The basic Solow model
2.6. The case with endogenous savings

- An optimal consumption rule
- What happens when the rate of time preference decreases?
- The modified golden rule
- Exogenous or endogenous savings?

2.7. The Solow Residual

2.8. Discussion: the Solow model and the growth question

Appendix 2.1: The optimal consumption path in a simple 2-period model

Problems and Exercises

Key concepts

Essay questions

Exercises
The basic Solow model

“A thrifty society will, in the long run, be wealthier than an impatient one, but it will not grow faster” [Robert Lucas Jr.]

Learning Goals:

- Acknowledge the distinctive feature of capital, as compared to those of the inputs labour and land.
- Understand the extent to which capital accumulation can overcome the diminishing returns to labour.
- The mechanics of the Solow model.
- Evaluation of the Solow model in light of the Kaldor facts.
- Distinguish the effects of change in the saving rate from those of an exogenous change in technology.
- Discuss the optimality of the saving rate in the context of the Solow model.

2.1. Introduction

The neoclassical theory of economic growth was pioneered by two independent authors, the American economist and Nobel Laureate Robert Solow and the Australian economist Trevor Swan. The main innovation of the Neoclassical model in respect to the Malthusian Model is the replacement of land by “capital” in the production function. By “capital”, we mean machinery, buildings, and other equipment.

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This modification is more than a mere change in form: contrary to land, which is available in finite supply, capital can be produced and accumulated. This opens an avenue to overcome the diminishing returns to labour: by allowing the capital stock to expand, the Solow model avoids the negative relationship between productivity and the size of population that plagues the Malthus model. Accordingly, in the Solow model, the long run is no longer characterized by a low wage equilibrium trap. Still, because capital itself faces diminishing returns, capital accumulation alone cannot generate long term growth: in the Solow model, a higher investment rate translates into a higher level of per capita income, but it does deliver faster economic growth.

This chapter presents the Solow model in its simplest formulation. In Section 2.2, we describe the basic model and its equilibrium. In Section 2.3, we compare the predictions of the model with the main stylized facts of Modern Economic Growth. In Section 2.4 we discuss how the main endogenous variables of the model respond to changes in the key exogenous parameters. Section 2.5 discusses the optimality of the saving rate, from a social point of view. Section 2.6 extends the model to the case in which the saving rate is the result of optimizing decisions by individual agents. Section 2.7 introduces a growth accounting exercise to illustrate a fundamental limitation of this simple version of the Solow model. Section 2.8 concludes.

2.2. The Solow model

The neo-classical production function

In the Malthus model, with everything else constant, an expansion of the labour force leads to a decline in per capita income. This is a direct consequence of the Law of Diminishing Returns: because since land remains in fixed supply, its higher intensive use translates into a declining productivity of labour.

The Solow model retains the assumption of diminishing returns, but explores another property of the neoclassical production function: the property of Constant
Returns to Scale (see box): if capital is set to expand at the same rate as population, then total output will grow at the very same rate and per capita income will remain constant, rather than declining towards a low level subsistence trap.

To see this, let’s assume that the aggregate production function in the economy is as follows:

\[ Y_t = AK_t^\beta N_t^{1-\beta} , \quad 0 < \beta < 1 \] (2.1)

where \( K \) denotes for the economy’ capital stock, \( N \) for the size of the labour force, \( Y \) for output and \( A \) is Total Factor Productivity.

Since we are interested in the well being of the average person, we focus on per capita income. Dividing the production function (2.1) by the size of the workforce \( N \), one obtains a new expression, which relates per capita output to the availability of capital per worker:

\[ y_t = A \left( \frac{K}{N} \right)^\beta = Ak_t^\beta , \] (2.2)

In (2.2), \( y=Y/N \) denotes for per capita income and \( k=K/N \) is the capital-labour ratio. Equation (2.2) is called the production function in the intensive form and stresses the role of the capital-labour ratio as a main driver of per capita income. Thus, as long as the capital labour ratio is constant, per capita output will be constant, regardless the population growth.

The interesting feature of the Solow model is that we do not need to postulate \( K \) to grow at the same rate as the labour force. As we will see next, the properties of the model are such that the capital stock, despite being endogenously determined, will end up growing at the same rate as the labour force in the long run, assuring a constant level of per capita income.

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2 The Solow model is consistent with more general specifications for the production function. The required assumptions are constant returns to scale and diminishing returns on both inputs. The Cobb-Douglas production function is assumed for mathematical convenience.
Box 2.1 Constant returns to scale

The key assumption of the neoclassical growth model is that of constant returns to scale (CRS). CRS means that if one increases the use of all inputs by a positive proportional factor, output will rise in the same proportion. For instance, duplicating the use of labour and capital, output will double. The CRS property can easily be checked in equation (2.1): for any $q \neq 0$, $A(qK)^\theta(qN)^{-\beta} = qY$.

Note that this is not inconsistent with the LDR: the Law of Diminishing Returns states that increasing the use of one input while holding the other input constant, output will grow less than proportionally. The Constant Returns to Scale property applies when all inputs increase in the same proportion at the same time.

To see this, we refer to Figure 2.1. The figure displays the average product of labour as a function of the size of the labour force. The Law of diminishing returns is illustrated by the move from point A to B: all else constant (including the capital stock), an increase in the use of labour from $N_0$ to $N_1$ implies a decline in the output per worker from $y_0$ to $y_1$. This is the basic mechanism of the Malthusian model. In the Solow model, in contrast, the second input, capital, is allowed to expand along with the labour force (actually, the steady state of the model will be such that the capital stock grows exactly in the proportion of the labour force). In that case, we see from (2.2) that per capita income remains constant. In terms of Figure 2.1, this is illustrate by a move from A to C: if, when labour expands from $N_0$ to $N_1$, the capital stock also increases, and exactly in the same proportion (from $K_0$ to $K_1$), then the capital labour ratio remains constant and so will be per capita income, $y_0$. 
Figure 2.1. Diminishing returns versus constant returns to scale

The Figure displays the average product of labour as a function of the size of the labour force. The curve is negatively sloped because of diminishing returns (from A to B). When the capital stock increases, the average product of labour shifts up. If the capital stock and population increase in the same proportion, the average product of labour will remain constant (from A to C).

In general, production functions may also exhibit decreasing returns to scale (in which case output grows less than proportionally than inputs) and increasing returns to scale (when output grows more than proportionally than inputs). It is believed that decreasing returns to scale are unlikely: if we managed to increase all inputs in a given proportion, there should be no reasons for output to respond less than proportionally. The case with increasing returns will be addressed later in this book.

Main assumptions of the Solow model

Consider a closed economy with no government with a large number of small firms producing a single homogeneous good, $Y$, using two inputs: labour ($N$), and capital ($K$), which can be produced and accumulated. Population and the labour force are the same. Inputs are hired from households, who are also the owners of the firms and the consumers in this economy. Households spend a fraction (1-s) of their income on
consumption, and save the remaining, to buy new capital. The capital stock depreciates at a constant rate, $\delta$. Perfect competition and flexible prices are assumed, so full employment holds each moment in time. In this model, the ability to accumulate capital (via savings) prevents output per capita from declining when population increases. Hence, one no longer needs to worry with Malthusian barriers and subsistence wages that limit population growth. Instead, it is assumed that population expands at an exogenous and constant rate, $n$.

The production function of each firm $i$ takes the following form:

$$Y_i = A_i K_i^{\beta} N_i^{1-\beta}. \quad (2.3)$$

This production function exhibits Constant Returns to Scale in labour and capital and decreasing marginal returns to each of those inputs (i.e. the LDR applies to each of them).

The parameter $A$ stands for the level of technology and is assumed exogenous to the firm. Throughout this chapter it will be assumed that the level of technology is constant over time:

$$A_t = A. \quad (2.4)$$

Assuming that all firms are equal and face the same relative prices, the aggregate production function in this economy becomes (2.1), where $Y = \sum_i Y_i$, $K = \sum_i K_i$ and $N = \sum_i N_i$ stand, respectively for the aggregate levels of output, capital and labour in the economy.

Households use their income either for consumption expenditures or for savings. In the aggregate, this implies that:

$$Y_t = C_t + S_t \quad (2.5)$$

where, $C$ and $S$ denote aggregate consumption and aggregate savings, respectively.
The Solow model combines the neoclassical features of perfect competition and flexible prices with a Keynesian consumption function. In particular, it is assumed that savings are proportional to current income:\(^3\):

\[ S_t = sY_t \]  \hspace{1cm} (2.6)

Given the constant saving rate, \( s \), the flow equilibrium in this economy is given by:

\[ sY_t = I_t \]  \hspace{1cm} (2.7)

where \( I \) denotes gross investment\(^4\).

With the passage of time, capital wears out or becomes obsolete. This process is called \textit{depreciation} and implies that some investment is needed every year just to replace the deprecating capital. In this model, it is assumed that the depreciation rate (\( \delta \)) is exogenous and constant over time. Hence:

\[ \dot{K}_t = I_t - \delta K_t \]  \hspace{1cm} (2.8)

Equation (2.8) states that the change in the capital stock (net investment) is equal to gross investment minus depreciation.

Finally, the population growth rate is:

\[ n = \dot{N}_t / N_t \]  \hspace{1cm} (2.9)

The above equations describe the basic Solow model.

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\(^3\) In the real world, savings depends on many other factors, such as the age structure of population, income inequality, financial deepening, taxation, macroeconomic and political stability, culture, etc. Setting the saving rate as exogenous is of course an important simplification that we should keep in mind.

\(^4\) Equation (2.7) implicitly postulates the price of the capital good to be the same as that of output. As an example, think that the only output in this economy was potatoes: potatoes can be either consumed or planted (invested) to grow more potatoes. If however total investment included a plough, in that case a given amount of saved output would translate into more or less capital accumulation, depending on how many potatoes would be necessary to buy a plough. In that case, equation (2.7) should be divided by the relative price of capital. The implications in changes in the relative price of capital are examined in chapters 4 and 11.
**Factor prices and factor income shares**

In this model, firms are price-takers both in the product market and in factor markets. When this is so, we know that profit maximization delivers demands for inputs equal to the respective marginal products.

Formally, the profit function of each individual firm each moment in time is given by:

\[
\pi_{it} = Y_{it} - (r_t + \delta)K_{it} - w_iN_{it}
\]  

(2.10)

In this equation, \(w\) is the real wage rate, \(r\) denotes for the real interest rate and \(\delta\) is the rate of depreciation of the capital stock (the sum of these two terms is labelled the “user cost” of capital).

The first order conditions of profit maximization imply:

\[
\frac{\partial \pi_{it}}{\partial N_{it}} = (1 - \beta)K^{\beta}i^{1-\beta} - w_i = (1 - \beta)\frac{Y_{it}}{N_{it}} - w_i = 0
\]

\[
\frac{\partial \pi_{it}}{\partial K_{it}} = \beta K^{\beta-1}i^{1-\beta} - (r_t + \delta) = \beta \frac{Y_{it}}{K_{it}} - (r_t + \delta) = 0
\]

Since firms are all alike, this leads to the following aggregate demand functions for labour and capital, respectively:

\[
w_i = (1 - \beta)\frac{Y_i}{N_i} \quad (2.11)
\]

\[
r_t + \delta = \beta \frac{Y_t}{K_t} \quad (2.12)
\]

**Factor income shares**

Equations (2.11) and (2.12) imply that the income shares of capital and labour, \((r + \delta)K/Y\) and \(wN/Y\), are constant and equal to \(\beta\) and \(1-\beta\), respectively. That is, even though the prices and quantities of capital and labour may vary, changes are such that the
shares of national income paid out to each factor of production remain constant. This is a direct implication of assuming perfect competition and a Cobb-Douglas production function.

**The flow income chart**

The flow income chart of this economy is displayed in Figure 2.2.

*Figure 2.2: The flow income chart in the basic Solow model*

The figure displays the flow of payments in this economy. Factor incomes are paid to households, who spend on consumption and save buying securities from the financial system. The money raised by sales of securities is entirely used by firms in purchases of new capital (there are no losses or frictions in the financial market).

**The Fundamental Dynamic Equation**

To understand how the model works, note that, in the absence of technological progress, the main determinant of per capita output (2.1) is the capital-labour ratio. The later is given each moment in time (pre-determined), by the stock of capital and the size of population.
The capital-labour ratio may change over time, depending on the size of investment relative to capital depreciation and population expansion. Formally, let’s take the time derivative on \( k \), to obtain:

\[
\dot{k} = \left( \frac{\dot{K}N - \dot{N}K}{N^2} \right) = \frac{\dot{K}}{N} - \frac{\dot{N}}{N} k \quad (2.13)
\]

After some substitutions using (2.2) and (2.7)-(2.9), we obtain the so-called Fundamental Dynamic Equation of the Solow model:

\[
\dot{k}_t = s\dot{A}k_t^\beta - (n + \delta)k_t \quad (2.14)
\]

This equation states that the capital-labour ratio (i.e. the amount of capital available per worker) increases with per capita saving (\( sy = s\dot{A}k_t^\beta \)) and decreases with the depreciation rate (\( \delta \)) and the population growth rate (\( n \)).

The term \((n + \delta)\) in (2.14) may be interpreted as the rate of depreciation of the capital-labour ratio: on one hand, the depreciation rate reduces \( k \) by causing the capital stock \( K \) to wear out; on the other hand, population growth results in less capital available for each worker (this negative effect of population growth on capital per worker is often called capital dilution). According to equation (2.14), the change in the capital-labour ratio is positive whenever per capita savings (investment) exceed the depreciation of the capital-labour ratio, and conversely.

A graphical illustration

Figure 2.3 describes the dynamics and the equilibria of the model, as implied by equation (2.14). The uppermost curve is the production function in per capita terms (2.2). The figure also depicts the two terms in the right hand side of (2.14): per capita savings (\( sy \)), and the locus \((n + \delta)k\). The latter is known as the break-even investment line: it depicts, for each level of capital per worker, the exact amount of gross savings that will be necessary to offset the corresponding capital depreciation and capital dilution.
To see how the capital-labour ratio evolves over time, assume first that initially the capital-labour ratio was equal to $k_0$. In that situation, per capita income would be $y_0$, of which $QR$ devoted to consumption and $k_0Q$ devoted to savings. Since in this case per capita savings exceed the “break even investment” (given by $k_0P$), from equation (2.14) it follows that the change in the capital-labour ratio will be positive. In words, since the economy generates savings (and hence new investment) larger than the amount needed to keep the amount of capital per worker constant, the capital-labour ratio will increase.

Figure 2.3. Dynamics and equilibria in the Solow Model

The figure displays the production function in the intensive form, per capita savings and the break even investment line. The steady state occurs when the break even investment line crosses the schedule of per capita savings. If the economy starts out on the left (right) of the steady state, per capita savings will exceed (be less than) the required to keep the capital labour ratio constant, so the capital will increase (decrease).

However, as $k$ progressively approach the point $k=k^*$, the distance between the two locus decreases. The reason is again diminishing returns: since income per worker grows less than proportionally than the stock of capital per worker, savings cannot grow as fast as depreciation. And a moment will come when the two schedules cross each other.
other: at $k=k^*$, the amount of savings per capita is just the needed (but no more) to equip the new entrants into the labour force and to replace the depreciating capital. This is the steady state (equilibrium) of the model\textsuperscript{5}.

\textbf{The steady state}

Formally, the equilibria of the model are obtained solving (2.14) for $\dot{k} = 0$. This equation has only two solutions, the trivial one ($k=0$) and:

$$k^* = \left( \frac{sA}{n + \delta} \right)^{1-\beta}. \quad (2.16)$$

Because the model predicts that the economy converges to the steady state (2.16) from any departing point in its neighbourhood, this equilibrium is said to be stable\textsuperscript{6}.

Substituting (2.16) in (2.2), one obtains the steady state level of per capita income:

$$y^* = A^{1-\beta} \left( \frac{s}{n + \delta} \right)^{\frac{\beta}{1-\beta}}. \quad (2.17)$$

Since parameters $A$, $s$, $n$ and $\delta$ are all constant, equations (2.16) and (2.17) imply that, in the steady state, capital per worker and per capita income are also constant.

Note that this outcome is in full conformity with the CRS property: if labour and capital are set to grow at rate $n$ (for the capital-labour ratio to remain constant), then output will also grow at rate $n$ (implying a constant level of per capita income). This is why the CRS assumption plays a key role in this model.

\textsuperscript{5} We invite the reader to use a similar reasoning to explain why the capital-labour ratio converges to the steady state departing from $k_i$.

\textsuperscript{6} Formally, the equilibrium described by $k^*$ is locally stable because the condition $\frac{\partial \dot{k}}{\partial k} < 0$ holds for any point in its neighbourhood. The reader may verify that the same condition does not hold in the neighbourhood of the trivial steady state, $k=0$. The later is an unstable equilibrium.
2.3. The Solow model and the facts of economic growth

The Solow's quest

Robert Solow developed his famous model with the main purpose being a better understanding of the growth performance of the US economy in the twentieth century. He was particularly interested in explaining the long-run tendency for output and capital to grow at the same rates – a statistical regularity first documented for the U.S. economy by the Russian-American economist, Simon Kuznets. Solow also wrote his model with the so-called “Kaldor’s facts” in mind. These are six “remarkable historical constancies” (empirical regularities) that the British economist Nicholas Kaldor identified as characterizing modern economic growth.

In particular, the Kaldor stylized facts are:

1. Output per worker grows over time at a sustained rate
2. The capital stock per worker grows over time at a sustained rate
3. The capital-output ratio exhibits no clear trend over time;
4. The real return to capital is relatively constant over time;
5. The shares of labour and of capital on national income are roughly constant over time;
6. There are wide differences in the growth rate of productivity across countries.

Kaldor did not claim that these facts hold each moment in time. For instance, per capita output falls during recessions and the real interest rate fluctuates significantly in the short run. Over long periods of time, however, these facts tend to show up in the statistical data, so they provide a natural benchmark to confront models focusing on long term growth.

Confronting the model with the stylized facts

As we just saw, equations (2.16) and (2.17) imply that, in the long run, per capita income and the capital labour ratio do not grow at all. Hence, the basic Solow model does not account for the Kaldor Stylized facts 1, 2, and 6.

To check whether Fact 3 is met, let’s divide (2.16) by (2.17), to obtain output capital ratio in the steady state:

\[
\left( \frac{Y}{K} \right)^* = \frac{n + \delta}{s}
\]  

(2.18)

This ratio is constant, because all three parameters on the right hand side are also constants (in Figure 2.3, this ratio is measured by the slope of the ray OS). Thus, the model accounts for Fact number 3: the long run tendency for capital (K) and output (Y) to grow at the same rate.

An important implication of a constant average product of capital in the steady state is that the interest rate will also be constant in the steady state (remember equation 2.12). Hence, the Kaldor’s stylized fact 4 is also implied by the model. Regarding fact number 5, we already saw that it holds in this model (equations 2.11 and 2.12).

Savings, population and per capita income in the real world

Other implications of the Solow model include the relationship between per capita income and saving rates and population growth rates. According to equation (2.17), countries with high saving rates and with slow population expansions should enjoy higher standards of living than countries with low saving rates and fast population expansions.

Figures 2.3 and 2.4 check how these two predictions of the model go along with the real world facts. The figures plot the level of GDP per capita in the year 2000 with, respectively (i) gross investment as a share of GDP and (ii) population growth rates.
Figure 2.4 reveals a positive correlation between investment rates and per capita incomes. Figure 2.5 reveal a negative correlation between population growth and per capita incomes. Both figures are in broad accordance with the Solow model.

Note, however, that these figures tell us nothing about the direction of causality. For example, it could be that the low saving rates in the poorest countries were explained by the fact that people living at the margin of subsistence cannot afford to save. On the other hand, poorest countries may exhibit faster population expansions simply because they still didn’t make their demographic transition (see Box 2.2). Thus, while this evidence is in accordance to the Solow model, it does not prove that the Solow model is the right one to capture this evidence.

Figure 2.4: Per capita GDP and gross Investment 1950-2000

Figure 2.5: Per capita GDP and population growth rates, 1950-2000

![Graph showing the relationship between average population growth and log of GDP per capita in 2000](image)

Source: same as Figure 2.4. The negative relationship between the growth rate of population and per capita incomes is in accordance to the Solow model.

**Box 2.2 Population dynamics and poverty traps**

The model outlined above postulates a constant rate of population expansion, $n$. The departure from the Malthusian assumption looks sensible to address the growth problem of economies that already made their demographic transition. However, it may be interesting to investigate how the model predictions change when one allows the rate of population expansion to be determined endogenously.

The stylized fact of the demographic transition is that a country’s population is expected to expand at moderate rates both in cases of extreme poverty (high birth rates and high death rates) and when living standards are high (low birth rates and low death rates). Yet in intermediate levels of development, the growth rate of population is expected to be high, because the birth rate is still high while the death rates are already low (Figure 1.6).

Adding a non-linear relationship between population growth and per capita income to the Solow model, one obtains a break-even investment curve that is non-linear too, as depicted in Figure 2.6. In this case, the model may display multiple equilibria.
In the figure, there are four equilibria: origin, L, A, and H. The equilibrium represented by H corresponds to a low population growth rate and a high level of per capita income. Equilibrium A is an intermediate equilibrium, characterised by fast population growth. Equilibrium L is characterised by a low rate of population growth and a low level of per capita income.

*Figure 2.6. Illustration of a Poverty trap*

In the figure the basic Solow model is extended, accounting for a population growth rate that depends non-linearly on per capita income. The possibility of the growth. In this case, there are three non-trivial equilibria, L and H stable and A unstable. Because equilibrium L is dominated by H, it is called a poverty trap.

The equilibria described by H and L are both stable: like in the basic Solow model, departing from any point in its left (right), per capita savings exceed (are less than) the break even investment and hence the capital labour ratio will increase (decrease) until reaching the steady state. The equilibrium described by A is unstable: if, departing from this equilibrium, the capital stock decreases (raises) by a small amount, then the saving rate becomes lower (higher) than the break even investment and the capital labour ratio will decrease (increase), until the low (high) income steady state is
reached. Because the equilibrium L is stable and dominated by another possible outcome (equilibrium H), it is called a “poverty trap”.

A key question in models with multiple equilibria is which equilibrium will prevail. In the case at hand, the stability properties of the model imply that history plays a critical role in equilibrium selection: if an economy starts out in the bad equilibrium L, it will remain in the bad equilibrium. If the economy starts out in the good equilibrium, H, it will remain in the good equilibrium.

Moreover, a policy designed to move the economy out of the poverty trap may fail to do so, unless it is powerful enough to push the economy to the right side of the break even investment line. To see this, suppose that the economy was initially at point L and then it received some external aid to improve its capital stock. As you can easily check in the figure, unless international assistance was large enough to move the economy to the right of the threshold point A, the impact would be merely temporary: the initial rise in output per capita would lead to an acceleration of population expansion, which in turn would require a higher saving rate just to stand still. Since the saving rate is constant, the capital-labour ratio would fall back, driving the economy again to the poverty trap. In that case, the “demographic transition” would not take place because the saving rate was too low to induce the required increase in the capital-labour ratio during the intermediate stage. In the end, the extra capital obtained was “spent” in an increasing population, rather than in improving living standards. If, however, the donation was large enough so that the critical point A was bypassed, then the economy would be able to escape the trap, moving towards the high income steady state, H.

This version of the model suggests that that a relatively small level of international assistance to a poor country trapped in a Post-Malthusian regime will not deliver long-run economic performance. In contrast, a temporary boost in savings may have a long run effects. This conclusion has important policy implications: it suggests that external aid to a developing country will be useless if small and permanent, while it can produce long-lasting effects, even if temporary, provided it is large enough for the economy to overcome the trap. This idea is known as the Big-Push theory.
Note however that a large investment in physical capital is not the only alternative for this economy to escape the trap: if a policy of birth control and family planning was successful in reducing the gap between the birth rate and the death rate in the intermediate stage, then the break-even investment line would eventually approach the straight line, eliminating the multiple equilibria and the poverty trap.

2.4. Transitional Dynamics

An important feature of the Solow model is that, if the economy is not in the steady state, it will converge to the steady state. But the economy will not jump instantaneously from one steady state to the other: since capital accumulation is bounded by the availability of savings, there will be an adjustment period, during which the economy approaches the new steady state.

This point is very important because any real world individual economy that we study for a particular period may be precisely in that adjustment process, rather than in the steady state: e.g. perhaps because the savings rate has risen recently but not yet been reflected fully in higher per capita income. In the light of the Solow model, that economy will be experiencing a transitory growth, reflecting the adjustment of the economy from one steady state to the other. The following sections address specifically the issue of transition dynamics.

What Happens if the Savings Rate Increases?

Figure 2.7 shows how the model adjusts to a rise in the saving rate. When the saving rate rises from \( s_0 \) to \( s_1 \), the curve measuring per capita savings shifts upwards, moving the steady state from \( k_0^* \) to \( k_1^* \).
Thus, the economy engages in an expansion process until the new equilibrium is met. In the long run, the rise in the saving rate produces a *level effect*: that is, in the new steady state, the economy will enjoy a higher *level* of output per worker than in the steady state before. During the transition from one steady state to the other, per capita output increases. But, because of diminishing returns, this increase in output per capita is no more than a *temporary* phenomenon.

It is worth mentioning that the average product of capital (Y/K) in the new steady state is lower than in the old steady state. This can be checked by reference to equation (2.18), where the saving rate enters in the denominator. Visually, in Figure 2.7 you see that the slope of the ray that departs from the origin and crosses the production function at point $I$ is lower than the one corresponding to the ray that crosses the production function at point $0$. Referring to (2.12), this also means that the *interest rate has declined*: intuitively, one consequence of there being relatively more capital per worker available in the economy is that capital becomes relatively cheaper.

The implication of what we just learned is that, in low-income countries with low savings (say $s=10\%$), a growth surge could eventually be achieved by raising the saving rate. However, once the economy reached the new steady state, per capita income would stagnate again. Thus, in order to achieve further increases in output per worker, one would need to raise the saving rate again and again. And clearly, there are limits in exploring this avenue: the maximum level of the savings rate observed in countries in the real world is around 30-35%. Rates much higher than this obviously eat into available consumption and so the current standard of living. The conclusion is that it will be impossible to achieve a continuous growth of per capita income by increasing the saving rate.
Figure 2.7. A higher saving rate raises the steady state level of per capita income but only boosts growth rates temporarily.

The Figure shows how the steady state in the Solow model changes with an exogenous increase in the saving rate. In the new steady state (point 1), the capital labour ratio and per capita output are higher than before, but the average product of capital \((Y/K)\) is lower, due to diminishing returns. In the long term, per capita income is again constant (the increase in the saving rate produced a “level effect”).

**What happens if Population Growth or the Depreciation rate Decline?**

From equation (2.17) we see that a fall in the population growth rate has a similar effect to that of a rise in the savings rate. In terms of the Figure 2.3, the difference is that the change in the steady state will be caused by a downward shift of the break-even investment line. Thus, both output per worker and capital per worker will increase, but this will happen only during the transition from one steady state to the other. In the new steady state, the average product of capital - and interest rate - will be lower than in the initial steady state.

**What happens if the level of technology improves?**
Figures 2.8 describes how the model adjust to a once-and-for-all improvement in Total factor Productivity. In the figure, when TFP changes from $A_0$ to $A_1$, both the production function and the curve measuring per capita savings shift upwards, moving the steady state from $k^*_0$ to $k^*_1$.

Figure 2.8. An increase in technology raises the steady state level of per capita income but leaves the output capital ratio unchanged

In contrast to the case of an increase in the saving rate, in this case there is an initial jump in per capita output: the productivity increase means that more production is achieved out of the initial capital labour ratio.
The paths of output per worker, the average product of capital, and of the interest rate are described in Figure 2.9. As shown in the figure, at the time of the shock ($t_0$), all the three variables jump up. During the adjustment to the new steady state, the average product of capital declines again and so will do the interest rate. In the new steady state (after $t_1$), the average productivity of capital and the interest rate are the same as before the shock (you may confirm this by observing that the long run level of $Y/K$ (equation 2.18), does not depend on the level of technology, $A$).

All in all, the improvement in technology allowed the economy to move from one steady state to a new one with more capital per worker, without any decline in interest rate. Changes in TFP face no diminishing returns.

Figure 2.9. The time paths of per capita output, the capital–labour ratio, the output capital-ratio and the interest rate, following an improvement in technology

Following a technological improvement, per capita income first jumps (with a constant $k$), and then starts growing slowly until the new steady
state is reached. The average product of capital also jumps at the impact, to decline to the level previous to technological change in the new steady state.

2.5 The Golden Rule

The Golden Rule of capital accumulation

In the Solow model, a higher saving rate delivers a higher level of per capita income in the steady state. Does this mean that any increase in the saving rate is desirable? From the household point of view, the answer to this question is negative.

The see this, remember that households care with consumption, not with income. Thus, while a higher saving rate brings a higher per capita income in the steady state, a higher proportion of the later will be foregone consumption. The final impact on consumption will depend on the balance between these two opposing effects.

Mathematically, the saving rate that maximizes the level of per capita consumption in the steady state can be found in the following manner:

$$\max_k c = y - sy, \text{ where subject to } \dot{k} = 0.$$ 

where $c = C/N$ denotes for per capita consumption. In the steady state, this is equivalent to choose $k$ so as to maximise:

$$c = Ak^\beta - (n + \delta)k.$$  \hspace{1cm} (2.19)

The first order condition of this problem leads to:

$$\frac{\partial y}{\partial k} = n + \delta$$ \hspace{1cm} (2.20)

---

8 Note the in the steady state $sy^*=(n+\delta)k^*$. The symbol "*" - which refers to the steady state - is suppressed to simplify the algebra. An alternative avenue is to replace (2.17) in the maximization problem (2.19) and take the derivative in order to $s$, obtaining directly the saving rate that maximizes the per capita consumption in the steady state.
This condition is called the "Golden Rule of Capital Accumulation." It states that the steady state level of per capita consumption is maximised when the slope of the production function (i.e. the marginal product of capital) is equal to the slope of the break-even investment line.

Geometrically, this problem is illustrated in Figure 2.10. The Golden Rule is met at point $k^G$. Whenever the steady state level of capital per worker is below this level (that is, when $k^* < k^G$), the rise in output that results from a possible increase in $k^n$ more than offsets the rise in the amount of savings that is necessary to sustain such equilibrium, implying that more resources are available for consumption. Conversely, whenever the steady-state capital-labour ratio is higher ($k^* > k^G$) the rise in output that would result from any further raise in $k^n$ is less than the required increase in savings, implying that less resources become available for consumption.

Algebraically, the golden rule level of $k^*$ is given by:

$$k^G = \left(\frac{\beta A}{n+\delta}\right)^{1-\beta}.$$ (2.21)

The value of $s$ that turns $k^G$ into a steady state is called the “golden rule” saving rate. Comparing (2.21) with the general solution for steady states (2.16), we conclude that the golden rule saving rate is:

$$s^G = \beta$$ (2.22)

That is, the golden-rule saving rate is equal to the share of capital in total income.

Using taxes and subsidies to achieve the golden rule

---

Now suppose you were a benevolent central planner wanting to maximize the steady state level of per capita consumption of your citizens. How would you achieve this objective?

One possibility would be to use taxes and subsidies. To illustrate this, assume that you had the ability of imposing a tax $\tau$ (subsidy if negative) on production and that tax proceedings were returned to households in the form of a lump-sum transfer, $T$ (e.g., a transfer made after households decided the amount of consumption and savings; if negative, this implies a confiscation of part of the household consumption).

The government budget is assumed to be balanced, that is $T = \tau Y$. The flow income chart of this economy is as described in Figure 2.11.

*Figure 2.10: Illustration of the Golden Rule*
Assuming, as before, a constant saving rate \( s \), total savings in this economy will be given by:

\[
S = s(1 - \tau)Y = \dot{K} + \delta K .
\]

To solve the model in the new version, just note that \( s(1 - \tau) \) shall be used instead of \( s \) in the fundamental dynamic equation. Proceeding as before, the steady state level of per capita income will now be given by:

\[
y^* = A^{\frac{1}{1-\beta}} \left( \frac{s(1 - \tau)}{n + \delta} \right)^{\frac{\beta}{1-\beta}} . \tag{2.23}
\]

The corresponding steady state level of per capita consumption is:

\[
c^* = (1 - s)(1 - \tau)y^* + \frac{T}{N} = [1 - s(1 - \tau)]y^* = [1 - s(1 - \tau)]A^{\frac{1}{1-\beta}} \left( \frac{s(1 - \tau)}{n + \delta} \right)^{\frac{\beta}{1-\beta}} . \tag{2.24}
\]

Maximizing (2.24) with respect to \( \tau \), one obtains an intuitive result:

\[
\tau = 1 - \frac{\beta}{s} . \tag{2.25}
\]
According to (2.25), the golden rule tax rate on output depends on the gap between the actual saving rate $s$ and the golden rule saving rate $\beta$: the optimal tax will be negative (subsidy) if the saving rate falls below the golden rule; it will be positive (tax) if the saving rate is higher than the golden rule; and it will be zero if the saving rate satisfies exactly the golden rule.

**Dynamic inefficiency**

The “golden rule” saving rate is the one that delivers the highest level of per capita consumption in the steady state. This is not to say that the society will always be better off approaching the golden rule.

To analyse this question, consider first the case in which the economy starts out in a steady state on the left of the golden rule (that is, initially $k^* < k^G$). In this case, reaching the golden rule will require the saving rate to increase. In other words, agents will have to sacrifice consumption today to enjoy more consumption in the future. This case raises an important policy question: if the decentralized economy delivers a saving rate that is lower than the golden rule, should a benevolent planner intervene, forcing the economy’ saving rate to increase (for instance, subsidizing savings, as illustrated in equation 2.25)?

As a general principle, as long as saving rates are decided by optimizing agents, any policy altering their choices will make them worse off. So, unless there are good reasons to believe that some kind of impediment prevents consumers from optimally
deciding their saving rates, or that some kind of market failure turns individual decisions socially unacceptable, there will be no case for intervention\textsuperscript{10}.

A different case occurs when the initial steady state lies beyond the golden rule (that is, if initially $k^* > k^G$). In that case, by reducing savings today, consumption would increase both today and tomorrow. Since a “free lunch” is readily available, this case is labelled as “dynamically inefficient”. By contrast, the case in which the saving rate is lower than the golden rule saving rate is "dynamically efficient", because no “free lunch” is readily available. Under dynamic inefficiency, there would be a gain for the society as a whole if a central planner forced the current generation to save less\textsuperscript{11}.

A case of dynamic inefficiency looks at odds with the principle that agents are optimizers: if households were saving too much, they should realize that reducing the saving rate today, they would be increasing their consumption both today and tomorrow. However, at least theoretically, it is possible to figure out cases in which individuals end up saving more than they desire: forced savings occur, for instance, when individuals have income available to spend but no goods to buy (some authors contend that this was the case with widespread rationing in the ex Soviet Union).

\textsuperscript{10} A tricky problem arises, in that private choices impact on the inter-generational distribution of income: in a world where individuals have finite lives, any impatience of the current generation (reflected in low saving rates) may be regarded as a kind of selfish behaviour, which comes at the cost of future generations. In principle, there is nothing wrong with the fact that individuals are impatient: if individuals are willing to pay a cost in terms of future consumption to consume more today, they are in their own right. Still, a planner could see reasons to force the current generation to save more, so as to make future generations better off. Such policy would be equivalent to a transfer between generations, a balance between conflicting interests which economic theory has little to say about. What we know for sure is that such an intervention would not be Pareto improving.

\textsuperscript{11} Phelps, E., 1965. Second Essay on the Golden Rule of Accumulation, American Economic Review 55, 793-814. Formally, a capital path is said to be dynamically inefficient if the path of savings can be changed so as to strictly increase consumption at some point in time without lowering it at any point in time.
Another possibility is individuals with finite lives optimally deciding individual saving rates that prove excessive from the social point of view: for instance, individuals saving for the retirement age may opt to accumulate too much capital, simply because this is the only way of transferring resources to the future, implying very low returns. In that case, the society would be better off if the current generation consumed more today and the future generation transferred some of the implied gain to the current generation in the future, so that both generations would be better off.\textsuperscript{12} Thus, at least theoretically, it is possible to find examples in which forcing the current generation to save more would constitute a Pareto improvement.

In the real world, we observe that the shares of capital in national income vary from 0.3 and 0.4. According to the model formulation, this corresponds to the contribution of capital to output, $\beta$. Since real world saving rates are, in general, lower than 30\%, one may conclude that “dynamic inefficiency” is not at all a general case.

### 2.6. The case with endogenous savings

\textit{An optimal consumption rule}

\textsuperscript{12} In that case, the competitive equilibrium would be dynamically inefficient, with too much capital accumulation.
In the Solow model, the saving rate is assumed exogenous. The neoclassical model can however be extended, to account for the case in which individuals optimally decide their saving rates\(^{13}\).

It is not in the scope of this book to solve complicated dynamic optimization models. So, in the following – and throughout the book – we will refer to a very simple formula, which can be obtained in a simple two-period framework (see Appendix 2.1).

\[
\gamma_t = r_t - \rho 
\]  
(2.26)

In (2.26), \(\gamma_t = \dot{c}/c\) denotes for the growth rate of per capita consumption, and \(\rho\) is the rate of time preference (that is, the rate at which individuals are willing to trade one unit of utility today for one unit of utility in the future).

According to (2.26), as long as the interest rate is higher than the rate of time preference, individuals will optimally increase consumption over time. If however the interest rate falls below the rate of time preference, individuals will optimally reduce consumption over time. When \(r_t = \rho\), the optimal level of consumption will be constant.

**What happens when the rate of time preference decreases?**

To see the implication of replacing an exogenous saving rate by (2.26) in the neoclassical growth model, remember that in the competitive equilibrium the interest rate is determined by the marginal product of capital (equation 2.12). The later, in turn, is a

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\(^{13}\) The problem of how much an individual should save was first addressed in an inter-temporal optimizing framework by a mathematician from Cambridge UK called Frank Ramsey (Ramsey, F., 1928. A mathematical theory of savings. Economic Journal, 38, No 152, 543-559). Ramsey died at the age 26 and his seminal contribution remained obscure for long time by the economics profession, because at that time most economists were not familiar to dynamic optimization. His work was re-discovered four decades later, by Cass and Koopmans, who used it to characterize the optimal saving paths in the context of the neoclassical growth model. (Cass, D. , 1965. Optimum growth in an aggregative model of capital accumulation. Review of Economic Studies 32 (3), 233-240. Koopmans, T., 1965. On the concept of optimal growth. In The Econometric Approach to Development Planning. Rand-McNally, Chicago).
negative function of the capital-labour ratio (as implied by the Law of Diminishing returns).

Referring to Figure 2.12, suppose that initially the rate of time preference was equal to the real interest rate \( r_0 = \rho_0 \), so that per capita consumption was constant over time (in the Solow model, we know that a constant level of per capita consumption holds in a steady-state; so you may interpret this initial situation as corresponding to point 0 in Figure 2.7). In Figure 2.12, this initial situation is described by point A, with the steady state capital labour ratio being equal to \( k_0^* \).

Now suppose that the rate of time preference falls to \( \rho_1 \). This means that individuals will demand a lower return to postpone consumption. Hence, at an unchanged interest rate, savings will increase and the consumption level will fall instantaneously at the time of the shock.

Since more savings translate into more investment, the implication is that the capital labour ratio starts increasing, inducing a temporary growth of per capita income and of consumption\(^\text{14}\). Then, the economy will move slowly, from A to C. As the stock of capital per worker increases, the marginal product of capital declines, and so will do the interest rate. At the time the interest rate becomes equal to the rate of time preference again, the desired consumption becomes constant over time (eq. 2.26) and the process of capital accumulation stops (point C). From C, any further investment in physical capital would bring a return that is lower than the new rate of time preference, so the individual consumer will prefer not to save. In the new steady state, both consumption and per capita income are higher than in the old steady state, but they will be constant again (just like in point 1 of Figure 2.7).

\(^\text{14}\) In terms of equation (2.26), because the rate of time preference falls below the interest rate - determined by the capital-stock - the growth rate of per capita consumption jumps initially to \( \gamma_1 = r_0 - \rho_1 \) (distance AB), declining slowly to zero afterwards.
This example reveals why the neoclassical model cannot generate sustained growth of per capita income, even when savings result from unrestricted optimization decisions: as the stock of capital per worker increases, its marginal product declines and so will do the interest rate. At the time the interest rate equals the discount rate, the desired consumption becomes constant over time (eq. 2.26) and the process of capital accumulation stops\(^{15}\).

\textit{The modified golden rule}

\(^{15}\) Note the similarity with the Malthus model: instead of a model where population expands whenever labour productivity is higher than a subsistence wage, you now have a capital stock that expands whenever its productivity is higher than the rate of time preference. In both cases, the growing process stops because of diminishing returns.
A question that arises is how the endogenous saving rate, determined in a competitive equilibrium where agents face no borrowing constraints (as implied by 2.26) compares to the golden rule saving rate.

Before addressing this question, it is important to note that in the model with endogenous savings, the saving rate is not in general constant along the transition to the steady state (that is, the transition from B to C in Figure 2.12 is not exactly the same as the transition from 0 to 1 in Figure 2.7). When the economy reaches the steady state, however, both the per capita consumption and the per capita income become constant over time, so the saving rate will become constant as well. Thus, steady states are comparable. It is the value of the endogenous saving rate in the steady state that we want to compare with the golden rule of the Solow model.

To obtain the steady state saving rate in the model with endogenous savings, let's first substitute the market interest rate (2.12) in (2.26), to obtain the growth rate of per capita consumption each moment in time (including during the transition dynamics):

$$\gamma_t = \beta \frac{Y_t}{K_t} - \delta - \rho$$

In the steady state the growth rate of consumption is zero and the capital output ratio is given by (2.18). Imposing these conditions in the equation above, and solving for the saving rate, one obtains:

$$s = \beta \frac{n + \delta}{\rho + \delta} < \beta$$ \hspace{1cm} (2.27)

The saving rate (2.27) is often labelled as the “modified golden rule”. It can be shown that the term inside brackets is less than one, so this saving rate is in general lower than the “golden rule” saving rate, $s=\beta$. Intuitively, the impatience reflected in the rate of
time preference means that an infinitely lived consumer will, in general, prefer a steady state consumption level that is lower than the maximum possible\(^{16}\).

**Exogenous or endogenous savings?**

One should keep in mind that the consumption rule (2.26) is obtained assuming that individuals in the economy are all alike and infinitely lived, that their instantaneous utility function is logarithmic, and that they all have full access to a frictionless financial market, whereby they can borrow or lend any amount of income at a given interest rate \( r \).

Acknowledging these assumptions is very important, for qualification purposes. For instance, we all know that financial markets are far from perfect, especially in poor countries. Households without collateral, in particular, will find it difficult to borrow from the banking system against future incomes. And whenever consumers face borrowing constraints, they are fated to consume at most their current income, no matter how impatient they are.

This means that a consumption function depending only on current income and with an exogenous saving rate, as assumed in the basic Solow model, may be fare more appropriate to describe the households’ behaviour in emerging economies with underdeveloped financial markets than rule (2.26).

\(^{16}\) Technically, the saving rate in the “modified golden-rule” is lower than the golden-rule saving rate because \( \rho > n \). The reader is not supposed to guess this. Intuitively, the condition is imposed to prevent consumers from choosing an infinite consumption level financed with an explosive debt (demanding students are invited to read a technical discussion in Romer 1996, p. 40).
2.7. The Solow Residual

In the sections above, we saw that a country’s per capita output can increase both because the capital-labour ratio increases or due to technological change. Empirically, a technique to disentangle the contribution of these two factors to economic growth using observable data is “growth accounting”.

Basically, the technique departs from a log-differentiation of the production function (2.1):

$$\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \beta \frac{\dot{K}}{K} + (1 - \beta) \frac{\dot{N}}{N} = \dot{A} + \beta \dot{K} + (1 - \beta)n$$ (2.28)

This equation states that the growth rate of output equals the growth rate of total factor productivity (“A”) plus a weighted average of the growth rates of physical capital and labour, where the weights are the corresponding elasticities in the production function. In the intensive form, the equivalent is:

$$\dot{y} = \dot{A} + \beta \dot{k}$$ (2.28a)

If factor markets are competitive, as assumed in the Solow model, the parameter $\beta$ shall be equal to the share of capital in national income (remember 2.11 and 2.12). In the real world, it has been observed that the share of capital in national income ranges from 0.3 to 0.4. Thus, based on observed variables one can estimate the unobservable growth rate of technology as the difference between the actual growth of output and the growth implied by factor accumulation (hence the label Solow residual)$^{17}$:

$$\dot{A} = \dot{y} - \beta \dot{K} - (1 - \beta)n$$ (2.29)

By definition, the Solow residual measures the part of actual output growth that is not accounted for by factor accumulation. In the intensity form, the Solow residual is obtained as:

$$\hat{A} = \hat{y} - \beta \hat{k}$$, \hspace{1cm} (2.30)

with $\hat{y} = \dot{Y} - n$ and $\hat{k} = \dot{K} - n$.

As an example, the growth rate of GDP in the US along the first half of the twentieth century was roughly 3% per annum, on average. Its capital stock also expanded at about 3% per annum in that same period, whereas its labour input (hours worked) expanded at only about 1% per annum. Assuming a capital share in national income of one third and a labour share of two thirds, the implied Solow residual is:

$$\hat{A} = 3\% - \left(\frac{1}{3}\right) 3\% - \left(\frac{2}{3}\right) 1\% = 1.3\%$$.

That is, labour and capital together accounted for about 1.7 percent per annum to the total GDP growth of 3 percent. The residual balance of 1.3 percent per annum is accounted for by “technological change”.

Using the intensive form (2.30), the conclusion is even more startling: 2/3 of the change in per capita income is accounted for TFP:

$$\hat{A} = 2\% - \left(\frac{1}{3}\right) 2\% = 1.3\%.$$.

This evidence points to a fundamental limitation of the basic Solow model: by assuming that A is constant, this model ignores a critical ingredient of economic growth: technological progress\(^{18}\).

\(^{18}\) The figures above are from the World Bank, World Development Report 1991, Washington. In his original paper, Solow (1957, op. cit) found out that only one eight of the growth rate of output per hour worked in the U.S. economy along 1909-1949 could be attributed to the increase in capital intensity, $k=K/N$. The remaining seven-eights were attributed to “technical change”.

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Note that computed in such a way, the TFP term captures much more than “technological progress” in narrow sense (changes in the “efficiency” with which the existing technology and inputs are combined): it also captures unmeasured changes in the quality of inputs (skill increments in the labour force, quality of land, climate) and aggregation errors. In any case, the accounting exercise points to the unrealism of assuming that the state of technology is constant over time.

2.8. Discussion: the Solow model and the growth question

The basic Solow model rightly accounts for the role of capital in production and stresses the key role of saving in generating the resources that are necessary to invest in new capital. This, in turn, exactly offsets the diminishing returns due to the growing labour force. The model also provides a sensible story about why historical ratios of capital to output and real interest rates appear to be relatively stable in the long run. Finally, it offers the credible suggestion that countries with high savings rates and low population growth rates should expect higher levels of per capita income in the long run than countries with low saving rates and rapid population expansion.

In its current form the model fails, however, to explain the most basic fact of modern economic growth, that per capita income tends to increase over time: in the Solow model, any growth in per capita income has a merely transitory nature, reflecting the adjustment of the economy from one steady state to the other. The reason is diminishing returns on physical capital.

Of course, continuous growth of per capita income would be obtained in the context of the Solow model if saving rates rose continuously over time. If however sustained growth of per capita income in the real world was really accounted for by successive rises in the saving rate, interest rates should exhibit declining trends. Since this is not a real world fact, the conclusion is that sustained growth of per capita incomes as we observe in the real world is not accounted by successive raises in the saving rates.
The key to overcome this limitation of the basic Solow model follows from our discussion around Figure 2.8: if we allow the level of technology to expand over time, then capital per worker (and per capita output) will increase over time while the capital-output ratio (and the interest rate) remain unchanged. In that case, the model will be consistent with all the stylized facts reported by Kaldor. This extension of the model will be addressed in the next chapter.
Appendix 2.1: The optimal consumption path in a simple 2-period model

Consider an individual who lives only two periods and whose life-time utility function is given by

$$U = u(c_1) + u(c_2)/(1 + \rho), \quad (a2.1)$$

where $u(c_t)$ is a concave function, $c_t$ is real consumption in period $t=1,2$, and $\rho$ is a given rate of time preference.

Assume that this individual has full access to financial markets, so he can borrow or lend any amount of income at the interest rate $r$. His problem is to maximize the lifetime utility function, subject to $c_1 + c_2/(1 + r) = \Omega$, where $\Omega$ denotes lifetime wealth.

From the first order conditions of the maximization problem one obtains the so-called Euler equation:

$$r u'(c_2) = u'(c_2) (1 + \rho)/(1 + r). \quad (a2.2)$$

This equation states that the marginal utility of consumption in the next period must be equal to the marginal utility of consumption in the current period, weighted by the ratio of the rate of time preference to the market discount rate. In other words, this rule implies that the consumption level each period must be such that an extra unit of consumption would make the same contribution to lifetime utility no matter to what period is allocated.

In the main text, we stick with the convenient assumption of logarithmic preferences, that is $u(c_t) = \ln c_t$. In this case, the Euler equation simplifies to:

$$c_2/c_1 = (1 + r)/(1 + \rho). \quad (a2.2)$$

Denoting by $\gamma$ the growth rate of per capita consumption, the later expression becomes equal to:

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$1 + \gamma = \frac{(1 + r)}{(1 + \rho)}$,

Which, by approximation gives:

$\gamma \approx \frac{\dot{c}}{c} = r - \rho$. (a2.3)

This is the optimal consumption rule we will use throughout the book, whenever the saving rate is not assumed exogenous.

To see how the implied saving rate compares to that of the golden rule, let’s consider again equation (2.19). This equation relates the steady state level of per capita consumption to the capital labour ratio, and is depicted in Figure A2.1 (this corresponds to the difference between per capita output and break even investment in Figure 2.10). As we already know, the maximum of this curve corresponds to the golden rule level of $k$, 

$$\frac{\dot{c}}{\delta k} = 0$$

given by (2.21).

The steady state level for capital per worker implied by the Ramsey rule (a2.3) is obtained setting $\dot{c} = 0$ and using (2.12), which gives

$$k^R = \left[ \frac{\beta A}{\rho + \delta} \right]^{\frac{1}{\beta - 1}}$$ (a2.4)

In Figure A2.1, the Ramsey capital-labour ratio is described by a vertical line, as it does not depend on per capita consumption. The optimal level of consumption in model with endogenous savings is less than the golden rule, because the consumer is assumed to be impatient, as reflected in parameter $\rho$.

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19 We know that it is less than the golden rule capital-labour ratio (2.21) because of the transversality condition of the maximization problem, $n < \rho$. 

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Since in the steady state the capital-labour ration is constant (2.14), the saving rate is \( sy = sAk_t^\beta \). Using (a2.14) and (2.2), the saving rate corresponding to the Ramsey rule (2.27) is obtained.

\[
c = \frac{C}{N} = \frac{\dot{c}}{c} = 0
\]

\[
c = Ak^\beta - (n + \delta)k
\]
Key ideas of Chapter 2

- The Solow model explores the assumption of Constant Return to Scale to overcome the limitations imposed by diminishing returns. The model combines the neoclassical assumptions of perfect competition and flexible prices with a Keynesian consumption function, whereby consumption is a linear function of current income.

- The properties of the model are such that in the long run the capital stock will grow at the same rate as population, implying a constant level of per capita income.

- The model predicts that economies with higher saving rates or with lower population growth rates should enjoy a higher level of per capita income in the steady state than economies with low saving rates or with fast population expansions.

- The model basically accords to the real world facts that the capital-output ratio, the interest rate and the shares of labour and capital in per capita income are roughly constant over time, but it fails to deliver the most basic of the stylized facts of economic growth, namely that output per capita (and real wages) tends to grow over time: in this model, any growth in per capita income is merely temporary, reflecting the adjustment in the economy from one steady state to the other.

- In this model, an exogenous improvement in technology leads to a higher level of per capita income without causing any decline in the interest rate. This suggests an avenue to overcome the main limitations of the basic model.

- In the context of the Solow model, there is a saving rate that maximizes the level of per capita consumption in the steady state. This is not to say that a change in the saving rate towards that level will be Pareto improving: this will only be the case if the starting situation is dynamically inefficient.

- Making saving endogenous does not rescue the model from its main limitation, that the long run growth rate of per capita output is zero.

- Measuring a country’ productivity change by the difference between output growth and the “contribution” of inputs to this growth is called “growth accounting”. In general, growth accounting exercises reveal that total factor productivity expands over time. This evidence points to the need of enriching the model so as to allow technology to expand over time.
Problems and Exercises

Key concepts

- Gross investment vs. net investment
- Break even investment
- The Kaldor facts
- Poverty trap
- Dynamic inefficiency (a la Phelps)
- The modified golden rule
- The Solow residual

Essay questions:

a) Explain how the steady state in the Solow model relates to the CRS property
b) To which extent the basic Solow model is capable of describing real world facts?
c) Why can’t the Solow model generate a sustained growth of per capita income?
d) Is the Golden Rule saving rate an optimal saving rate?
Exercises

2.1.

Consider an economy where the aggregate production function $Y = AF(K, N)$ exhibits Constant Return to Scale, positive and decreasing marginal productivity and unit elasticity of substitution between factors. Admitting that the saving rate, the population growth rate, technology and the rate of capital depreciation are all constant and exogenous:

a) Describe in a graph the steady state of this economy. Is it a stationary steady state? Why?

b) Describe in a graph the effects of the following changes on the long run level of per capita output:
   i. An increase in the population growth rate;
   ii. An earthquake that destroys part of the capital stock.

c) Describe the effects of a rise in the saving rate in the time paths of the following variables:
   iii. Capital per worker;
   iv. Per capita income;
   v. Per capita consumption.

d) Describe the effects of a rise in the level of technology on the time paths of the following variables:
   vi. Per capita output;
   vii. Capital per worker;
   viii. Interest rate.

e) In light of the Solow model, is there a tendency for per capita output levels in different countries to approach each other in the long term? Why?

2.2.

Consider an economy where the production function is given by: $Y_t = 20K_t^{1/3}N_t^{2/3}$, where $N_t$ is the number of workers in period $t$. In this economy, 25% of income is saved, the labour force grows at 2.5% and capital depreciates at 2.5%. We also know that in this economy there is perfect competition, and wages and prices are fully flexible.
2.3. Consider an economy where the production function is given by \( Y = AK^{0.5}N^{0.5} \).

a) Write down the main assumptions of the Solow model and find out the expression that describes the dynamics of the capital-labour ratio.

b) Assume that: \( A = 1 \), the saving rate is 20%, the capital depreciation rate is 8% and that the population grows at 2% per year. Find out the steady state levels of: output per capita, capital per worker, real wages and interest rate.

c) What is the growth rate of output in the steady state?

d) Suppose that the productivity level increased to \( A = 2 \). Describe the impact on per capita output, wages and the real interest rate.

2.4. Consider two economies, A and B, sharing the same technology, given by \( Y = K^{0.5}N^{0.5} \). Assume that the saving rates in A and B are, respectively 10% and 20% and that the sum \( n + \delta \) is equal to 10% in both countries.

a) Suppose that initially the capital-labour ratio was equal to 2 in both countries. What will be the corresponding initial levels of per capita consumption and per capita income?

b) Starting from the position described in a), compare the evolution of per capita income in both economies as time goes by. Discuss.

2.6. Consider an economy where the production function is given by \( Y_t = 0.2K_{t}^{1/3}N_{t}^{2/3} \). In that economy, 25% of income is saved, capital depreciation is 5% and population is constant and equal to 1000 inhabitants.

a) Find out the steady state values of per capita income, per capita consumption, real wages and the interest rate.
b) Find out the saving rate that would maximize $C/N$ in steady state, where $C$ is consumption. Illustrate with the help of a graph the adjustment dynamics of $Y/N$ and $C/N$ admitting that the saving rate actually changed to that level.

c) Suppose you were a benevolent planner who could coerce firms to pay a tax on production $\tau$, and transfer the proceedings consumers without distorting the saving-consumption decisions. What would be the level of $\tau$ if you wanted to maximize the steady state level of per capita consumption?

2.6.

Consider an economy where the labour income share is 75%. What would be the Solow residual, if both output and capital were growing at 3% per year and the labour force was expanding at 1.5%?