of time.\(^{21}\)

Second, the AK model is much easier to solve than the Solow model. Because of this, from the expositional point of view, it is often more convenient to study the impact of particular policies in the context of the AK model than in the context of the Solow model, especially when the math becomes too complex. Of course, in doing so, one shall take into account that any conclusion regarding the impact of the policy at hand on

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\(^{21}\) Easterly (2005) calibrated a simple neoclassical growth model with a share of total capital equal to 2/3 (which accords to MRW) and with other reasonable values for the remaining parameters. He found that a tax decrease from 30% to zero raises per capita income by a factor of 2.25 times. The author also showed that immediately after the change in policy, the growth rate of the economy shoots up by almost 8 percentage points relative to its steady state. Only in the very long run (more than 5 decades after), the growth effect wears off and the growth rate returns to its long run level. The author concluded that policies have significant effects in the neoclassical model, too (pp. 1024-1026).
growth has to be spelled out in term of level effects, when adapted to the context of the Solow model. In some of the upcoming chapters, we will follow this approach.

Last, but not at all the least, the basic AK model illustrates how linearity avoids the basic problem of diminishing returns, generating long-term growth. Linearity is a basic feature of most endogenous growth models, including those focusing on technological change. This is precisely what the Usawa model illustrates.

The AK model can therefore be interpreted as a toy version of more complex endogenous growth models, whereby knowledge expands through investments in R&D. Knowledge shares with capital the characteristic that it can be built over time by sacrificing some of today’s consumptions. So, interpreting investment as foregone consumption in a broad sense (that is, including physical assets, human capital and R&D), one can interpret the AK model as a general framework to think the mechanics of economic growth through technological change.
Key ideas of Chapter 5

• The AK mode reveals in a simple manner that getting rid of diminishing returns, factor accumulating alone can generate continuous growth of per capita income.

• In the context of the AK model, changes in the saving rate produce “growth effects” rather than “level effects”.

• The predecessor of the AK model, was the Harrod Domar model. Since this model assumes unemployment of labor, the main constraint to economic growth is capital availability. A similar story was formulated by Sir Arthur Lewis. For the context of developing countries.

• Extending the AK model to the case with endogenous savings, the direct effect of aggregate efficiency on growth is reinforced by an indirect effect via a higher return on savings. The implication is that, wherever financial markets are more developed so that households can smooth consumption over time, the impact of policy changes on growth is more dramatic.

• The model with endogenous savings appeals to the distinction between proximate causes of growth and fundamental causes of growth. As for the proximate causes, we are explaining differences in economic performance by differences in investment rates and differences in aggregate efficiency. As for fundamental causes, one wants to deepen the analyses so to understand why some countries have higher investment rates and better efficiency than others.

• Extending the model to include human as well as physical capital does not change its properties, as long as there are constant returns to scale on reproducible factors. This extension suggests that one can interpret K in the simple AK model in a broad sense, including different types of capital and even technology.
• A branch in the literature explored the possibility of obtaining unceasing growth in the context of the neoclassical framework. For this to happen, one only need to assume that the marginal product of capital is bounded below by a positive constant that is higher than the rate of time preference. This will prevent the interest rate from following to a point where households desire consumption to be constant, as is happens in the Solow model.

• Another hybrid model between the AK and the Solow model was proposed by Usawa. In this model, linearity arises in the relationship between innovation effort and technological change. In this model, changes in research intensity produce growth effects, but changes in the saving rate produce level effects. Like the Solow model, this model has a transition dynamics.

• The Harrod Domar model inspired the idea that complementing low domestic savings in poor countries by foreign aid would be a key to generate economic growth. In practice, however, the impact of external aid on the growth varied significantly across countries, depending on the quality of domestic policies, institutions, and geography. This suggests, once again, a key role for aggregate efficiency in determining growth performances.

• The empirical evidence of conditional convergence suggests that the neoclassical model, and the implied hypothesis of level effects and conditional convergence, fits better the reality than the simple AK model. Still, the AK structure underlies most models of endogenous growth based on technological change.
Appendix 5.1 Unbalanced growth in the HD model

Because prices play no role in the Harrod-Domar model, in general the model will unemployment of labour or of capital.

To make the point, let’s consider first the knife hedge case in which the parameters of the model are such that population and capital grow exactly at the same rate (that is, \( sA - \delta = n \)). In that case, the capital-labour ratio and per capita income remain constant. In terms of figure 5.1, this exceptional case occurs when the break-even investment line coincides with the saving line. Thus, any starting point will be a steady state.

Note however that even in this exceptional case, the economy will not in general grow evolve at full employment. To see this, consider Figure A5.1: if, for instance, the economy started out with labour surplus (point S), then it would move along a path with a constant capital-labour ratio \( k_s \) in the figure), but with increasing unemployment. The only case in which the economy evolves along the full employment locus is when \( sA - \delta = n \) and simultaneously the economy starts out without surplus labour (point R).

A less fortunate scenario occurs when \( sA - \delta < n \). In that case, the economy does not save the enough to keep the capital labour ratio unchanged. In terms of (5.5), we see that in this case, the growth rate of per capita income is negative, implying a rising labour surplus and chronic underproduction (path \( k_u \) in Figure A5.1).

The best scenario in the HD model occurs when the parameters in the economy are such that the capital stock grows faster than population (that is, when \( sA - \delta > n \)). In this case, per capita income increases until the surplus labour is completely eliminated. Still, the mechanics of the model is such that per capita income cannot growth indefinitely. The reason is that at the time the full employment line is crossed (point R in Figure A5.1), the binding constraint in production becomes the availability of labour (that
is, the relevant segment of the production function in (5.11) shifts to \( Y = BN \). Hence, beyond this point output will be bound to expand at the same rate as population, implying a constant level of per capita income thereafter.

To see how the steady state in that case looks like, let’s divide both terms of \( Y = BN \) by \( K \), and substitute this for \( Y/K \) in the Harrod-Domar equation (5.5). The growth rate of the economy in this segment becomes:

\[
\gamma = sB/k, - (n + \delta).
\]

Since this expression depends negatively on \( k \) (that is, as \( k \) rises, its growth rate declines), this segment of the model has a stable equilibrium. Solving for \( \gamma = 0 \), one obtains the steady state level of capital per worker:

\[
k^* = \frac{sB}{n + \delta}.
\]

The implication is that, after crossing the full employment locus, the economy will evolve along the path \( k = k^* \), with unemployment of capital (path \( k^* \) in Figure A5.1).
Problems and Exercises

Key concepts

Surplus labour
Proximate versus fundamental causes of economic growth
Broad concept of capital
Neoclassical models of endogenous growth
Two sector model of endogenous growth
Cross-country growth regressions

Essay questions:

a) Referring to the Harrod-Domar equation, compare the AK model and the Solow model in respect to the variables that are exogenous and endogenous. In particular, examine the impact of an increase in the saving rate in light of the two models.

b) Comment: “The proof that the AK model is not true is that foreign assistance to poor countries failed to deliver faster economic growth”.

c) Comment: “Poor countries, with underdeveloped financial markets, are more likely to tolerate bad policies than rich countries with developed capital markets”.

d) Explain why the Usawa model is hybrid. In the context of this model, which policies could influence the rate of economic growth?
Exercises

5.1.

Consider an economy where the production function is given by $Y = AK$. In this economy, the saving rate is $s$, the population grows at rate $n$ and the capital depreciation rate is $\delta$.

a) Does this production function satisfy the usual neoclassical properties? Why?

b) Describe analytically and graphically the dynamics of per capita income in this economy. Is there any stable equilibrium?

c) Does this model predict convergence of per capita incomes across economies?

d) Describe, comparing with the Solow model, the impact of: (i) a fall in the population growth rate; (ii) An increase in $A$.

5.2.

Consider an economy, where the production function is given by $Y=0.2K$, the population grows at 2% per year, the capital depreciates at 3% and the saving rate is 25%.

a) Find out the growth rate of per capita income in this economy.

b) What will be the effect of $A$ increasing to 0.25?

c) Now assume that the saving rate was endogenous, as implied by the following optimal consumption rule: $\gamma_t = r_t - 0.17$. Analyse in this case the implications of an increase in efficiency from 0.2 to 0.25.

d) Comparing the two models, find out the expression that relates the saving rate to efficiency ($A$). Explain why a change in the efficiency parameter ($A$) has a larger impact when savings are endogenous.

5.3.

In Micronésia, the aggregate production function is given by $Y = K^{0.2}H^{0.5}$, where $H=hn$, $N$ is the number of workers, and $h$ measures the amount of human capital per worker. In this economy, the saving rate is given by $s=25\%$, the population is constant and the rate of depreciation of physical capital is equal to 5%.
a) Assume for the moment that \( h=1 \). Find out the equilibrium values of \( k=K/N \) e \( y=Y/N \). Explain the dynamics of the model with the help of a graph.

Assume now that \( \dot{h} = s_h y - \delta h = 0.25 y - 0.05 h \).

b) Explain this specification.

c) In this case, the properties of the neoclassical model are satisfied? Why?

d) Find out the growth rate of per capita income in this economy.

e) Compare, in the light of both models: (i) the short run and the long run effects of a rise in the saving rate (ii) The convergence hypothesis

5.3.

Consider the following production function and law of motion of per capita consumption:

\[ Y_t = A_t K_t^{\alpha} N_t^{\beta} H_t^{1-\alpha-\beta}, \text{ with } \alpha, \beta \leq 1 \]
\[ \gamma = r - \rho. \]

Assume that the depreciation rate is identical for the two capital types and that population does not grow over time.

f) Suppose that \( \alpha+\beta=1, \alpha, \beta \neq 0 \).
   i. Explain if it is possible to obtain sustained growth of per capita income in the long-run through factor accumulation.
   ii. Describe the impact of an increase in \( \rho \) in the interest rate and in per capita income.

h) Finally, suppose that \( \alpha=0, \beta<1 \).
   iii. Explain if it is possible to obtain sustained growth in the long-run through factor accumulation.
   iv. Describe the impact of an increase of \( \rho \) in the interest rate and in per-capita income.
5.4.

Consider a closed economy without government, where population is equal to one thousand inhabitants, and constant over time. In this economy, the relevant production function is given by $Y = 0.5K$, capital deteriorates at the rate of $\delta = 0.04$ per year, and consumption is a linear function of income, according to $C = 8000 + 0.84Y$.

Suppose the initial capital stock in this economy was exactly $K = 200,000$. In this case, per capita income: (i) will be growing over time; (ii) will be decreasing over time; (iii) will be stagnant.

Suppose that, due to some external support, the capital stock in this economy jumped temporarily to $K = 250,000$. In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04.

Suppose this economy started out with a capital stock equal to $K = 250,000$, but the production function was actually given by $Y = \min\{0.5K, 150N\}$. In this case, the long run growth rate of per capita income will approach: (i) 0; (ii) 0.008; (iii) 0.04.

In the conditions set out in (c), the long run value of the capital-labour ratio will be: (i) decreasing over time; (ii) $k=300$; (iii) $k=400$; (iv) increasing over time.

5.5.

Consider an economy where the production function is given by $Y_t = u^{1/2}K_t^{1/2}(Nh_t)^{u/2}$, were $u=0.25$ is the fraction of time devoted to production. In this economy, the saving rate is 15%, the population is constant and capital does not depreciate. Human capital per worker accumulates at the rate $\gamma = \dot{h}/h = b(1-u)$, where $b=0.02$.

a) Explain the equation describing the accumulation of human capital.

b) Using the equation describing the change in $\bar{k} = K/H$ (the fundamental dynamic equation), find out the steady state values of $\bar{y}$ and $\bar{k}$.

c) Examine the implications of an increase in the saving rate from $s=15\%$ to $s=18\%$. In particular, compute the new equilibrium values of $\bar{y}$ and $\bar{k}$. Describe the change in a graph and explain what will happen to the interest rate.
d) Returning to the initial figures, examine the implication of an increase in $b$ to $b=0.04$. In particular, compute the new equilibrium values of $\bar{y}$ and $\bar{k}$. Describe the change in a graph and explain what will happen to the interest rate.

e) Compare the effects on the path of per capita income, $y=Y/N$, of the changes described in c) and in (d).